SAMIULLAH

Based on National Curriculum 2022-23

Textbook of

Mathematics

GRADE

10

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A Textbook of Mathematics for Grade 10
Based on National Curriculum of Pakistan (NCP) 2022-23

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and promotional purposes, benefiting the public interest.

PREFACE

Textbook of Mathematics for Grade 10 has been developed by NBF according to the National Curriculum of Pakistan 2022- 2023. The aim of this experimentation skills is to enhance learning abilities through inculcation of logical thinking in learners, and to develop higher order thinking processes by systematically building upon the foundation of learning from the previous grades. A key emphasis of the present experimentation skills is on creating real life linkages of the concepts and methods introduced. This approach was devised with the intent of enabling students to solve daily life problems as they go up the learning curve and for them to fully grasp the conceptual basis that will be built upon in subsequent grades.

After amalgamation of the efforts of experts and experienced author, this book was reviewed and finalized after extensive reviews by professional educationists. Efforts were made to make the contents student friendly and to develop the concepts in interesting ways.

The National Book Foundation is always striving for improvement in the quality of its books. The present book features an improved design, better illustration and interesting activities relating to real life to make it attractive for young learners. However, there is always room for improvement and the suggestions and feedback of students, teachers and the community are most welcome for further enriching the subsequent editions of this book.

May Allah guide and help us (Ameen).

Dr. Kamran Jahangir
Managing Director

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01

Complex Numbers

After studying this unit students will be able to:

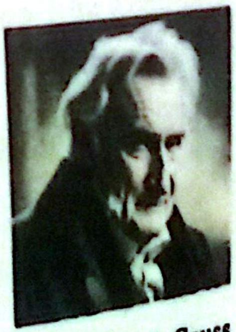
- Recognize complex numbers, conjugate and modulus of a complex numbers.
- Apply basic operations on complex numbers.
- Define commutative laws, associative laws and distributive laws for complex numbers.
- Recognize additive and multiplicative identity and find additive and multiplicative inverse of complex numbers.
- Find real and imaginary parts of $\left(\frac{x_1+iy}{x_2+iy}\right)^n$ for $n=\pm1, \pm2, \ldots$
- Find solution of equations having complex values.
- Apply the geometric interpretation of a complex numbers.
- Solve daily life problems involving complex numbers.

Until the sixteenth century, mathematicians were puzzled by square roots of a negative numbers. For example, the solutions of $x^2-2=0$ are $\sqrt{2}$, and $-\sqrt{2}$. But what are the solution of $x^2+2=0$? To find the solution of this equation, imaginary numbers were defined.

A French mathematician René Descartes was the first to emphasize the imaginary nature of numbers, positing that "one can imagine as many (numbers) as already mentioned in each equation, but sometimes, there is no quantity that matches what we imagine." However, in 16th century an Italian mathematician Gerolamo Cardano, proved that having a negative term inside a square root can lead to the solution of an equation. Later, in the 18th century, mathematician Carl Friedrich Gauss consolidated Cardano's premises, in addition to developing a treatise on complex numbers in a plane and thereby established the modern bases of the term.

René Descartes

Gerolamo Cardano



Carl Friedrich Gauss

Imaginary and Complex Numbers

Negative numbers do not have square roots in the real number system. However, a larger number system that contains the real number system is designed so that negative numbers do have square roots. This system is called the complex number system and it makes use of a number that is a square root of -1. We call this new number i.

Let us find the solution of:

$$x^2 + 1 = 0 \qquad \text{(i)}$$

From (i), $x^2 = -1$ which implies $x = \pm\sqrt{-1} = \pm i$

The number i is the solution of $x^2 + 1 = 0$, and is defined as:

$i = \sqrt{-1}$ where $i^2 = -1$

Since the square of a real number is not negative, therefore i is not a real number and is called the imaginary unit.

Using i we can define square root of any negative number.

Interesting Information

Girolamo Cardano began his career as a doctor and studied, taught and wrote mathematics as a side line. He held important positions at The University of Pavia and Bologna in Italy and wrote many works on arithmetic, astronomy, physics and medicine.

To express the roots of negative numbers in terms of i, we can use the fact that in the complex numbers, $\sqrt{-x} = \sqrt{-1}\sqrt{x} = i\sqrt{x}$. When x is a positive real number and i is the imaginary number, the number $i\sqrt{x}$ is called a pure imaginary number.

Example: Express each number in terms of i.

a. $\sqrt{-5}$ b. $\sqrt{-9}$ c. $\sqrt{-17}$

d. $-\sqrt{-64}$ e. $3i \times 5i$ f. $\sqrt{-5} \times \sqrt{-20}$

Solution:

a. $\sqrt{-5} = \sqrt{-1 \times 5} = \sqrt{-1} \times \sqrt{5} = i\sqrt{5} = \sqrt{5}i$

b. $\sqrt{-9} = \sqrt{-1 \times 9} = \sqrt{-1} \times \sqrt{9} = i \times 3 = 3i$

We prefer to write ia as ai, where a is rational number.

c. $-\sqrt{-17} = -\sqrt{-1 \times 17} = -\sqrt{-1} \times \sqrt{17} = -i\sqrt{17} = -\sqrt{17}i$

d. $-\sqrt{-64} = -\sqrt{-1 \times 64} = -\sqrt{-1} \times \sqrt{64} = -i\sqrt{64} = -i8 = -8i$

e. $3i \times 5i = 15i^2 = 15 \times (-1) = -15$ where $(i^2 = -1)$

f. $\sqrt{-5} \times \sqrt{-20} = \sqrt{-1 \times 5} \times \sqrt{-1 \times 20} = (\sqrt{-1} \times \sqrt{5}) \times (\sqrt{-1} \times \sqrt{20})$

$= i\sqrt{5} \times i\sqrt{20} = i^2\sqrt{5 \times 20} = -1\sqrt{100} = -10$

Imaginary Number

An imaginary number is a number that can be written $a + bi$, where a and b are real numbers, $b \neq 0$.

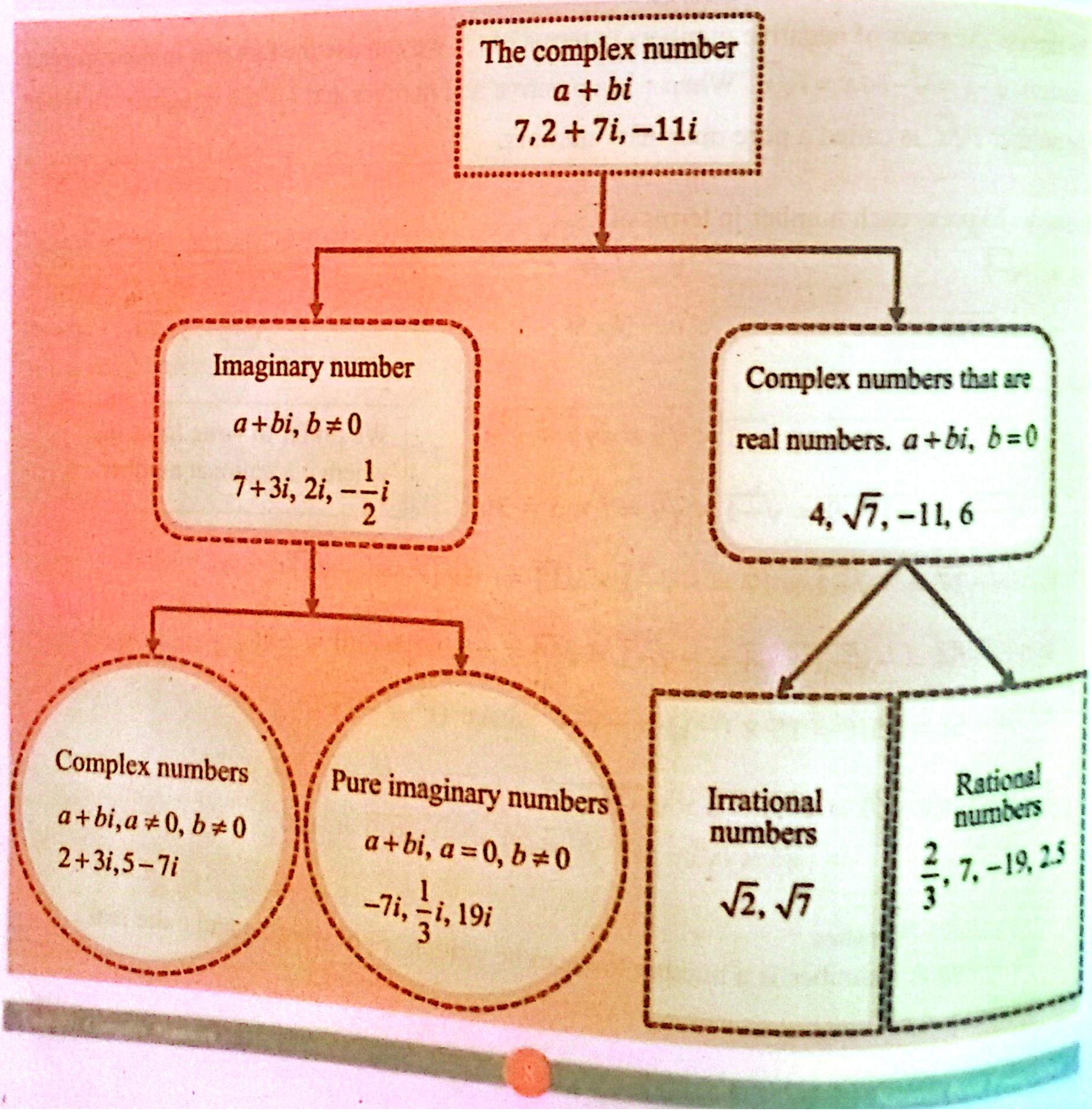
Complex Numbers

When a and b are real numbers and b is allowed to be 0, the number $a+bi$ is said to be complex

A complex number is any number that can be written in the form of $a+bi$, where a and b are any real numbers and both a and b can be zero.

Key Facts

- The numbers $2+5i$, $7-11i$, $18i$ are imaginary as $b \neq 0$ in these numbers. But they are also complex numbers.
- Complex number like $5i$ in which $a=0$ and $b \neq 0$ are imaginary number with no real part. Such numbers are called pure imaginary numbers.
- As 6 can be written as: $6+0i$. Therefore, every real number is a complex number with 0 as imaginary part.
- The set of complex numbers is denoted by C and R ⊆ C.



Addition and Subtraction of Complex Numbers

The complex numbers obey the commutative, associative and distributive law. Thus, we can add and subtract them as we do binomials. Moreover, the sum and difference of two complex numbers is also a complex number. If $z_1 = a + bi$ and $z_2 = c + di$ are two complex numbers then their sum and difference is defined as:

Sum: Add real part in real part and imaginary part in imaginary part.

$$z_1 + z_2 = (a + bi) + (c + di) = (a + c) + (b + d)i$$

Difference: Subtract real part in real part and imaginary part in imaginary part.

$$z_1 - z_2 = (a + bi) - (c + di) = (a - c) + (b - d)i$$

Example: Simplify:

a. $(7+5i)+(4+3i)$

b. $(5+6i)-(7-4i)$

Solution:

a. $(7+5i)+(4+3i)$

$=7+5i+4+3i$

$=7+4+5i+3i$ (combine real and

$=11+8i$ imiginary parts)

b. $(5+6i)-(7-4i)$

$=5+6i-7+4i$

$=5-7+6i+4i$ (combining real and

$=-2+10i$ imiginary parts)

Multiplication of Complex Numbers

If $a + bi$ and $c + di$ are two complex numbers then their product can be found by using distributive law.

$$(a + bi)(c + di) = a(c + di) + bi(c + di) = ac + adi + bci + bdi^2$$

$$= ac + adi + bci - bd \quad = (ac - bd) + (ad + bc)i \quad \text{..... (as } i^2 = -1)$$

Therefore, the product of two complex numbers is also a complex number.

For complex numbers, the property $\sqrt{a}\sqrt{b}=\sqrt{ab}$ does not hold in general, but it does when a is negative and b is a non-negative number.

To express square root of negative numbers, we first express them in terms of i. For example:
$\sqrt{-3}\times\sqrt{-7}=\sqrt{-1}\sqrt{3}\times\sqrt{-1}\sqrt{7}=i\sqrt{3}\times i\sqrt{7}=i^2\sqrt{21}=-\sqrt{21}$: correct.

But, $\sqrt{-2}\times\sqrt{-5}=\sqrt{-(2)(-5)}=\sqrt{10}$ is wrong.

Example: Multiply and simplify:

a. $\sqrt{-2}\sqrt{-11}$

b. $-2i.5i$

c. $-3i(2-8i)$

d. $\sqrt{-9}\sqrt{-36}$

e. $(3-2i)(1+5i)$

f. $(5-i)(9-3i)$

Solution:

a. $\sqrt{-2}\sqrt{-11}=\sqrt{-1}\sqrt{2}\times\sqrt{-1}\sqrt{11}=i\sqrt{2}\times i\sqrt{11}=i^2\sqrt{2\times 11}=-\sqrt{22}$

b. $-2i.5i=-10(i^2)=-10(-1)=10$ (as $i^2=-1$)

c. $-3i(2-8i) = -6i + 24(i^2) = -6i + 24(-1) = -6i - 24 = -24 - 6i$

d. $\sqrt{-9}\sqrt{-36} = \sqrt{-1\times 9}\sqrt{-1\times 36} = \sqrt{-1}\sqrt{9} \times \sqrt{-1}\sqrt{36} = i3 \times i6 = 18i^2 = -18$

e. $(3-2i)(1+5i) = 3 + 15i - 2i - 10i^2 = 3 + 13i - 10(-1) = 13 + 13i$

f. $(5-i)(9-3i) = 45 - 15i - 9i + 3i^2 = 45 - 24i + 3(-1) = 42 - 24i$

Powers of *i*

We now want to simplify certain expressions involving higher powers of *i*. simplifying powers of *i* can then be done by using the fact that $i^2 = -1$ and expressing the given power of *i* in terms of i^2. Consider:

$i^3 = i^2.i = (-1).i = -i$

$i^5 = i^4.i = (i^2)^2.i = (-1)^2.i = i$

$i^7 = i^6.i = (i^2)^3.i = (-1)^3.i = -i$

$i^4 = (i^2)^2 = (-1)^2 = 1$

$i^6 = (i^2)^3 = (-1)^3 = -1$

$i^8 = (i^2)^4 = (-1)^4 = 1$

The powers of *i* cycle themselves through the values of *i*, −1, −*i* and 1.

Example: Simplify:

a. i^{11} b. i^{22} c. i^{61} d. i^{100}

Solution:

a. $i^{11} = (i^2)^5 i = (-1)^5 i = (-1)i = -i$

b. $i^{22} = (i^2)^{11} = (-1)^{11} = -1$

c. $i^{61} = (i^2)^{30} i = (-1)^{30} i = (1)i = i$

d. $i^{100} = (i^2)^{50} = (-1)^{50} = 1$

Imaginary numbers appear in the field of engineering and physical sciences.

Example: Write the expressions in the form of $a+ib$.

a. $2+i^3$ b. $19+7i^2$ c. $i^{12}+i^{15}$ d. $4i^3+8i^6$

Solution:

a. $2+i^3 = 2+i^2 i = 2+(-1)i = 2-i$

b. $19+7i^2 = 19+7(-1) = 19-7 = 12+0i$

c. $i^{12}+i^{15} = (i^2)^6 + (i^2)^7 i = (-1)^6 + (-1)^7 i$
$= 1+(-1)i = 1-i$

d. $4i^3+8i^6 = 4i^2 i + 8(i^2)^3 = 4(-1)i + 8(-1)^3$
$= -4i + 8(-1) = -4i - 8 = -8 - 4i$

Conjugates of Complex Numbers

Conjugate of complex number is a complex number obtained by changing the sign of the imaginary part. In simple words, conjugate of a complex number is a number that has the same real part as the original complex number, and the imaginary part has the same magnitude but opposite sign.

The conjugate of a complex number $a + ib$ is $a - ib$.

For example, the conjugate of $5+6i$ is $5-6i$ and that of $3-11i$ is $3+11i$.

Commonly, the complex number is denoted by *z* and its conjugate by $\overline{z}$.

For example, if $z = 11+7i$ then $\overline{z} = \overline{11+7i} = 11-7i$

Example: Find the conjugate:

a. $5+9i$ b. $13-5i$ c. $-5-7i$ d. $11i$

Solution:

a. $5+9i$ the conjugate is $5-9i$ b. $13-5i$ the conjugate is $13+5i$

c. $-5-7i$ the conjugate is $-5+7i$ d. $11i$ the conjugate is $-11i$

Division of Complex Numbers

Dividing complex numbers is a little more complicated than addition, subtraction, and multiplication of complex numbers because it is difficult to divide a number by an imaginary number.

For dividing complex numbers, we need to find a term by which we can multiply the numerator and the denominator that will eliminate the imaginary part of the denominator so that we end up with a real number in the denominator.

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are the two complex numbers, then division of z_1 by z_2 is written as:

$$\frac{z_1}{z_2} = \frac{x_1+iy_1}{x_2+iy_2} \quad \ldots\ldots \text{(i)}$$

To eliminate the imaginary part in the denominator, we multiply the numerator and denominator of (i) by the conjugate of $x_2 + iy_2$ i.e. by $x_2 - iy_2$

$$\frac{z_1}{z_2} = \frac{x_1+iy_1}{x_2+iy_2} \times \frac{x_2-iy_2}{x_2-iy_2} \quad \ldots\ldots \text{(ii)}$$

After that, we simplify the expression (ii).

Steps for Dividing Complex Numbers

To divide the two complex numbers, follow these steps:

- First, calculate the conjugate of the complex number that is at the denominator of the fraction.
- Multiply the conjugate with the numerator and the denominator of the complex fraction.
- Apply the algebraic identity $(a+b)(a-b) = a^2 - b^2$ in the denominator and substitute $i^2 = -1$.
- Apply the distributive property in the numerator and simplify.
- Separate the real part and the imaginary part of the resultant complex number.

Example: Divide and simplify to the form $a+ib$.

a. $\frac{-7+3i}{2-5i}$ b. $\frac{11+5i}{10i}$ c. $\frac{1-7i}{3+4i}$

Solution:

a. $\frac{-7+3i}{2-5i}$

$= \frac{-7+3i}{2-5i} \times \frac{2+5i}{2+5i}$ multiplying and dividing by conjugate of denominator

$$=\frac{(-7+3i)(2+5i)}{(2-5i)(2+5i)}=\frac{-14-35i+6i+15i^2}{(2)^2-(5i)^2}$$

$$=\frac{-14-35i+6i+15(-1)}{4-25i^2}=\frac{-14-29i-15}{4-25(-1)}$$

$$=\frac{-29-29i}{4+25}=\frac{-29-29i}{29}=\frac{29(-1-i)}{29}$$

$$=-1-i$$

b. $\frac{11+5i}{10i}$

$\frac{11+5i}{10i}\times\frac{-10i}{-10i}$ multiplying and dividing by conjugate of denominator $(10i=0+10i)$

$$=\frac{(11+5i)(-10i)}{(10i)(-10i)}=\frac{-110i-50i^2}{-100i^2}=\frac{-110i-50(-1)}{-100(-1)}$$

$$=\frac{-110i+50}{100}=\frac{50-110i}{100}=\frac{50}{100}-\frac{110}{100}i$$

$$=\frac{1}{2}-\frac{11}{10}i$$

c. $\frac{1-7i}{3+4i}$

$=\frac{1-7i}{3+4i}\times\frac{3-4i}{3-4i}$ multiplying and dividing by conjugate of denominator

$$=\frac{(1-7i)(3-4i)}{(3+4i)(3-4i)}=\frac{3-4i-21i+28i^2}{(3)^2-(4i)^2}=\frac{3-25i+28(-1)}{9-16i^2}$$

$$=\frac{3-25i-28}{9-16(-1)}=\frac{-25-25i}{9+16}=\frac{-25-25i}{25}=\frac{25(-1-i)}{25}$$

$$=-1-i.$$

Exercise 1.1

1. Simplify and express in terms of iy.
    - i. $\sqrt{-3}$
    - ii. $6\sqrt{-4}$
    - iii. $\sqrt{-\frac{4}{9}}$
    - iv. $-\sqrt{-20}$
    - v. $4-\sqrt{-60}$
    - vi. $\sqrt{-8}\sqrt{-2}$
2. Simplify:
    - i. $(4-i)+(5+5i)$
    - ii. $(7-6i)-(5-6i)$
    - iii. $(-2+8i)-(7+3i)$
    - iv. $(4-2i)-(5-2i)$
    - v. $(2+4i)(1+2i)$
    - vi. $(1-4i)(2-3i)$
    - vii. $-8i(2-2i)$
    - viii. $(3+2i)^2$
    - ix. $(3-6i)(3+6i)$

x. $(-5-3i)^2$ xi. $(1+\sqrt{2}i)(1-\sqrt{3}i)$ xii. $(\sqrt{2}+i)(\sqrt{2}-i)$

Simplify:

i. i^9 ii. i^{13} iii. i^{28}

iv. $(-i)^{21}$ v. $(3i)^3$ vi. $(-2i)^4$

4. Simplify in the form of $a+bi$.

i. $9+i^6$ ii. $-17+i^5$ iii. i^4-13i

iv. i^5+21i v. i^5+i^7 vi. $i^{74}-i^{100}$

5. Divide and simplify in the form of $a+bi$.

i. $\dfrac{3}{4-i}$ ii. $\dfrac{3i}{6+5i}$ iii. $\dfrac{3-i\sqrt{5}}{3+i\sqrt{5}}$ iv. $\dfrac{2+7i}{5i}$

v. $\dfrac{4+5i}{4-5i}$ vi. $\dfrac{3+2i}{2+i}$ vii. $\dfrac{5+i}{1+2i}$ viii. $\dfrac{a+ib}{a-ib}$

ix. $\dfrac{1+i}{(1-i)^2}$ x. $\dfrac{(2+2i)^2}{(1+i)^2}$

Developing skilled knowledge

- ✓ What is an imaginary number?
- ✓ Is an imaginary number real number?
- ✓ Provide examples of numbers that are pure imaginary numbers.
- ✓ What is the pattern formed by the powers of i?
- ✓ Can we say product of $(2-3i)(2-3i)$ is imaginary?
- ✓ If conjugate of z is $\overline{z}$, then is $z\overline{z}$ a real number?

Properties of Complex Numbers

Let $z=a+bi$, $w=c+di$ and $v=e+fi$ be any three complex numbers, then the following properties hold.

- $z+w=w+z$ (commutative law of addition)
- $z\times w== w\times z$ (commutative law of multiplication)
- $z+0=0+z=z$ (existence of additive identity where $0=0+0i$)
- $z\times 1=1\times z=z$ (existence of multiplicative identity where $1=1+0i$)
- $(z+w)+v=z+(w+v)$ (associative law of addition)
- $(z\times w)\times v=z\times(w\times v)$ (associative law of multiplication)
- $z(w+v)=zw+zv$ (distributive law of multiplication over addition)
- $z(w-v)=zw-zv$

Challenge

Prove the above properties by taking, $z=1+2i$, $w=3-i$ and $v=2+2i$

Additive Inverse of Complex Numbers

If $z = a + bi$ is a complex number then its additive inverse is $-z = -a - bi$.

In the same way, additive inverse of $-z = -a - bi$ is $z = a + bi$.

Sum of a complex number and its additive inverse is zero.

$$z + (-z) = (a + bi) + (-a - bi) = a + bi - a - bi = 0 = 0 + 0i$$

Example:

Find the additive inverse of $3 - 2i$.

Solution:

Let $a + bi$ be the additive inverse of $3 - 2i$.

Then by additive inverse rule:

$$(3 - 2i) + (a + bi) = 0 + 0i$$
$$(3 + a) + (-2i + bi) = 0 + 0i$$
$$(3 + a) + (-2 + b)i = 0 + 0i$$

Comparing real and imaginary parts of both sides, we get:

$3 + a = 0; -2 + b = 0 \quad \Rightarrow \quad a = -3; b = 2$

Thus, additive inverse of $3 - 2i$ is $-3 + 2i$.

Equality of Complex Numbers

If two complex numbers $a + bi$ and $c + di$ are equal then:

$$a = c \text{ and } b = d$$

Direct Method:

We can apply the rule that additive inverse is the opposite sign of $3 - 2i$ that is $-3 + 2i$.

Similarly the additive inverse of $-7 + 11i$ is $7 - 11i$.

Multiplicative Inverse of Complex Numbers

The multiplicative inverse of $a + bi$ is $\frac{1}{a+bi}$.

Example: Find the multiplicative inverse of $7 - 5i$.

Solution: The multiplicative inverse of $7 - 5i$ is $\frac{1}{7-5i}$.

Now, simplifying the expression $\frac{1}{7-5i} = \frac{1}{7-5i} \times \frac{7+5i}{7+5i}$ multiplying by conjugate

$$= \frac{7+5i}{(7)^2 - (5i)^2} \Rightarrow \frac{7+5i}{49 - 25i^2} \Rightarrow \frac{7+5i}{49 - 25(-1)}$$
$$= \frac{7+5i}{49+25} = \frac{7+5i}{84} = \frac{7}{84} + \frac{5}{84}i$$
$$= \frac{1}{12} + \frac{5}{84}i$$

Verification: $(7 - 5i) \times \left(\frac{1}{12} + \frac{5}{84}i\right) = (7 - 5i)\left(\frac{7+5i}{84}\right)$

$$= \frac{(7)^2 - (5i)^2}{84} = \frac{49 - 25i^2}{84} = \frac{49 - 25(-1)}{84}$$
$$= \frac{49+25}{84} = \frac{84}{84} = 1$$

Key Fact

The product of a complex number and its multiplicative inverse is 1.

Properties of Conjugate of Complex Number

a. $z\bar{z}$ is a real number.

Consider $z = 7 - 6i$ then $\bar{z} = \overline{7 - 6i} = 7 + 6i$

$z\bar{z} = (7 - 6i)(7 + 6i) = (7)^2 - (6i)^2 = 49 - 36i^2$

$= 49 - 36(-1) = 49 + 36 = 85$

b. $\overline{z_1 z_2} = \bar{z}_1 . \bar{z}_2$

Consider $z_1 = 1 - 2i$ and $z_2 = 4 + 3i$, then:

$\bar{z}_1 = 1 + 2i$ and $\bar{z}_2 = 4 - 3i$

Now, LHS $= z_1 . z_2 = (1 - 2i)(4 + 3i) = 4 + 3i - 8i - 6i^2$

$= 4 - 5i - 6(-1) = 4 - 5i + 6 = 10 - 5i$

$\overline{z_1 . z_2} = \overline{10 - 5i} = 10 + 5i$

RHS $= \bar{z}_1 . \bar{z}_2 = (1 + 2i)(4 - 3i) = 4 - 3i + 8i - 6i^2$

$= 4 + 5i - 6(-1) = 4 + 5i + 6 = 10 + 5i$

Hence, $\overline{z_1 . z_2} = \bar{z}_1 . \bar{z}_2$

c. $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

Consider $z_1 = -1 + 5i$ and $z_2 = 2 + i$

Then, $\bar{z}_1 = -1 - 5i$, $\bar{z}_2 = 2 - i$

Taking $\frac{z_1}{z_2} = \frac{-1+5i}{2+i} = \frac{-1+5i}{2+i} \times \frac{2-i}{2-i} = \frac{(-1+5i)(2-i)}{(2+i)(2-i)}$

$= \frac{-2+i+10i-5i^2}{(2)^2-(i)^2} . = \frac{-2+11i-5(-1)}{4-(-1)} = \frac{-2+11i+5}{4+1} = \frac{3+11i}{5} = \frac{3}{5} + \frac{11}{5}i$

LHS $= \overline{\left(\frac{z_1}{z_2}\right)} = \frac{3}{5} - \frac{11}{5}i$

RHS $= \frac{\bar{z}_1}{\bar{z}_2} = \frac{-1-5i}{2-i} = \frac{-1-5i}{2-i} \times \frac{2+i}{2+i} = \frac{(-1-5i)(2+i)}{(2-i)(2+i)}$

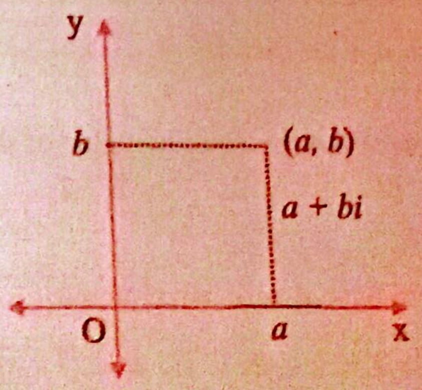
$= \frac{-2-i-10i-5i^2}{(2)^2-(i)^2} = \frac{-2-11i-5(-1)}{4-(-1)} = \frac{-2-11i+5}{4+1} = \frac{3-11i}{5} = \frac{3}{5} - \frac{11}{5}i$

Hence, $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

The complex number $a + ib$ can be written as (a, b), a is a real part and b is imaginary part.

Graphical Representation of Complex Number

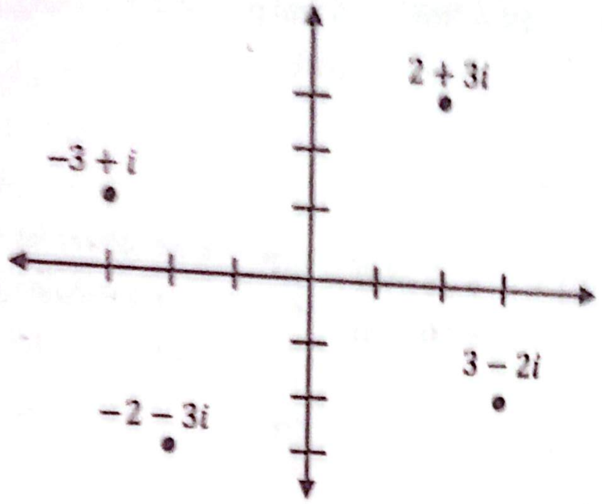
In complex number $z = a + ib$, there are two parts of z, a is a real part and b is an imaginary part associated with i. On the graphical representation of complex number z, the real part of z is taken along x-axis and imaginary part is taken along y-axis and considered as to real axis and imaginary axis respectively. The coordinate plane itself is called the complex plane or z-plane.



Example: Represent the numbers in the complex plane.

$2+3i,\ 3-2i,\ -3+i,\ -2-3i$

Solution:



Key Fact:

The complex plane is also called "The Argand Diagram" after the French-Swiss mathematician Jean Robert Argand (1768-1822) who was the first to represent a complex number geometrically as point in the plane.

We observe that the complex numbers appear in all the four quadrants due to the negative and positive signs of their real and imaginary parts.

Key Fact:

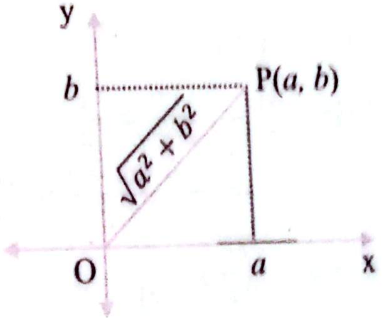
If $x=0$, then $x+yi=yi$ is an imaginary number located on vertical axis.
If $y=0$, then $x+yi=x$ is a real number located on horizontal axis.

Absolute Value or Modulus of a Complex Number

Let $z=(a,b)=a+ib$ be a complex number.
The absolute value or modulus of z, denoted by $|z|$ is defined as:

$$|z|=\sqrt{a^2+b^2}$$

which is a real number. From the diagram, it is clear that the modulus of a complex number is the distance OP of complex number from the origin.



Key Fact:

If, $z=a+ib$, then $|z|=\sqrt{(\text{real part})^2+(\text{img part})^2}$ = Real Number

Example: Find the absolute value of the complex numbers:

a. $5i$ b. 2 c. $4-7i$ d. $-3+2i$

Solution:

a. $z=5i=0+5i=|z|=\sqrt{0^2+5^2}=\sqrt{25}=5$

b. $z=2=2+0i=|z|=\sqrt{2^2+0^2}=\sqrt{4}=2$

c. $z = 4 - 7i \Rightarrow |z| = \sqrt{(4)^2 + (-7)^2} = \sqrt{16 + 49} = \sqrt{65}$

d. $z = -3 + 2i \Rightarrow |z| = \sqrt{(-3)^2 + (2)^2} = \sqrt{9 + 4} = \sqrt{13}$

Properties of Modulus

If z is a complex number, then following properties holds:

a. $|z| = |-z| = |\bar{z}| = |-\bar{z}|$ *b.* $\bar{z}.z = |z|^2$

a. Consider a complex number $z = 2 - 5i$

$|z| = \sqrt{(2)^2 + (-5)^2} = \sqrt{4 + 25} = \sqrt{29}$......(i)

$-z = -(2 - 5i) = -2 + 5i$

$|-z| = \sqrt{(-2)^2 + (5)^2} = \sqrt{4 + 25} = \sqrt{29}$......(ii)

$-\bar{z} = -2 - 5i$

$|\bar{z}| = \sqrt{2^2 + 5^2} = \sqrt{4 + 25} = \sqrt{29}$ ... ...(iii)

$\bar{z} = \overline{2 - 5i} = 2 + 5i$

$|-\bar{z}| = \sqrt{(-2)^2 + (-5)^2} = \sqrt{4 + 25} = \sqrt{29}$... (iv)

From (i), (ii), (iii), and (iv): $|z| = |-z| = |\bar{z}| = |-\bar{z}|$

b. Consider $z = 2 - 5i$

$\bar{z} = 2 + 5i$ then,

$\bar{z}.z = (2 + 5i)(2 - 5i)$

$= (2)^2 - (5i)^2 = 4 - 25i^2$

$= 4 - 25(-1) = 4 + 25$

$= 29$ .....(i)

$|z| = \sqrt{(2)^2 + (-5)^2}$

$= \sqrt{4 + 25} = \sqrt{29}$

$|z|^2 = (\sqrt{29})^2 = 29$ ... ... (ii)

From (i) and (ii),

$\bar{z}.z = |z|^2$

Example: Find the real and imaginary parts of the complex numbers:

a. $(2-3i)^2$ b. $(1+2i)^{-2}$ c. $\frac{(\sqrt{2}+i\sqrt{3})^2}{1+i}$ d. $\frac{(1-i)^2}{(1+i)^2}$

Solution:

a. $(2-3i)^2 = (2)^2 + (3i)^2 - 12i = 4 + 9i^2 - 12i = 4 + 9(-1) - 12i$

$= 4 - 9 - 12i = -5 - 12i$. Real part $= -5$, Imiginary part $= -12$

b. $$(1+2i)^{-2} = \frac{1}{(1+2i)^2} = \frac{1}{(1)^2 + (2i)^2 + 4i} = \frac{1}{1 + 4i^2 + 4i} = \frac{1}{1 + 4(-1) + 4i}$$

$$= \frac{1}{1-4+4i} = \frac{1}{-3+4i} = \frac{1}{-3+4i} \times \frac{-3-4i}{-3-4i}$$

$$= \frac{-3-4i}{(-3)^2 - (4i)^2} = \frac{-3-4i}{9-16i^2} = \frac{-3-4i}{9-16(-1)}$$

$$= \frac{-3-4i}{9+16} = \frac{-3-4i}{25} = \frac{-3}{25} - \frac{4}{25}i.$$ Real part $= \frac{-3}{25}$, Imiginary part $= \frac{-4}{25}$

c. $\frac{(\sqrt{2}+i\sqrt{3})^2}{1+i} = \frac{(\sqrt{2})^2+(i\sqrt{3})^2+2i\sqrt{2}\sqrt{3}}{1+i} = \frac{2+3i^2+2i\sqrt{3\times 2}}{1+i}$

$= \frac{2+3(-1)+2i\sqrt{6}}{1+i} = \frac{2-3+2i\sqrt{6}}{1+i} = \frac{-1+2i\sqrt{6}}{1+i}$

$= \frac{-1+2i\sqrt{6}}{1+i} \times \frac{1-i}{1-i} = \frac{(-1+2i\sqrt{6})(1-i)}{(1)^2-(i)^2} = \frac{-1+2i\sqrt{6}+i-2i^2\sqrt{6}}{1-i^2}$

$= \frac{-1+(2\sqrt{6}+1)i+2\sqrt{6}}{1-(-1)} = \frac{-1+2\sqrt{6}+(2\sqrt{6}+1)i}{2} = \frac{-1+2\sqrt{6}}{2} + \frac{(2\sqrt{6}+1)}{2}i$

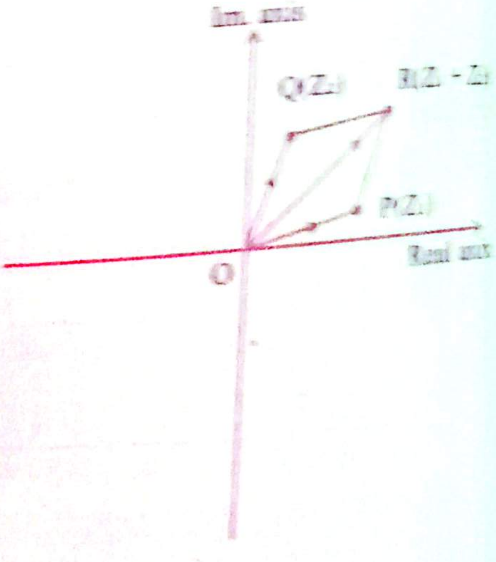
Real part $= \frac{-1+2\sqrt{6}}{2}$, Imiginary part $= \frac{(2\sqrt{6}+1)}{2}$

d. $\frac{(1-i)^2}{(1+i)^2} = \frac{(1)^2+(i)^2-2i}{(1)^2+(i)^2+2i} = \frac{1-1-2i}{1-1+2i} = \frac{-2i}{2i} = -1$

Real part $= -1$, Imiginary part $= 0$

Geometrical Representation of Algebraic Operations

The graphical representation of algebraic operations in the complex number system makes it easier to understand the concepts.



(i) Geometrical Representation of Addition

Geometrically, addition of two complex numbers z_1 and z_2 can be visualized as addition of the vectors by using the *parallelogram law*. The vector sum $z_1 + z_2$ is represented by the diagonal of the parallelogram formed by the two original vectors.

If two points P and Q represent complex numbers z_1 and z_2 respectively in the Argand Plane, then the sum $z_1 + z_2$ is represented by the point R of the diagonal OR of parallelogram OPRQ having OP and OQ as two adjacent sides.

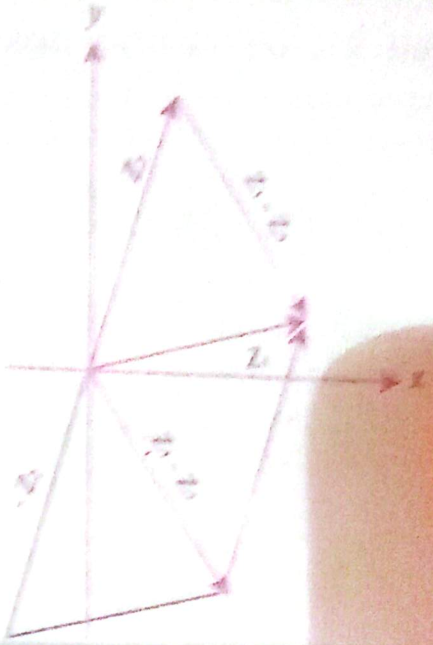
(ii) Geometrical Representation of Subtraction

The representation of the difference of two complex numbers, is slightly complicated than the addition of the complex numbers. The easiest way of presenting the subtraction is to think of addition of a negative vector.

If we want to represent $z_1 - z_2$, then the easiest way of representing it would be to think of adding a negative vector $z_1 + (-z_2)$.

The negative vector is the same as the positive one, the only difference being that the negative vector points in the opposite direction.

The difference vector $z_1 - z_2$ is represented in the adjoining figure.



Key Fact:

It is important to note here that the vector representing the difference of the vectors $z_1 - z_2$ may also be drawn joining the end point of z_2 to the tip of z_1 instead of the origin. This kind of representation does not alter the meaning or interpretation of the difference operator.

Check Point:

Can you think about a geometric interpretation of the addition of three or more complex numbers?

Example:

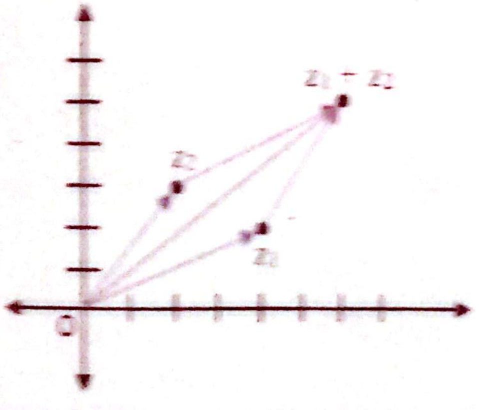
Represent sum and difference of $z_1 = 4 + 2i$ and $z_2 = 2 + 3i$ graphically.

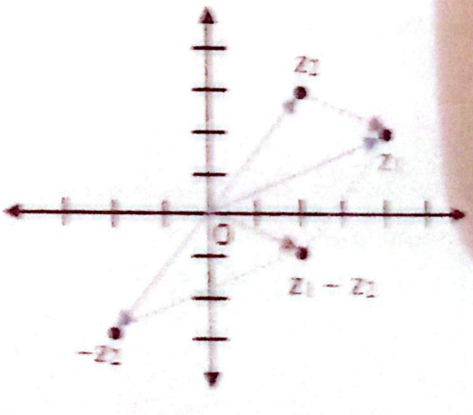
Solution:

Given, $z_1 = 4 + 2i = (4, 2)$ and $z_2 = 2 + 3i = (2, 3)$

Sum $= z_1 + z_2 = (4 + 2i) + (2 + 3i) = 6 + 5i = (6, 5)$

Difference $= z_1 - z_2 = (4 + 2i) - (2 + 3i) = 2 - i = (2, -1)$





(iii) Geometrical Representation of Multiplication and Division

Consider, $z_1 = 4 + 3i$ and $z_2 = 2 - 2i$.

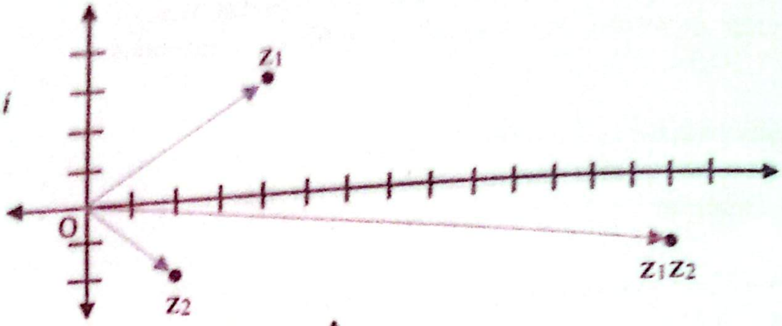
Now, $z_1 z_2 = (4 + 3i)(2 - 2i)$

$= 8 - 8i + 6i - 6i^2$

$= 8 - 2i + 6 = 14 - 2i$

$= (14, -2)$

Product $z_1 z_2$ is shown in the adjoining diagram.



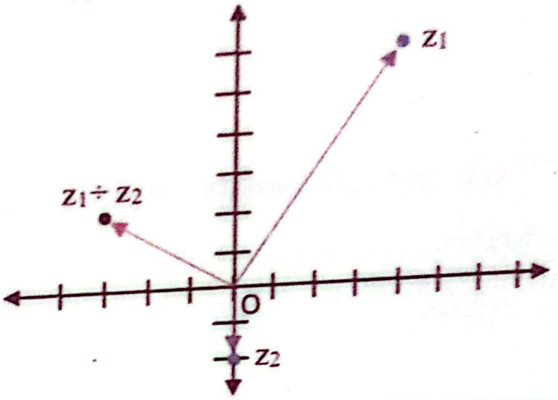
Again, consider $z_1 = 4 + 6i$

and $z_2 = -2i$ then:

$z_1 \div z_2 = (4 + 6i) \div (-2i)$

$= -3 + 2i$

The result is shown in the diagram.



Exercise 1.2

1. Find the additive inverse of each complex number.

a. $-4+5i$ b. $-3-3i$ c. $5-5i$ d. $4i$

2. Show that each pair of complex numbers are multiplicative inverse of each other.

a. $2+3i, \frac{2-3i}{13}$ b. $5-4i, \frac{5+4i}{41}$ c. $6+8i, \frac{3-4i}{50}$

3. Find the multiplicative inverse of each complex number.

a. $1+i$ b. $7-3i$ c. $10-12i$

d. $\frac{2}{5-i}$ e. $\frac{-i}{2-3i}$ f. $a-bi$

4. Find the product of each complex number and its conjugate.

a. 4 b. $1-i$ c. $7i$

d. $6-2i$ e. $10+9i$ f. $-4-11i$

5. If $z_1 = 1 - 2i$ and $z_2 = 2 + i$:

a. Show that:

i. $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$

ii. $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$

iii. $|z_1| = |-z_1| = |\overline{z_1}| = |-\overline{z_1}|$ iv. $z_1.\overline{z_1} = |z_1|^2$

b. Find

i. $|z_1 + z_2|$ ii. $|z_1 z_2|$ iii. $\left|\frac{z_1}{z_2}\right|$

6. Represent the numbers in the complex plane.

a. $-1-3i$ b. $2+4i$ c. $-3+2i$ d. $2-3i$

e. $2i$ f. $-3i$ g. 2

7. Separate into real and imaginary parts of each complex number.

a. $(\sqrt{2}-\sqrt{3}\,i)^2$ b. $(\sqrt{2}+i)^2$ c. $\frac{(2+3i)^2}{1-3i}$ d. $\left[\frac{1}{2}+\frac{\sqrt{3}}{2}i\right]^3$

e. $\frac{1-i}{(i)^2}$ f. $\frac{1}{i(1-i)^2}$ g. $\frac{(1+i)^2}{(1-2i)^2}$

8. Taking any complex number and show that:

a. $z.\overline{z}$ is a real number. b. $z^2 + (\overline{z})^2$ is a real number.

c. $(z-\overline{z})^2$ is a real number. d. $|z|$ and $|\overline{z}|$ are real numbers.

e. $z^2 - (\overline{z})^2$ is an imaginary number.

9. Represent sum and difference of complex numbers graphically.

(i) $z_1 = 5+3i$ and $z_2 = 2-3i$ (ii) $z_1 = -3+2i$ and $z_2 = 4+3i$

10. Represent product of complex numbers graphically.

(i) $z_1 = 4+2i$ and $z_2 = -2+3i$ (ii) $z_1 = -2+4i$ and $z_2 = 3-i$

11. Represent $z_1 + z_2$ graphically when:

(i) $z_1 = 6-4i$ and $z_2 = 3$ (ii) $z_1 = -4-6i$ and $z_2 = 1+i$

Applications of Complex Numbers

Complex numbers are used in many fields in real life. Major Fields where complex numbers are used are:

Electronics	Electromagnetism	Computer science engineering
Civil engineering	Mechanical engineering	Control systems

In modelling and problem solving, we find many equations whose roots are complex numbers. Dealing such kind of problems, we must have conceptual understanding of complex numbers. Practically, negative numbers do not have square roots in the number system. However, a large number system that contains the real number system is designed so that negative numbers do have square roots.

Example: Solve $x^2+5=0$

Solution: $x^2+5=0$

$x^2=-5$

$x=\pm\sqrt{-5}=\pm\sqrt{-1\times5}=\pm\sqrt{-1}\times\sqrt{5}$

$=\pm\sqrt{5}i$

The solution $+\sqrt{5}i$ and $-\sqrt{5}i$ are both pure imiginary numbers.

CHECK POINT

$x^2+5=0$

Solution is $\pm\sqrt{5}i$.

Verify the solution.

Sometimes we need factorization of the equation, but in real number system in some expressions making factors are not possible. By using the complex numbers properties, we can make factorization possible.

Example: Factorize:

a. z^2+9

b. $3z^2+5$

Solution:

a. z^2+9

$=(z)^2-9i^2$

$=(z)^2-(3i)^2$

$=(z-3i)(z+3i)$

The factors of z^2+9 are $(z-3i)(z+3i)$

b. $3z^2+5$

$=(\sqrt{3})^2z^2-i^2(\sqrt{5})^2$

$=(\sqrt{3}z)^2-(i\sqrt{5})^2$

$=(\sqrt{3}z-i\sqrt{5})(\sqrt{3}z+i\sqrt{5})$

The factors are $(\sqrt{3}z-i\sqrt{5})(\sqrt{3}z+i\sqrt{5}$

Example: Mr. Waqas is an electrical engineer designing the electrical circuits for a new office building. There are three basic things to be considered in an electrical circuit. The flow of the electric current I, the resistance to that flow Z, called impedance, and electromotive force E, called voltage. Their quantities are related in the formula $E=IZ$. The current of the circuit Waqas is designing is to be $(35-40J)$ amp. Electrical engineers use the letter J to represent the imaginary unit. Find the impedance of the circuit if the voltage is to be $(430-330J)$ volts.

Solution: We know that $E=IZ$ ……(i)

We know it is given that $I=35-40J$, $E=430-330J$

Substituting the values in (i) $E=IZ$

and $Z=\dfrac{E}{I}$

$$Z=\frac{430-330J}{35-40J}$$

$$=\frac{430-330J}{35-40J}\times\frac{35+40J}{35+40J}$$

$$=\frac{15050+12200J-11550J-13200J^2}{1225-1600J^2},\qquad J^2=-1$$

$$=\frac{2825+5650J}{2825}=1+2J$$

The impedance will be $1+2J$ ohms.

Example: If $z_1 = 5.5 + 5i, z_2 = 7.7 + 7i$, then calculate

a. $z_1 + z_2$ b. $z_1 z_2$ c. $\frac{z_1}{z_2}$

Solution:

a. $z_1+z_2=5.5+5i+7.7+7i=13.2+12i$

b. $z_1z_2=(5.5+5i)(7.7+7i)=42.35+38.5i+38.5i+35i^2$
$=7.35+77i$

c. $$\frac{z_1}{z_2}=\frac{5.5+5i}{7.7+7i}=\frac{5.5+5i}{7.7+7i}\times\frac{7.7-7i}{7.7-7i}=\frac{(5.5+5i)(7.7-7i)}{(7.7)^2-(7i)^2}$$

$$=\frac{42.35-38.5i+38.5i-35i^2}{53.9-49i^2}=\frac{77.35}{102.9}$$

Solution of System of Equations with Complex Coefficients

In this section, we shall find solution of different equations in complex variables either with real or complex coefficients. Consider the following equation:

$$pz+qw=r......(\text{i})$$

Where p, q and r are complex numbers (every real number is a complex number). The equation (i) is called a linear equation in two complex variables z and w.

$$p_1z+q_1w=r_1$$
$$p_2z+q_2w=r_2$$

These two equations together form a system of linear equations in two variables z and w. The linear equations in two variables are also called simultaneous linear equations.

Example: Solve:

$$z+w=3i$$
$$2z+3w=2$$

Solution: Given equations are:

$$z+w=3i.........(\text{i})$$
$$2z+3w=2......(\text{ii})$$

Multiply equation (i) by 2 to equate the coefficient of z with equation (ii) and subtracting

$$\begin{array}{r} 2z+2w=6i \\ \pm 2z \pm 3w = \pm 2 \\ \hline -w=6i-2 \end{array}$$

$$w=2-6i \quad \text{(Substitute in (i) to find z)}$$

$$z+2-6i=3i$$
$$z=9i-2$$

Solution is: $z=9i-2,\ w=2-6i$

Exercise 1.3

Solve:

1. $x^2+7=0$
2. $x^2+9=0$
3. $x^2+100=0$

Determine whether the given complex number is a solution of the equation.

4. $1+2i,\ x^2-2x+5=0$
5. $1-2i,\ x^2-2x+5=0$
6. $1-i,\ x^2+2x+2=0$
7. $i,\ x^2+1=0$

Factorize the expressions:

8. x^2+16
9. a^2+b^2
10. x^2+25y^2

Solve the following system of linear equations.

11. $z-4w=3i$
    $2z+3w=11-5i$
12. $3z+(2+i)w=11-i$
    $(2-i)z-w=-1+i$

13. In an electrical circuit, the flow of the electric current I, the impedance Z and the voltag E, are related by the formula $E=IZ$.

    a. Find I, given the values:

       i. $\mathrm{E}=(70+220J)$ volts, $Z=(16+8J)$ ohms

       ii. $\mathrm{E}=(85+110J)$ volts, $Z=(3-4J)$ ohms

    b. Find Z given the value:

       i. $\mathrm{E}=(-50+100J)$ volts, $I=(-6-2J)$ amp

       ii. $\mathrm{E}=(100+10J)$ volts, $I=(-8+3J)$ amp

    c. Evaluate $\dfrac{1}{z-z^2}$ when $z=\dfrac{1-i}{10}$

I have Learnt

- Recognizing complex numbers, and conjugate and modulus of a complex numbers.
- Applying basic operations on complex numbers.
- Defining commutative laws, associative laws and distributive laws for complex numbers.
- Recognizing additive and multiplicative identity and finding additive and multiplicative inverse of complex numbers.
- Finding real and imaginary parts of $\left(\frac{x_1+iy_1}{x_2+iy_2}\right)^n$ for $n=\pm1, \pm2, \ldots$
- Finding solution of equations having complex values.
- Applying the geometric interpretation of a complex numbers.
- Solving daily life problems involving complex numbers.

MISCELLANEOUS EXERCISE-1

1. Tick the correct option.

i. $\sqrt{-1}$ is equal to:

a. 1 b. -1 c. i d. $-i$

ii. If $x<0$, then $\sqrt{x}$ is :

a. Real b. Complex c. Irrational d. Rational

iii. Conjugate of $\sqrt{x}-i\sqrt{y}$ is:

a. $\sqrt{x}+i\sqrt{y}$ b. $x-iy$ c. $x-y$ d. $x+iy$

iv. If $z=x+iy$, then $z\bar{z}$ is :

a. Imaginary b. Complex

c. Non-negative number d. Negative number

v. $\sqrt{-25}+\sqrt[3]{8}$ is equal to:

a. $-5+\sqrt{8}$ b. $2+5i$ c. $-5+2i$ d. $2\sqrt{2}+5i$

vi. $1+(-i)^9=?$

a. $1+i$ b. $1+\sqrt{-1}$ c. $1-i$ d. $-i$

vii. $\frac{2}{1-i}=?$

a. $\frac{1+i}{2}$ b. $\frac{(1+i)^2}{2}$ c. $1-i$ d. $1+i$

viii. $(-xi)^{19}=?$

a. $-x^{19}i$ b. $x^{19}i$ c. $-i^{19}$ d. $-x^{19}$

ix. If $z = 3 + 4i$, then $|z|^2$ is:

a. 5 b. $\sqrt{5}$ c. 25 d. 16

x. The solution of $x^2 + 4 = 0$ is:

a. $2i$ b. $-2i$ c. ± 2 d. $\pm 2i$

2. Simplify:

   a. $(-2+4i)-(8-5i)$ b. $(-3+4i)+(-7i+4)$

3. Find the product:

   a. $(x+iy)(2+3i)$ b. $(-3+6i)(-6+3i)$

4. Write in the form of $a+bi$, then find its conjugate:

   a. $3\sqrt{2}-\sqrt{-7}$ b. $\sqrt{-2}$

5. Find $z.\bar{z}$:

   a. $z = \dfrac{-1}{2} + i$ b. $z = 14 - 7i$

6. Simplify:

   a. $\dfrac{-3-i}{-3+i}$ b. $\dfrac{1+3i}{i\sqrt{5}}$

7. Factorize:

   a. $2x^2 + 18$ b. $-x^2 - 25y^4$

8. Solve:

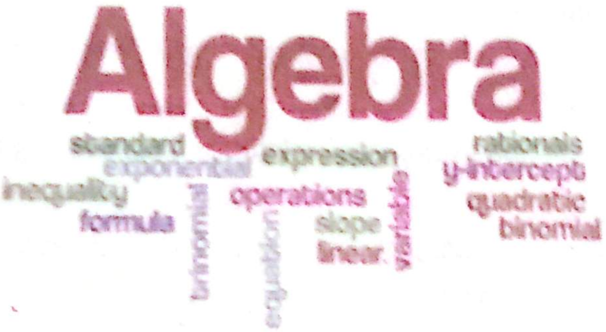
   a. $3x^2 + 15 = 0$ b. $6y^2 + 36 = 0$

02

Quadratic Equations

After studying this unit students will be able to:

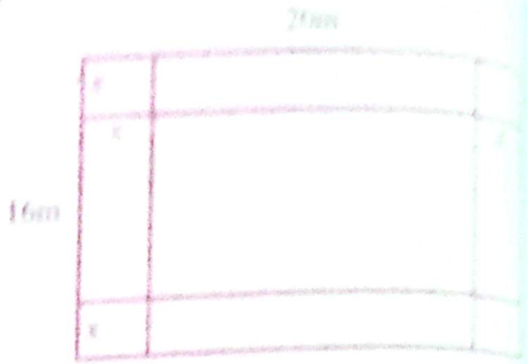
- Define quadratic equations.
- Solve quadratic equations by graphical method, factorization and completing squares.
- Derive quadratic formula and use it to solve quadratic equations.
- Solve different types of equations which are reducible to quadratic form.
- Solve simultaneous equations. When one is homogenous and one is linear.
- Know nature of roots of a quadratic equation.
- Find relation between roots and coefficients of a quadratic equation.
- Define and evaluate symmetric functions of roots of a quadratic equation.
- Form a quadratic equation when its roots are given.



Algebra is your trusted tool that helps you carry out of various activities of daily importance. There is hardly any line of work that does not employ the concept of algebra. We will look at the variables and equation, you surely will not be wondered about why you need to leave there in a nutshell. Algebra and equations prepare you for hardly all aspects of life and stays with you right from your infanthood to your adulthood.

Quadratic Equations

Saleem wants to build a swimming pool surrounded by a sidewalk of uniform width. He wants the dimensions of the pool and sidewalk to be 16 meters by 20 meters. The pool has an area of 192 square meters. How wide should the sidewalk be?



Let x meters be the width of the sidewalk.

The length of a pool is 20 – 2x meters.

The width of the pool is 16 – 2x meters.

The area of the pool can be expressed as the product of the length and width.

Area = Length × Width

$$= (20 - 2x)(16 - 2x)$$

$$= 4x^2 - 72x + 320$$

The area of the pool can be expressed as $4x^2 - 72x + 320$ square meters. Since the area of the pool is 192 square meters, replacing area with 192 results in the equation $192 = 4x^2 - 72x + 320$. Saleem would probably never use a quadratic equation like $192 = 4x^2 - 72x + 320$ to express area of a pool. However, in this unit we will study many other formulas in science and business that involve this type of equations.

Quadratic Equation

It is an equation that can be written in the form of $ax^2 + bx + c = 0$, where $a \neq 0$. We say that the greatest exponent of the variable is 2. We equations like this have a degree of 2. Since the greatest exponent of the variable is 2. We observe that this quadratic equation only has one variable and all the exponents are positive.

Standard Quadratic Equation

The equation $ax^2 + bx + c = 0$ is a standard form of the quadratic equation in one variable x. The equation $x^2 + 6x + 5 = 0$ is in standard form.

$(x+1)^2 = 4$ and $x^2 + 3x = 18$ are examples of quadratic equation in one variable x but they are not in standard form.

Example: Write the following quadratic equations in standard form.

(i). $ax^2 + bx = c$ (ii). $(3x - 1)^2 = 4$ (iii). $(x + 1)^2 - (2x - 1)^2 = 2$

Solution:

i. $ax^2 + bx = c$

$ax^2 + bx - c = 0$

ii. $(3x - 1)^2 = 4$

$9x^2 - 6x + 1 = 4$

$9x^2 - 6x + 1 - 4 = 0$

$9x^2 - 6x - 3 = 0$

$3(3x^2 - 2x - 1) = 0$, as $3 \neq 0$

$3x^2 - 2x - 1 = 0$

iii. $(x + 1)^2 - (2x - 1)^2 = 2$

$(x^2 + 2x + 1) - (4x^2 - 4x + 1) =$

$x^2 + 2x + 1 - 4x^2 + 4x - 1 - 2 =$

$-3x^2 + 6x - 2 = 0$

$-1(3x^2 - 6x + 2) = 0$

$3x^2 - 6x + 2 = 0$ as $-1 \neq$

Solution of Quadratic Equations

The values of variables for which an equation becomes a true sentence are called solutions or roots of the equation. i.e. if $P(a) = c$, then $x = a$ is root of the equation $P(x) = c$. The set of roots of an equation is called the solution set. A quadratic equation in one variable has two roots. These roots of the quadratic equation can be found in many ways. In this section we will study four methods.

i. Graphing method
ii. Factorization Method
iii. Completing Square Method
iv. Using Quadratic Formula

i. Graphical Method

One way to determine the roots of a quadratic equation is to graph the related quadratic function. Write the equation in general form like:

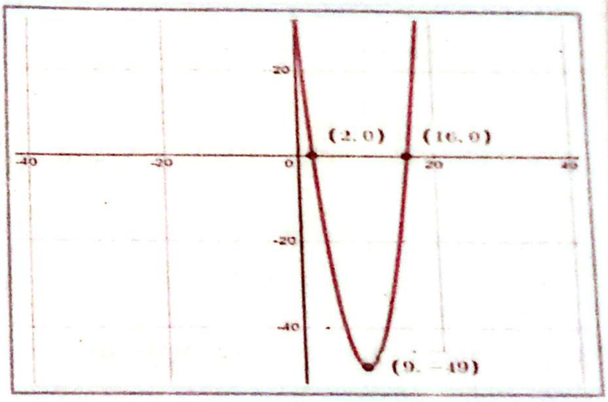
$$192 = 4x^2 - 72x + 320$$

$$0 = 4x^2 - 72x + 128 \text{ subtract 192 from each side.}$$

$$0 = x^2 - 18x + 32 \text{ dividing each side by 4.}$$

The related function for this equation is

$f(x) = y = x^2 - 18x + 32$.



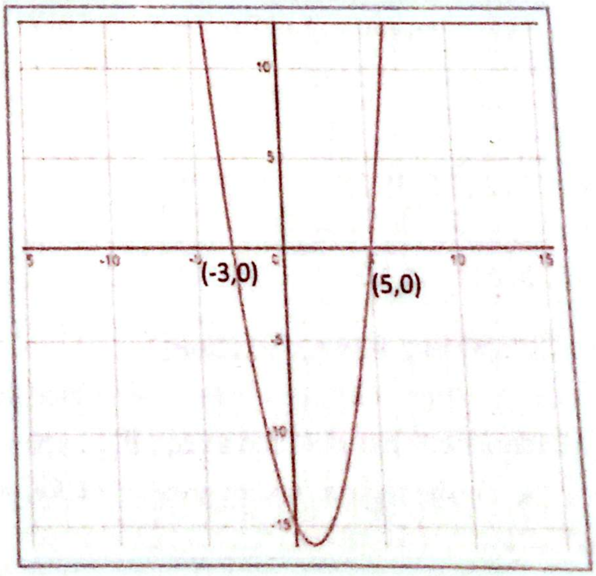
x	2	4	16	20
y	0	–24	0	72

Notice that the function crosses the x-axis twice and the function has a degree of 2. These intercepts points are called the zero of a function. The x values, 2 and 16 are the roots of the equation $x^2 - 18x + 32 = 0$

Example: Graph the related quadratic function $y = x^2 - 2x - 15$ by finding and graphing the ordered pairs that satisfy the function.

Solution:

x	–3	–1	1	5
y	0	–12	–16	0



The graphs tells that the solutions are

–3 and 5, where the curve intersects the x-axis.

ii. Factoring Method

Another way of solving a quadratic equation is by factoring. The factoring method depends on th zero-product property.

Zero Product Property:

For any real number a and b, if ab = 0, then either a = 0 or b = 0.

i. To solve the quadratic equation by factorization first write it in standard form, then factorize the polynomial on the left-hand side of the equation.

ii. Use zero product property.

iii. Now solve linear equations and write solution set.

Fr example, to solve $x^2 = 4$ by factorization we do the following steps:

$$x^2 = 4$$

$$x^2 - 4 = 0 \qquad \text{(standard form)}$$

$$(x)^2 - (2)^2 = 0$$

$$(x-2)(x+2) = 0 \qquad \text{(factorization)}$$

$$x-2 = 0 \; or \; x+2 = 0 \qquad \text{(linear equations)}$$

$$x = 2 \; or \; x = -2 \qquad \text{(roots)}$$

$$S.S. = \{-2, 2\} \qquad \text{(solutionset)}$$

Example: Factorize to solve the equations.

i. $x^2 - 18x + 32 = 0$

ii. $x^2 - 2x - 15 = 0$.

Solution:

i.
$$x^2 - 18x + 32 = 0$$
$$x^2 - 16x - 2x + 32 = 0$$
$$x(x-16) - 2(x-16) = 0$$
$$(x-2)(x-16) = 0$$
$$x - 2 = 0 \;\text{ or }\; x - 16 = 0$$
$$x = 2 \;\text{ or }\; x = 16$$
$$S.S = \{2, 16\}$$

ii.
$$x^2 - 2x - 15 = 0$$
$$x^2 - 5x + 3x - 15 = 0$$
$$x(x-5) + 3(x-5) = 0$$
$$(x-5)(x+3) = 0$$
$$x - 5 = 0 \text{ or } x + 3 = 0$$
$$x = 5 \text{ or } x = -3$$
$$S.S = \{-3, 5\}$$

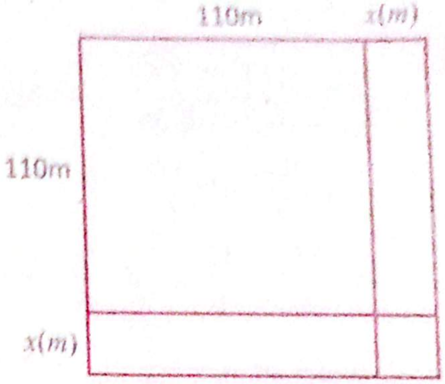
iii. Completing Square Method

In a city, a square field has been maintained for playing. Each side of the field is $110m$ long. The management has decided to equally expand two of the sides of the field for parking spaces so that the entire region has an area of 14000 m^2. What is the width of the parking strip being added?

Let x represents the width of each parking strip.
After expanding each side of the field would be $110+x$ meters long.
The equation expresses the area of the new field, which is still square.

$$(x+110)^2 = 14000$$
$$\sqrt{(x+110)^2} = \sqrt{14000}$$
$$x+110 = \pm\sqrt{14000}$$
$$x = -110 \pm \sqrt{14000}$$
$$x = 8.32m$$

Where the negative value is neglected.



Quadratic equations can be solved in the same way as long as one side of the equation contains a perfect square. When the equation does not contain a perfect square, we can use process called completing the square to create a perfect square.

For an expression of the form $x^2 + bx$, we can add a constant c in the expression so that the expression $x^2 + bx + c$ is a perfect square trinomial. This process is called completing square.

Procedure

Step I:	Write the equation in standard form.
Step II:	Shift constant term to the right hand side of the equation.
Step III:	To make the left hand side a complete square, add a suitable term to both sides of the equation. If the coefficient of x^2 is 1, then it is easy to complete the square. Just add the square of half the coefficient of x and get a complete square.
Step IV:	Take the square root on both sides and form two linear equations.
Step V:	Solve the linear equations and find roots.

Check Point:

Describe the error.

$$x^2 - 14x = 11$$
$$x^2 - 14x + 49 = 11$$
$$(x-7)^2 = 11$$
$$x - 7 = \pm\sqrt{11}$$
$$x = 7 \pm \sqrt{11}$$

Example:

Solve $3x^2 - 11x - 4 = 0$ by completing square.

Solution:

$$3x^2 - 11x - 4 = 0$$

$3x^2 - 11x = 4$ Shift 4 right side

$x^2 - \frac{11}{3}x = \frac{4}{3}$ dividing each side by 3.

$$x^2-\frac{11}{3}x+\frac{121}{36}=\frac{4}{3}+\frac{121}{36} \quad \text{adding } \frac{121}{36} \text{ on both sides.}$$

$$\left(x-\frac{11}{6}\right)^2=\frac{169}{36} \quad \text{factor}$$

$$\sqrt{\left(x-\frac{11}{6}\right)^2}=\sqrt{\frac{169}{36}} \quad \text{taking square root}$$

$$x-\frac{11}{6}=\pm\frac{13}{6}$$

$$x-\frac{11}{6}=\frac{13}{6} \text{ or } x-\frac{11}{6}=-\frac{13}{6}$$

$$x=4 \text{ or } x=-\frac{1}{3} \qquad S.S.=\{-\frac{1}{3},4\}$$

Example:

Solve $3x^2+6x-9=0$ by the completing square method.

Solution:

$3x^2+6x-9=0$

$3x^2+6x=9$

$x^2+2x=3$ (dividing by 3)

$x^2+2x+1=3+1$ (adding 1 to both sides)

$(x+1)^2=4$

$x+1=\pm\sqrt{4}$ (taking square root)

$x+1=\pm2$

$x=\pm2-1$

$S.S.=\{1,-3\}$

iv. Quadratic Formula

We have learnt several ways to solve quadratic equations. Each has its limitations. The question we think is that is there any formula that will work for any quadratic equation? The answer is "yes" and the formula is called the quadratic formula. The formula is derived from solving the general form of a quadratic equation for x.

The solution of a quadratic equation of the form $ax^2+bx+c=0$ with $a\neq0$ is

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Derivation of the Quadratic Formula

Standard form of quadratic equation is: $ax^2+bx+c=0$

$$ax^2+bx=-c$$

$$x^2+\frac{b}{a}x=-\frac{c}{a} \quad \text{(Divide by a both sides)}$$

$$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=\left(\frac{b}{2a}\right)^2-\frac{c}{a} \quad \text{(Add } \left(\frac{b}{2a}\right)^2 \text{ both sides)}$$

$$x^2 + 2x\frac{b}{2a} + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \text{ (Simplify the left side)}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \text{ (Taking square root both sides)}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This equation is known as quadratic formula.

Example: Solve $2x^2 + 7x = 4$ using quadratic formula.

Solution: $2x^2 + 7x = 4$

$2x^2 + 7x - 4 = 0$ General form

$a = 2, b = 7, c = -4$

We know that: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{-7 \pm \sqrt{(7)^2 - 4(2)(-4)}}{2(2)}$$

$$x = \frac{-7 \pm \sqrt{49 + 32}}{4}$$

$$x = \frac{-7 \pm \sqrt{81}}{4} \Rightarrow \frac{-7 \pm 9}{4}$$

$$x = \frac{-7 + 9}{4}, x = \frac{-7 - 9}{4}$$

$$x = \frac{2}{4}, x = \frac{-16}{4}$$

$x = \dfrac{1}{2}, x = -4$ S. S. $= \{-4, \dfrac{1}{2}\}$

Procedure

To solve a quadratic equation by using the quadratic formula, we proceed as follows:

Step I: Write the equation in standard form.

Step II: Compare it with $ax^2 + bx + c = 0$, to get the values of a, b and c.

Step III: Write the quadratic formula for x.

Step IV: Put values of a, b and c, then simplify.

Step V: Write solution set.

Exercise 2.1

1. Write the following quadratic equation in standard form:
   - i. $(x+2)(x-3) = 5$ ii. $(x-5)^2 - (2x+4)^2 = 7$ iii. $x = x(x-1)$
2. Solve the following equations by factoring method.
   - i. $(x-1)(x-4) = 0$ ii. $x^2 - 2x + 1 = 0$

iii. $x^2-7x-8=0$ iv. $x^2-4x+4=(2x-7)^2$

v. $\left(2x+\frac{7}{4}\right)^2=\frac{48x^2+529}{16}$

3. Solve the following equations by completing the square method:

i. $x^2+4x-32=0$ ii. $x^2+8x=0$ iii. $x^2+6x-9=0$

iv. $3x^2+12x+8=0$ v. $x^2+x+1=0$ vi. $4x^2-8x-5=0$

4. Solve the following equations by quadratic formula:

i. $x^2-9=0$ ii. $2x^2+5x+1=0$

iii. $x^2-23x-24=0$ iv. $(x+1)^2=(2x-1)^2$

v. $\frac{x+1}{2}-\frac{x(x+2)}{3}=0$ vi. $(x-2)(x-6)=(2x+1)(x+1)$

5. Solve $x^2+6x=-9$ by graphing and factoring method.

6. Graph the function $y=x^2+2x+4$ and verifying solution by completing square meth

7. Explain each term:

i. Solution ii. root

iii. Zero of a function iv. x-intercept

8. Can a quadratic equation have more than two solutions? Why or why not?

9. Plans for rectangular ice-skating rink that is 30m by 60m have to be revised to double the area of the playground by adding strips of the same width to a side and an end of the area to form a rectangle. Find the width of the strips. What are the dimensions of the playground now?

10. If a car has an initial speed of 20m/s and a constant acceleration of 2m/s², determine the amount of time it takes to travel 145m. (Use: $S=v_it+\frac{1}{2}at^2$)

Equation Reducible to Quadratic Form

Some equations are not quadratic, but to solve them we can reduce them to quadratic form.

The example below illustrates solving an equation that is not a quadratic equation itself, but conta a factor that is a quadratic. The method we use to solve the quadratic part of that equation.

Example: A cuboid box has a base that is x cm long. The width of the base is 2 cm less than length. The height is 3 cm greater than the length of the base. The measure of the volume is 6 time the measure of the length of the base. Find the dimensions of the box.

Solution:

The volume of the rectangular box is the product of the length, width and height of box.

Here, Length = x, Width = $x - 2$, Height = $x + 3$

$\therefore$ The volume is:

V = Length × Width × Height = $l\,w\,h$

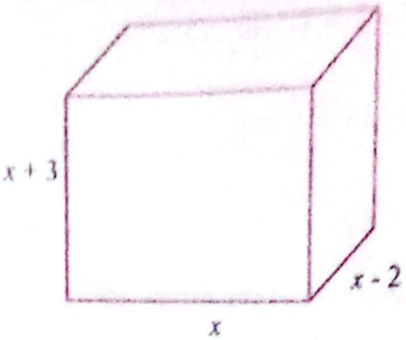
V = 6 × the measure of the length of the base ..... (i)

As, $V = x(x-2)(x+3)$ .... (ii)

$\therefore \quad 6x = x(x-2)(x+3)$ (from (i) and (ii))

$6x = x^3 + x^2 - 6x \quad \Rightarrow \quad x^3 + x^2 - 12x = 0$

$x(x^2 + x - 12) = 0 \quad \Rightarrow \quad x(x-3)(x+4) = 0$



We use the zero-product property

$x = 0\,,\ x + 4 = 0\,,\ x - 3 = 0$

$x = 0\,, x = -4\,, x = 3$

Since we are dealing with the dimensions of a box, we ignore 0 and – 4, the solution is:

length $x = 3$

width $= x - 2 = 3 - 2 = 1$

height $= x + 3 = 3 + 3 = 6$

Hence, the dimensions of the box are $3\text{cm} \times 1\text{cm} \times 6\text{cm}$.

We have many types of equations that are not quadratic, but we can write in a form that resembles a quadratic equation. For example, the equation $9x^4 - 8x^2 - 1 = 0$ can be written as: $9(x^2)^2 - 8x^2 - 1 = 0$. Equations that can be written in this way are said to be equations reducible to quadratic form.

Some of equations that can be reduced to quadratic equations, are given below.

Equations of the Form $ax^4 + bx^2 + c = 0$

It is an equation of degree 4. It can be reduced to quadratic form by using the substitution $x^2 = y$. In this way the above equation becomes $ay^2 + by + c = 0$. Solve it by any of the three methods discussed earlier. Replace y with its substitute x^2. Find values of x by taking the square root and write the solution set.

Example: Solve $9x^4 - 8x^2 - 1 = 0$.

Solution:

$9x^4 - 8x^2 - 1 = 0$

Put $x^2 = y$ which gives $x^4 = y^2$

$\therefore\ 9y^2 - 8y - 1 = 0$

$9y^2 - 9y + y - 1 = 0$

$9y(y - 1) + 1(y - 1) = 0$

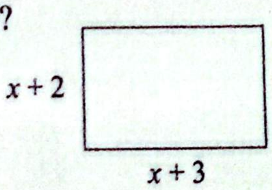
$(y - 1)(9y + 1) = 0$

$y - 1 = 0$ or $9y + 1 = 0$

Back substitution gives:

Challenge

The rectangle has an area of 42 square centimeters. What is the value of length of sides?



$x^2-3=0$ or $9x^2+1=0$

$x=\pm\sqrt{3}$ or $x=\pm\sqrt{\frac{-1}{9}}$

$x=\pm 1$ or $x=\pm\frac{i}{3}$

Equations of the Form: $aP(x)+\frac{b}{P(x)}=c$

To solve this type of equation, use a substitution $P(x)=y$. Multiply the equation by y and form a quadratic equation involving the variable y. Solve the new equation for y and replace y by P(x). In this way, you will get two simple equations involving variable x. Solve these equations and write the solution set.

Challenge

(i) Solve for real roots, $y^3-27=0$

(ii) Solve:

$$x^2+3x+7=\frac{6}{x^2+3x+2}$$

Example: Solve: $3x+\frac{4}{x}=7$

Solution: $3x+\frac{4}{x}=7$

$3x^2+4=7x$

$3x^2-7x+4=0$

$3x^2-3x-4x+4=0$

$3x(x-1)-4(x-1)=0 \quad \Rightarrow \quad x-1=0$ or $3x-4=0$

$x=1$ or $x=\frac{4}{3}$

$\therefore S.S.=\left\{1,\frac{4}{3}\right\}$

Example: Solve $2(x^2+1)-\frac{2}{x^2+1}=3$

Solution: $2(x^2+1)-\frac{2}{x^2+1}=3$

Put $x^2+1=y$

So, $2y-\frac{2}{y}=3 \quad \Rightarrow \quad 2y^2-2=3y$

$2y^2-3y-2=0 \quad \Rightarrow \quad 2y^2-4y+y-2=0$

$2y(y-2)+1(y-2)=0 \quad \Rightarrow \quad (y-2)(2y+1)=0$

$y-2=0$ or $2y+1=0 \quad \Rightarrow \quad y=2$ or $y=\frac{-1}{2}$

Back substitution: When $y=2$

$x^2+1-2=0$

$x^2-1=0$

$x=\pm 1$

When $y=\frac{-1}{2}$

$x^2+1=-\frac{1}{2}$

$2x^2=-3 \quad \Rightarrow \quad x^2=\frac{-3}{2}$

$x=\pm\sqrt{\frac{3}{2}}i$

Solution $= \pm 1, \pm\sqrt{\frac{3}{2}}i$

Reciprocal Equations

If an equation is not affected by the replacement of a variable by its reciprocal, then it is called a reciprocal equation. The reciprocal of a root of such an equation is also its root.

Consider the equation $ax^4+bx^3+cx^2+bx+a=0.$ → (i)

$$a\left(\frac{1}{x}\right)^4+b\left(\frac{1}{x}\right)^3+c\left(\frac{1}{x}\right)^2+b\left(\frac{1}{x}\right)+a=0 \qquad (x \text{ is replaced by } \tfrac{1}{x})$$

$$\frac{a}{x^4}+\frac{b}{x^3}+\frac{c}{x^2}+\frac{b}{x}+a=0$$

$$a+bx+cx^2+bx^3+ax^4=0 \qquad (\text{equation is multiplied by } x^4)$$

$$ax^4+bx^3+cx^2+bx+a=0. \qquad \to \text{(ii)}$$

Equations (i) and (ii) are same.

To solve the reciprocal equation (i), arrange it in the form

$$a\left(x^2+\frac{1}{x^2}\right)+b\left(x+\frac{1}{x}\right)+c=0$$

Put $y=x+\frac{1}{x}$ and find the value of $x^2+\frac{1}{x^2}=y^2-2$.

In this way equation (i) is reduced to quadratic form. Solve this equation for y and then replace y with its substitute $x+\frac{1}{x}$. Finally find values of x and write solution set.

Example: Solve the equation:

$$6x^4-35x^3+62x^2-35x+6=0.$$

Solution:

$$6x^4-35x^3+62x^2-35x+6=0.$$

$$6x^2-35x+62-\frac{35}{x}+\frac{6}{x^2}=0 \quad (\text{dividing by } x^2)$$

$$6(x^2+\frac{1}{x^2})-35(x+\frac{1}{x})+62=0 \quad \to \text{(i)}$$

Let $x+\frac{1}{x}=y$

$$\left(x+\frac{1}{x}\right)=y^2$$

$$x^2+\frac{1}{x^2}+2=y^2$$

$$x^2+\frac{1}{x^2}=y^2-2$$

Put in equation (i)

$$6\left(y^2-2\right)-35y+62=0$$

$$6y^2-12-35y+62=0$$

Critical Thinking

Example: Solve $y-7\sqrt{y}-8=0,\ y\in R$

Solution: $y-7\sqrt{y}-8=0$

Let $\sqrt{y}=z$

Then $y=z^2$

Equation will become:

$$z^2-7z-8=0$$

$$z=\frac{-(-7)\pm\sqrt{(-7)^2-4(1)(-8)}}{2(1)}$$

$$z=\frac{+7\pm\sqrt{49+32}}{2}=\frac{+7\pm\sqrt{81}}{2}$$

$$z=\frac{+7\pm9}{2}$$

$$z=\frac{+7+9}{2},\ z=\frac{+7-9}{2}$$

$$z=8,\ z=-1$$

Back substitution gives:

$\sqrt{y}=8,\ \sqrt{y}=-1$

$y=64$, (No real roots for $\sqrt{y}=-1$)

So, the solution is = 64

$6y^2 - 35y + 50 = 0$

$6y^2 - 15y - 20y + 50 = 0$

$3y(2y - 5) - 10(2y - 5) = 0 \quad \Rightarrow \quad 2y - 5 = 0 \text{ or } 3y - 10 = 0$

$(2y - 5)(3y - 10) = 0$

$2\left(x + \frac{1}{x}\right) - 5 = 0$ or $3\left(x + \frac{1}{x}\right) - 10 = 0$ (Substituting the value of y)

$2x + \frac{2}{x} - 5 = 0$ or $3x + \frac{3}{x} - 10 = 0$

$2x^2 + 2 - 5x = 0$ or $3x^2 + 3 - 10x = 0$

$2x^2 - 5x + 2 = 0$ or $3x^2 - 10x + 3 = 0$

$2x^2 - 4x - x + 2 = 0$ or, $3x^2 - 9x - x + 3 = 0$

$2x(x - 2) - 1(x - 2) = 0$ or $3x(x - 3) - 1(x - 3) = 0$

$(x - 2)(2x - 1) = 0$ or $(x - 3)(3x - 1) = 0$

x [illegible] $= 0$ or $2x - 1 = 0$ or $x - 3 = 0$ or $3x - 1 = 0$

$x = 2$ or $x = \frac{1}{2}$ or $x = 3$ or $x = \frac{1}{3}$ $\Rightarrow$ $S.S. = \{2, \frac{1}{2}, 3, \frac{1}{3}\}$

Exponential Equations

An equation involving expressions of the form k^x is called an exponential equation where k is a constant. In exponential equations the exponents are variables. We will discuss only those exponential equations which can be reduced to quadratic form.

Exponential equations reducible to quadratic form are of the type:

$ak^{2x} + bk^x + c = 0, a \neq 0.$

We use the substitution $k^x = y$ to reduce the above equation to quadratic form. Then we find y and replace it with k^x. In this way, roots of the exponential equations are found.

Example: Solve the following exponential equations

i. $2^{2x} - 2 \times 2^x + 1 = 0$

ii. $9^x - 3^{x+1} - 4 = 0$

Solution:

i. $2^{2x} - 2 \times 2^x + 1 = 0 \rightarrow$ (i)

Put $2^x = y$, so $2^{2x} = y^2$

$\therefore$ Equation (i) becomes:

$y^2 - 2y + 1 = 0$

$(y - 1)^2 = 0$

$y - 1 = 0$

$y = 1$

Replace y by its substitute 2^x

ii. $9^x - 3^{x+1} - 4 = 0 \rightarrow$ (i)

$3^{2x} - 3^x \times 3 - 4 = 0$

Put $3^x = y$, so $3^{2x} = y^2$

$\therefore$ Equation (i) becomes:

$y^2 - 3y - 4 = 0$

$y^2 - 4y + y - 4 = 0$

$y(y - 4) + 1(y - 4) = 0$

$(y - 4)(y + 1) = 0$

$y = 4$ or $y = -1$

$3^x = 4$ or $3^x = -1$

$2^x = 1 = 2^0$

$x = 0$

$S.S = \{0\}$

Converting $3^x = 4$ into logarithmic form, we get

$x = \log_3 4$

Solution of $3^x = -1$ is not possible.

$\therefore$ S.S. = $\{\log_3 4\}$

Equations of the Form $(x+a)(x+b)(x+c)(x+d) = k$; $a+b = c+d$

In this type, we multiply the pairs of factors after rearranging them in such a way that the variable terms of both products are the same or one of them is multiple of the other. Choose a substitute for the common terms and form a quadratic equation. Solve the new equation and then replace the substitute. Find the values of original variable involved in the given equation and write the solution set.

Example:

Solve the following equations by using suitable substitution.

$$(x+1)(x+2)(x+3)(x+4) = 24$$

Solution:

$$(x+1)(x+2)(x+3)(x+4) = 24$$

$$(x+1)(x+4)\ (x+2)(x+3) = 24$$

$$(x^2+5x+4)(x^2+5x+6) = 24$$

Put $x^2+5x = y$

$$(y+4)(y+6) = 24$$

$$y^2+10y+24 = 24$$

$$y^2+10y = 0$$

$$y(y+10) = 0$$

$y = 0$ or $y+10 = 0$

For $y = 0$

$$x^2+5x = 0$$

$$x(x+5) = 0$$

$x = 0$ or $x+5 = 0$

$x = 0$ or $x = -5$

For $y + 10 = 0$

$$x^2+5x+10 = 0$$

Here, $a = 1, b = 5$ and $c = 10$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$x = \frac{-5 \pm \sqrt{5^2-4(1)(10)}}{2(1)}$$

$$x = \frac{-5 \pm \sqrt{25-40}}{2}$$

$$x = \frac{-5 \pm \sqrt{-15}}{2}$$

$$x = \frac{-5 \pm \sqrt{15}\,i}{2}$$

$$S.S. = \{0, -5, \frac{-5 \pm \sqrt{15}\,i}{2}\}$$

Exercise 2.2

1. Reduce the following equations to quadratic form using suitable substitution.

i. $ax^4+bx^2+c=0$

ii. $9x^6-3x^3+7=0$

iii. $3x+\frac{4}{6x-2}=-1$

iv. $(x+1)^2+\frac{3}{(x+1)^2}=4$

v. $3\left(x^2+\frac{1}{x^2}\right)+8\left(x+\frac{1}{x}\right)+11=0$

vi. $3x^4+7x^3+5x^2-7x+3=0$

vii. $ak^{2x}+bk^x+c=0$

viii. $8\times4^x-7\times2^x-1=0$

ix. $(x-1)(x-2)(x+3)(x-6)=6$

x. $(2x-1)(2x-7)(x-3)(x-1)=8$

2. Solve the following equations by reducing them to quadratic form.

i. $x^4-20x^2+64=0$

ii. $x^4+16x^2-225=0$

iii. $x^{\frac{2}{5}}+5x^{\frac{1}{5}}+6=0$

iv. $3x^2+\frac{4}{x^2}=7$

v. $5(x+1)+\frac{3}{x+1}=8$

vi. $5x^2+\frac{36}{5x^2+4}=16$

vii. $2x^4-x^3-6x^2-x+2=0$

viii. $12x^4+11x^3-146x^2+11x+12=0$

ix. $4\left(x^2+\frac{1}{x^2}\right)-\left(x-\frac{1}{x}\right)-11=0$

x. $2^{2x}-34\times2^x+64=0$

xi. $3^{2x}-12\times3^x+27=0$

xii. $5^{2x}-150\times5^x+3125=0$

xiii. $(x+2)(x-3)(x+10)(x+5)=-396$

xiv. $x(x-1)(x+2)(x+3)=40$

xv. $(x-2)(x-6)(x+4)(x+8)+256=0$

3. Solve the following equations by factoring without using substitution.

i. $2x^4-3x^2+1=0$

ii. $8x^6-7x^3-1=0$ (Find only real roots)

iii. $x^2+\frac{1}{x^2}=2$

iv. $4\times2^{2x}-4\times2^x+1=0$

4. Write two examples of equations that are not quadratic, but can be written in quadratic form.
5. How would you solve the equation $(y-4)^2 - 7(y-4) = -6$? Write explanation and then solve the equation.
6. Solve $y^3 = 125$.

Nature of Roots of a Quadratic Equation

We have already derived the quadratic formula by solving the general form of quadratic equation for x. We know that the solutions of a quadratic equation of the form $ax^2 + bx + c = 0$ with $a \neq 0$, are given by this formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

To define discriminant, first we solve the following examples by using the quadratic formula:

Example A: Solve. $x^2 - 3x - 28 = 0$

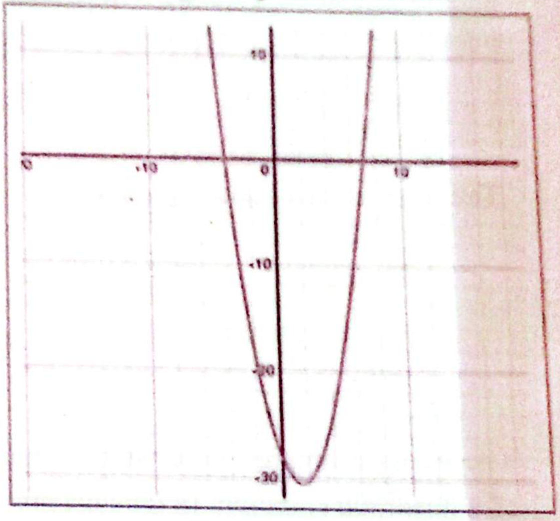
Solution: We have $a = 1$, $b = -3$ and $c = -28$

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-28)}}{2(1)}$$

$$= \frac{3 \pm \sqrt{121}}{2}$$

$$x = \frac{3 \pm 11}{2} = 7, -4$$

The roots of equation are rational and unequal.



Example B: Solve $x^2 - 8x + 16 = 0$

Solution: We have $a = 1$, $b = -8$ and $c = 16$

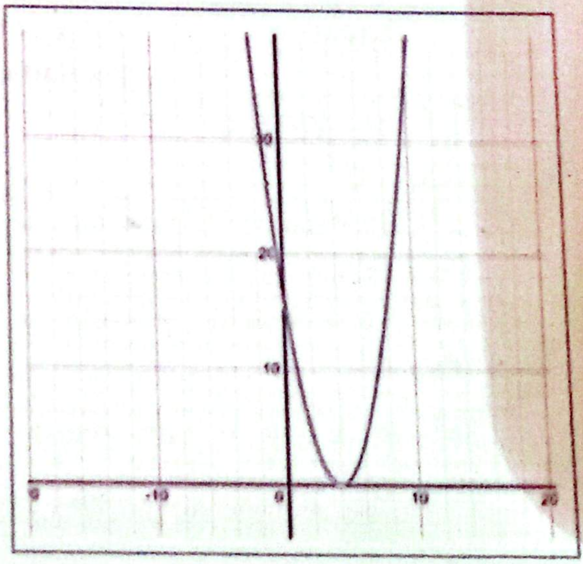
$$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(16)}}{2(1)}$$

$$= \frac{8 \pm 0}{2}$$

$$x = \frac{8+0}{2}, x = \frac{8-0}{2}$$

$x = 4, x = 4$

The roots of equation are rational and equal.



Example C: Solve $3y^2 - 5y + 9 = 0$

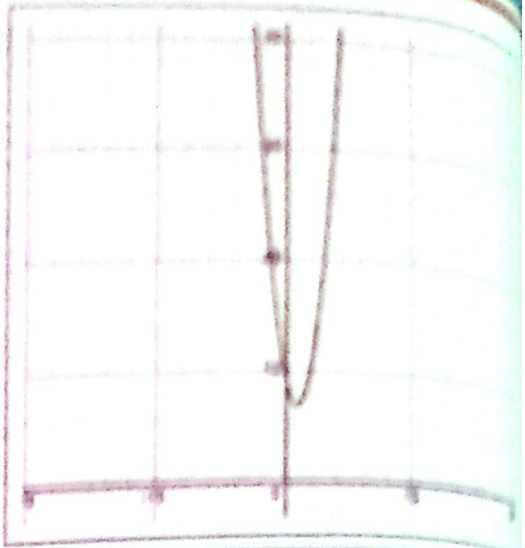
Solution: We have $a = 3$, $b = -5$ and $c = 9$

$$y = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(9)}}{2(3)}$$

$$= \frac{5 \pm \sqrt{-83}}{6} \qquad = \frac{5 \pm \sqrt{83}\,i}{6}$$

Since radical contains a negative value,

therefore, the roots of equation are imaginary.



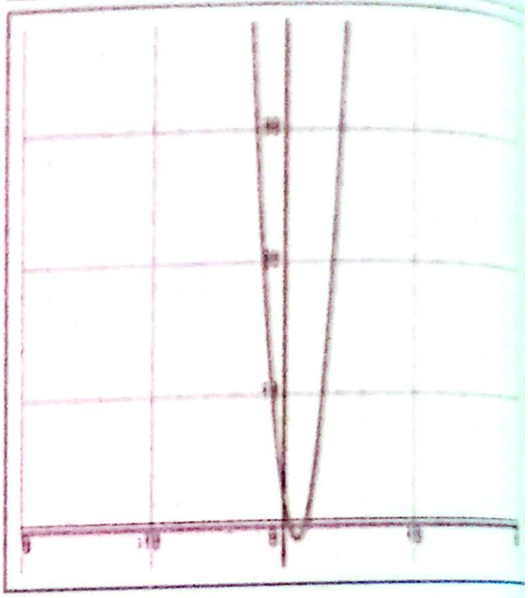
Example D: Solve $3t^2 - 6t + 2 = 0$

Solution: We have $a = 3$, $b = -6$ and $c = 2$

$$t = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(3)(2)}}{2(3)}$$

$$= \frac{6 \pm \sqrt{12}}{6}$$

The roots are irrational and unequal.



These four examples demonstrate a pattern that is useful in determining the nature of the roots of a quadratic equation. In the quadratic formula, the expression under the sign '$b^2 - 4ac$', is called the discriminant. The discriminant tells the nature of roots of a quadratic equation.

Equation	Value of the discriminant	Roots	Nature of Roots
$x^2 - 3x - 28 = 0$ (Exp. A)	$b^2 - 4ac = 121$	$7, -4$	Roots are rational and unequal.
$x^2 - 8x + 16 = 0$ (Exp. B)	$b^2 - 4ac = 0$	$4, 4$	Roots are rational and equal.
$3y^2 - 5y + 9 = 0$ (Exp. C)	$b^2 - 4ac = -83$	$\frac{5 \pm \sqrt{83}\,i}{6}$	Roots are imaginary/complex.
$3t^2 - 6t + 2 = 0$ (Exp. D)	$b^2 - 4ac = 24$	$\frac{3 \pm \sqrt{3}}{3}$	Roots are irrational and unequal.

The chart on the previous page shows that if the value of the discriminant is a perfect square or 0, the roots are real and rational. Other positive discriminant will yield irrational roots. A negative discriminant means roots will be imaginary/complex.

Example: Find the value of discriminant and describe the nature of roots.

a. $2x^2 + x - 3 = 0$ b. $x^2 + 8 = 0$

Solution:

a. $2x^2 + x - 3 = 0$ $a = 2, b = 1, c = -3$ $\text{Disc} = b^2 - 4ac$ $= (1)^2 - 4(2)(-3) = 25$ The value of discriminant is positive and a perfect square. So, the given equation has two real roots and they are rational and unequal.	b. $x^2 + 8 = 0$ $a = 1, b = 0, c = 8$ $\text{Disc} = b^2 - 4ac$ $= (0)^2 - 4(1)(8) = -32$ The value of discriminant is negative. So, the given equation has two imaginary roots.

Check Point:

Verify the nature of roots of $9x^2 - 12x + 4 = 0$, by solving it.

Example: Find the value of m, when $x^2 - 3x + m = 0$ has equal roots.

Solution: $x^2 - 3x + m = 0$

Here $a = 1, b = 3, c = m$

It is given that equation has equal roots (same root) so the discriminant is zero.

i.e. $\text{Disc} = b^2 - 4ac = 0$

$$(3)^2 - 4(1)(m) = 0$$

$$9 - 4m = 0 \quad \text{or} \quad 4m = 9 \quad \text{or} \quad m = \frac{9}{4}$$

Exercise 2.3

1. Find the discriminant of the following quadratic equations.
   (i) $x^2 + 6x - 27 = 0$ (ii) $x^2 - x - 12 = 0$
   (iii) $8x^2 + 2x + 1 = 0$ (iv) $12x^2 - 11x - 15 = 0$
2. Discuss the nature of roots of the following quadratic equations.
   (i) $x^2 - 2x - 15 = 0$ (ii) $x^2 + 3x - 4 = 0$ (iii) $12x^2 + x - 20 = 0$
   (iv) $x^2 + 2x + 8 = 0$ (v) $x^2 + 3x - 9 = 0$
3. For what value of k, $9x^2 - kx + 16 = 0$ is a perfect square?
4. If roots of $x^2 + kx + 9 = 0$ are equal, find k ?
5. Show that the roots of $2x^2 + (mx - 1)^2 = 3$, are equal if $3m^2 + 4 = 0$.

6. Find the value of "m" when roots of the following quadratic equations are equal.

(i) $x^2 - 6x + m = 0$ (ii) $m^2x^2 + (2m + 1)x + 1 = 0$

(iii) $(m + 3)x^2 + (m + 1)x + m + 1 = 0$

7. Show that the roots of the equation $(a^2 + b^2)x^2 + 2(ac + bd)x + c^2 + d^2 = 0$ are imaginary. Moreover, it shows repeated roots if $ad = bc$.

8. Show that the roots of the equation $(ax + c)^2 = 4bx$ will be equal, if $b = ac$.

9. Show that the roots of the following equations are real.

(i) $mx^2 - 2mx + m - 1 = 0$ (ii) $bx^2 + ax + a - b = 0$

10. Show that the roots of the following equation are real.

$(a + b)x^2 - ax - b = 0$

Roots and Coefficients of the Quadratic Equation

Let S_1, S_2 be the roots of $ax^2 + bx + c = 0$, then using the quadratic formula:

$$S_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad S_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$\text{Sum of roots} = S_1 + S_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-b + \sqrt{b^2 - 4ac} - b - \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2b}{2a} = \frac{-b}{a}$$

$$\text{and product of roots} = S_1S_2 = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)$$

$$= \frac{(-b)^2 - (\sqrt{b^2 - 4ac})^2}{4a^2}$$

$$= \frac{4ac}{4a^2} = \frac{c}{a}$$

$\therefore$ Sum of the roots $S_1 + S_2 = -\dfrac{b}{a} = -\dfrac{\text{coefficient of } x}{\text{coeficient of } x^2}$

and product of roots $S_1S_2 = \dfrac{c}{a} = \dfrac{\text{contant term}}{\text{coefficient of } x^2}$

Example: Write down the sum and product of the roots of quadratic equation:

(i) $x^2 - x + 2 = 0$ (ii) $x^2 - 7x + 5 = 0$

(iii) $3x^2 - 11x - 4 = 0$ (iv) $6s^2 + 2s + 3 = 0$

Solution:

(i) $x^2 - x + 2 = 0$ Here $a = 1, b = -1, c = 2$ Sum of the roots $= S_1 + S_2$ $= \frac{-b}{a} = \frac{-(-1)}{1} = 1$ Product of roots $= S_1 S_2$ $= \frac{c}{a} = \frac{2}{1} = 2$	(ii) $3x^2 - 11x - 4 = 0$ $a = 3, b = -11, c = -4$ Sum of roots $= S_1 + S_2$ $= \frac{-b}{a} = \frac{-(-11)}{3} = \frac{11}{3}$ Product of roots $= S_1 S_2$ $= \frac{c}{a} = \frac{-4}{3}$
(iii) $x^2 - 7x + 5 = 0$ $a = 1, b = -7, c = 5$ $S_1 + S_2 = \frac{-b}{a} = \frac{-(-7)}{1} = 7$ $S_1 S_2 = \frac{c}{a} = \frac{5}{1} = 5$	(iv) $6s^2 + 2s + 3 = 0$ $a = 6, b = 2, c = 3$ $S_1 + S_2 = \frac{-b}{a} = \frac{-(2)}{6} = \frac{-1}{3}$ $S_1 S_2 = \frac{c}{a} = \frac{3}{6} = \frac{1}{2}$

Example: If S_1, S_2 are the roots of $x^2 - 8x + 20 = 0$. Find the values of

(i) $S_1^2 + S_2^2$ (ii) $(S_1 - S_2)^2$ (iii) $\frac{1}{S_1+1} + \frac{1}{S_2+1}$

Solution: Here $a = 1, b = -8, c = 20$

Since S_1, S_2 are the roots of $x^2 - 8x + 20 = 0$,

$$\therefore \quad S_1 + S_2 = \frac{-b}{a} = \frac{-(-8)}{1} = 8 \quad \text{and} \quad S_1 S_2 = \frac{c}{a} = \frac{20}{1} = 20$$

(i) Now
$$\begin{aligned} S_1^2 + S_2^2 &= S_1^2 + S_2^2 + 2S_1S_2 - 2S_1S_2 \\ &= (S_1 + S_2)^2 - 2S_1S_2 \\ &= (8)^2 - 2(20) \\ &= 64 - 40 = 24 \end{aligned}$$

(ii)
$$\begin{aligned} (S_1 - S_2)^2 &= S_1^2 + S_2^2 - 2S_1S_2 \\ &= S_1^2 + S_2^2 + 2S_1S_2 - 4S_1S_2 \\ &= (S_1 + S_2)^2 - 4S_1S_2 \\ &= (8)^2 - 4(20) \\ &= 64 - 80 = -16 \end{aligned}$$

(iii)
$$\begin{aligned} \frac{1}{S_1+1} + \frac{1}{S_2+1} &= \frac{S_2 + 1 + S_1 + 1}{(S_2+1)(S_1+1)} \\ &= \frac{S_1 + S_2 + 2}{S_1S_2 + (S_1 + S_2) + 1} \\ &= \frac{8+2}{20+8+1} = \frac{10}{29} \end{aligned}$$

Note:
Instead of S_1, S_2, we can use α and β.

Formation of an Equation whose Roots are Given

We have solved several quadratic equations and found the roots. Here, we may know the roots of a quadratic equation but we do not know the equation itself. Consider the roots of a quadratic quadratic equation are 4 and $-\frac{1}{3}$.

let $x=4$ and $x=-\frac{1}{3}$ are roots and $x-4=0$ and $x+\frac{1}{3}=0$ are factors of required equation

$$(x-4)(3x+1)=0 \quad \text{or} \quad 3x^2+x-12x-4=0$$

$$3x^2-11x-4=0$$

Which is the required equation.

If S_1, S_2 are the roots of required quadratic equation, then

$$\text{Let } x=S_1 \text{ and } x=S_2$$

$$\Rightarrow x-S_1=0, x-S_2=0$$

$$(x-S_1)(x-S_2)=0$$

$$x^2-(S_1+S_2)x+S_1S_2=0,$$

$$S_1+S_2=\text{Sum of roots}=S \quad \text{and} \quad S_1S_2=\text{Product of roots}=P$$

Then, $x^2-Sx+P=0$.........(i)

Above formula (i) is used to form a quadratic equation when its roots are given.

Let us solve some examples to understand the concept.

Example: Form a quadratic equation whose roots are $\frac{1}{4}$ and -1.

Solution: The roots are $\frac{1}{4}$ and -1.

$$\text{Let } x=\frac{1}{4} \text{ and } x=-1$$

$$x-\frac{1}{4}=0 \text{ and } x+1=0$$

$$4x-1=0\,,\ x+1=0$$

Factors of quadratic equation are $(4x-1)$ and $(x+1)$, we have

$$(4x-1)(x+1)=0$$

$$4x^2+4x-x-1=0$$

$$4x^2+3x-1=0,\ \text{required quadratic equation.}$$

Example: Form a quadratic equation with roots:

(i) 5 and 6 (ii) $-\frac{5}{4}$ and $\frac{16}{5}$ (iii) $3+2i$ and $3-2i$

Solution:

(i) Since 5 and 6 are roots then

(iii) $3+2i$ and $3-2i$

S = sum of roots = $5+6=11$

P = product of the roots = $5(6)=30$

The required equation is

$x^2 - Sx + P = 0$ ....(i)

S = Sum = $3+2i+3-2i=6$

P = Product = $(3+2i)(3-2i)$

$= 9-4i^2 = 13$

The equation is:

$$x^2-6x+13=0$$

Substituting these values of S and P in (*i*), we have

$$x^2-11x+30=0$$

(ii) $-\frac{5}{4}$ and $\frac{16}{5}$

$$\text{Sum} = S = -\frac{5}{4}+\frac{16}{5}=\frac{13}{3}$$

$$\text{Product} = P = -\frac{5}{4}\times\frac{16}{5}=-4$$

The equation is: $x^2-\frac{13}{3}x+4=0$

$$\text{or} \quad 3x^2-13x+12=0$$

Example: If S_1, S_2 are the roots of $ax^2+bx+c=0$. Form the equation whose roots are twice to the roots of the given equation.

Solution: If S_1, S_2 are the roots of $ax^2+bx+c=0$ then

$$S_1+S_2=\frac{-b}{a} \quad , \quad S_1S_2=\frac{c}{a}$$

The new roots are $2S_1$ and $2S_2$

Sum of new roots = $S = 2S_1+2S_2$

$$= 2(S_1+S_2)=2(\frac{-b}{a}) = \frac{-2b}{a}$$

Product of new roots $= P = (2S_1)(2S_2) = 4S_1S_2$

$$=4\left(\frac{c}{a}\right)=\frac{4c}{a}$$

Now substituting the values of S and P in $x^2 - Sx + P = 0$, we get

$$x^2-\left(\frac{-2b}{a}\right)x+\frac{4c}{a}=0$$

$$\text{or} \quad ax^2+2bx+4c=0$$

Exercise 2.4

1. Find the sum and product of the roots of the quadratic equation.

(i) $x^2-5x+2=0$ (ii) $-4x^2-6x-2=0$ (iii) $5x^2-2x+2=0$

(iv) $-4x^2-8x-9=0$ (v) $16y^2-17y-12=0$ (vi) $0.3x^2-7.7x+1.8=0$

2. Form a quadratic equation with roots:

(i) $1, -\frac{8}{3}$ (ii) $\sqrt{3}, 2\sqrt{3}$

(iii) $2+\sqrt{3}, 2-\sqrt{3}$ (iv) $5i, -5i$

(v) $7+2i, 7-2i$

3. If S_1, S_2 are the roots of $3x^2-2x+4=0$, find the value of:

(i) $\frac{1}{S_1^2}+\frac{1}{S_2^2}$ (ii) $S_1^2+S_2^2$ (iii) $2S_1+2S_2+4$ (iv) $\frac{1}{S_1}+\frac{1}{S_2}$

(v) $\frac{S_1}{S_2}+\frac{S_2}{S_1}$ (vi) $S_1S_2^2+S_1^2S_2$ (vii) $S_1^3S_2+S_1S_2^3$ (viii) $(S_1-3)(S_2-3)$

4. if S_1, S_2 are the roots of $7x^2+10x+7=0$, form the equations whose roots are:

(i) S_1^2, S_2^2 (ii) $\frac{1}{S_1}, \frac{1}{S_2}$ (iii) $S_1^3S_2, S_1S_2^3$ (iv) $S_1-\frac{1}{S_1}, S_2-\frac{1}{S_2}$

(v) $2S_1+1, 2S_2+1$ (vi) $\frac{S_1}{S_2}, \frac{S_2}{S_1}$ (vii) $S_1+S_2, \frac{1}{S_1}+\frac{1}{S_2}$ (viii) S_1^2+1, S_2^2+1

(ix) $S_1^2+S_2, S_1+S_2^2$

5. If S_1, S_2 are the roots of $x^2+6x+3=0$, form the equation whose roots are $(S_1+S_2)^2, (S_1-S_2)^2$

6. If S_1, S_2 are the roots of $2x^2+6x-3=0$, form the equation whose roots are $S_1-\frac{3}{S_2^2}, S_2-\frac{3}{S_1^2}$

7. Find k if S_1 and S_1-5 are the roots of $x^2-3kx+5=0$.

8. Find k such that 3 is a root of $x^2+kx-21=0$.

Simultaneous Equations

A system of equations having a common solution is called a system of simultaneous equations. The set of all the ordered pairs (x, y) which satisfies the system of equations is called the solution of the system. System of two equation involving two variables.

System of two Equations Involving two Variables

Case I: When one Equation is Linear and One Quadratic

If one of the equations is linear, we can find the value of one variable in terms of the other variable from linear equation. Substituting this value of one variable in the quadratic equation, we can solve it. The procedure is illustrated through the following example.

Example: Solve the system of equations $x+y=7$ and $x^2-xy+y^2=13$

Solution: Given Equations are

$$x+y=7......(i) \qquad x^2-xy+y^2=13............(ii)$$

From equation (i)

$$x=7-y$$

Substituting in equation (ii)

$$(7-y)^2-(7-y)y+y^2=13$$

$$49-14y+y^2-7y+y^2+y^2=13$$

$$3y^2-21y+36=0$$

$$y^2-7y+12=0$$

$$y^2-4y-3y+12=0$$

$$y(y-4)-3(y-4)=0$$

$$(y-4)(y-3)=0$$

$$y-4=0 \text{ or } y-3=0$$

$$y=4, \; y=3$$

Putting $y=3$, in (i), we get $x=7-3=4$

Putting $y=4$, in (i), we get $x=7-4=3$

Hence solution set is $\{(4, 3), (3, 4)\}$

Case II: When both of the Equations are Quadratic

The equations in this case are classified as:

(i) Both the equations contain only x^2 and y^2 terms.

(ii) One of the equations, is homogeneous in x and y.

(iii) Both the equations are non-homogeneous.

The methods of solving these types of equations are explained through the following examples.

Example: Solve the system of equations $x^2+y^2=25$ and $2x^2+3y^2=66$

Solution: Given equations are

$$x^2+y^2=25 \quad (i) \qquad 2x^2+3y^2=66 \quad (ii)$$

Multiplying equation (i) by 2 and subtracting from equation (ii).

$$\begin{array}{r} 2x^2+3y^2=66 \\ \underline{\pm 2x^2 \pm 2y^2=50} \\ y^2=16 \end{array}$$

Putting y^2=16 in equation (i)

$$x^2+16=25$$
$$x^2=9$$

Now from $x^2=9$, we have $x=\pm 3$

and from $y^2=16$, we have $y=\pm 4$

Hence, solution set is $\{(\pm 3, \pm 4)\}$

Example:

Solve the system of equations $x^2+y^2=20$ and $6x^2+xy-y^2=0$

Solution:

Given equations are

$$x^2+y^2=20 \qquad \text{(i)}$$
$$6x^2+xy-y^2=0 \qquad \text{(ii)}$$

From equation (ii)

$$-y^2+xy+6x^2=0$$
$$y^2-xy-6x^2=0$$
$$y^2-3xy+2xy-6x^2=0$$
$$y(y-3x)+2x(y-3x)=0$$
$$(y-3x)(y+2x)=0$$
$$y-3x=0 \quad \text{or} \quad y+2x=0$$
$$y=3x\text{.......(iii)} \quad y=-2x\text{......(iv)}$$

Putting $y=3x$ in (i)

$$x^2+(3x)^2=20$$
$$x^2+9x^2=20$$
$$10x^2=20$$
$$x^2=2$$
$$x=\pm\sqrt{2}$$

When $x=\sqrt{2}$

From (iii) $y=3\sqrt{2}$

When $x=-\sqrt{2}$,

From (iii) $y=-3\sqrt{2}$

Putting $y=-2x$ in (i)

$$x^2+(-2x)^2=20$$
$$x^2+4x^2=20$$
$$5x^2=20$$
$$x^2=4$$
$$x=\pm 2$$

When $x=2$

From (iv) $y=-4$

When $x=-2$

From (iv) $y=4$

Hence, solution set is $\{(\sqrt{2}, \ 3\sqrt{2}), (-\sqrt{2}, \ -3\sqrt{2}), (2, -4), (-2, \ 4)\}$

Exercise 2.5

Solve the following simultaneous equations:

1. $2x+y=1; \quad x^2+y^2=10$
2. $3x-2y=1; \quad x^2+xy-y^2=1$
3. $3x+y+3=0; \quad (x+1)^2-4(x+1)-6=y$
4. $(x+3)^2-(y-2)^2=10; \quad x+y=4$
5. $2x^2-8xy+6y^2=0; \quad x^2+y^2=45$
6. $6x^2-5xy-y^2=0; \quad y^2+4xy=30$
7. $4x^2+4y^2=65; \quad 6x-2y=5$
8. $x^2+y=0; \quad x+y=-2$
9. $x^2+3y^2=14; \quad 3x^2+y^2=6$
10. $2x^2-5y^2=8; \quad x^2+2y^2=13$

Test Taking Tip:
Most standardized tests have a time limit, so you must budget your time carefully. Some questions will be much easier than others. If you cannot answer a question within a few minutes go on to the next one. If there is still time left when you get to the end of the test, go back to the ones that you skipped.

Real Problems Involving Quadratic Equations

There are many problems which lead to quadratic equations. To form an equation, we consider symbols for unknown quantities in the problems.

In order to solve problems, we must have

(i) Suppose the unknown quantities to be x or y.

(ii) Translate the problem into symbols and form the equation satisfying the conditions.

The method of solving the problem will be illustrated through the following examples.

Example: The length of a room is 3 meters greater than its breadth. If the area of the room is 180 square meters, find length and breadth of the room.

Solution: Let the breadth of room = x metres

and the length of room = $x+3$ metres

Area of the room = $x(x+3)$ square metres

By the conditions of the equation

$$x(x+3)=180 \qquad \text{or} \qquad x^2+3x-180=0$$
$$(x+15)(x-12)=0 \qquad \text{or} \qquad x=-15, x=12$$

As breadth cannot be negative, therefore, we take $x=12$.

Length $=x+3=12+3=15$

$\therefore$ Breadth of the room $=12m$ and length of room $=15m$

Example:

The sum of the Cartesian coordinates of a point is 6 and the sum of their squares is 20. Find
coordinate of the point.

Solution:

Let (*x*, y) be the coordinates of required point, then by the given condition we get

$$x+y=6 \qquad (i)$$

$$x^2+y^2=20 \qquad (ii)$$

From(i) $x+y=6 \quad \Rightarrow y=6-x$ putting in (ii)

$$x^2+y^2=20 \quad \Rightarrow x^2+(6-x)^2=20$$

$$x^2+36-12x+x^2=20$$

$$x^2-6x+8=0$$

$$x^2-4x-2x+8=0$$

$$x(x-4)-2(x-4)=0$$

$$(x-4)(x-2)=0$$

$$x-4=0, \quad x-2=0$$

$$x=4, \quad x=2$$

When $x=4$, $y=6-x=6-4=2$

When $x=2$, $y=6-x=6-2=4$

∴ The coordinates of the point are 4 and 2, the points may be (4, 2) and (2, 4).

Problem Solving Plan:	
• Explore the problem.	• Plan the solution.
• Solve the problem.	• Examine the solution.

Exercise 2.6

1. Product of two consecutive positive even numbers is 120. Find the numbers.
2. The difference of a positive number and its square is 380. Find the number.
3. The difference of cubes of two consecutive positive numbers is 91. Find them.
4. The sum of the Cartesian coordinates of a point is 9 and the sum of their squares is 45. Find the coordinates of the point.
5. The sum of two numbers is 11 and the product is 30. Find the numbers.
6. The sum of the squares of two consecutive positive odd integers is 34. Find the integers.
7. The sum of ages of a father and his son is 50 years. Ten years ago, the father was 9 times as old as his son. Find the present age of father and son.
8. A two-digit number is decreased by 45 when the digits are reversed. If the sum of the digits is 11, find the number.
9. Sum of two numbers is 20. Find the numbers if the sum of first number and square of other is 40.
10. The reciprocal of the sum of reciprocals of two numbers is $\frac{12}{5}$. Find the numbers if their sum is 10.

11. A group of 1025 students form two square patterns during morning assembly. One square pattern contains 5 more students than the other. Find the number of students in each pattern.

12. Sum of squares of two positive consecutive numbers is 145. The difference of their squares is 17. Find the numbers.

13. A ball is thrown upwards its height $h(t)$ (in meters) after t seconds is modeled by $d(t) = -5t^2 + 20t + 2$. Find the time when the ball hit the ground.

14. The stopping distance $d(x)$ (in meters) of a car traveling at x km/h is modeled by $d(x) = 0.05x^2 + 0.4x$. If the stopping distance is 30 meters, find the speed of the car.

15. A valuable stamp 4cm wide and 5cm long. The stamp is to be mounted on a sheet of paper that is $5\frac{1}{2}$ times the area of the stamp. Determine the dimensions of the paper that will ensure a uniform border around the stamp.

I have Learnt

- $ax^2 + bx + c = 0, a \neq 0$ is standard form of quadratic equation in one variable x.
- There are three methods for solving a quadratic equation.
  i. Factorization Method ii. Completing Square Method iii. The Quadratic Formula
- Some equations are not quadratic but they can be reduced in quadratic form.
- In the quadratic formula, the expression under the radical sign '$b^2 - 4ac$', is called the discriminant. The discriminant tells the nature of roots of quadratic equation.
- Sum of the roots of a quadratic equation = $S = -\frac{b}{a} = -\frac{\text{coefficient of } x}{\text{coeficient of } x^2}$
- Product of roots of a quadratic equation = $P = \frac{c}{a} = \frac{\text{contant term}}{\text{coefficient of } x^2}$
- Formula used to form a quadratic equation when its roots are given is $x^2 - Sx + P = 0$
- A system of equations having a common solution is called a system of simultaneous equations.
- The set of all the ordered pairs (x, y) which satisfies the system of equations is called the solution of the system.

NEO'S EXERC

1. Encircle the correct option in the following.
   i. Which of the following is a quadratic equation?
   (a) $ax + b = c$ (b) $ax^2 + bx + c$
   (c) $ax^2 + bx + c = 0, a \neq 0$ (d) $ax^2 + bx + c = 0, a = 0$

ii. How many roots of $(x-3)(x-2)=6$ exist?

(a) no (b) 0 (c) 1 (d) 2

iii. What should be added to x^2+x to make it a complete square?

(a) 1 (b) $\frac{1}{4}$ (c) $\frac{1}{2}$ (d) 4

iv. Solution set of $x^2-4=0$ is:

(a) $\{0, 4\}$ (b) $\{2, -2\}$ (c) $\{4, -4\}$ (d) $\{\ \}$

v. Roots of the equation $(x-1)^2=9$ are:

(a) $-2, 4$ (b) $2, 4$ (c) $-4, 2$ (d) $-2, -4$

vi. Solution set of $2^{2x}-2^{x+1}+1=0$?

(a) $\{0\}$ (b) $\{1\}$ (c) $\{0, 1\}$ (d) $\{0, -1\}$

vii. Solution set of $x+\frac{1}{x}=2$ is:

(a) $\{0\}$ (b) $\{-1\}$ (c) $\{-1, 1\}$ (d) $\{1\}$

viii. Which of the following is a reciprocal equation?

(a) $x^2+2x+2=0$ (b) $x^4+x^3+x^2+x+1=0$

(c) $\sqrt{2x+3}=0$ (d) $x^4+2x^3+x^2+4x=0$

ix. 2 and -3 are roots of:

(a) $(x-2)(x-3)=0$ (b) $(x+2)(x+3)=0$

(c) $(x-2)(x+3)=0$ (d) $(x+2)(x-3)=0$

x. The discriminant of $ax^2+bx+c=0$ is:

(a) b^2+4ac (b) b^2-4ac

(c) $4ac-b^2$ (d) $-b^2-4ac$

xi. If S_1, S_2 are the roots of $ax^2+bx+c=0$,then sum of roots is:

(a) $\frac{c}{a}$ (b) $\frac{a}{c}$ (c) $\frac{-b}{a}$ (d) $\frac{a}{b}$

xii. Roots of the equations $x^2-5x+5=0$ are:

(a) Imaginary (b) rational (c) equal (d) irrational

xiii. Sum and product of roots of a quadratic equation are respectively 2 and 5. The equation is:

(a) $x^2-2x+5=0$ (b) $x^2+2x+5=0$

(c) $x^2-2x-5=0$ (d) $x^2+2x-5=0$

2. Find all roots of $8x^6-7x^3-1=0$.

3. For what values of m the roots of the equation $(m-1)x^2+2mx+m+3=0$ are equal? Also solve the equation.

4. If S_1, S_2 are the roots of $ax^2+bx+c=0$, find the value of $(S_1-3)(S_2-3)$.

5. If roots of $25x^2-5ax-b=0$ are equal then find the values of a and b if $a^2+b=6$.

6. A rectangular chocolate box has volume as x^3+2x^2-5x-6. Find the length and width of the box if its height is $x-2$.

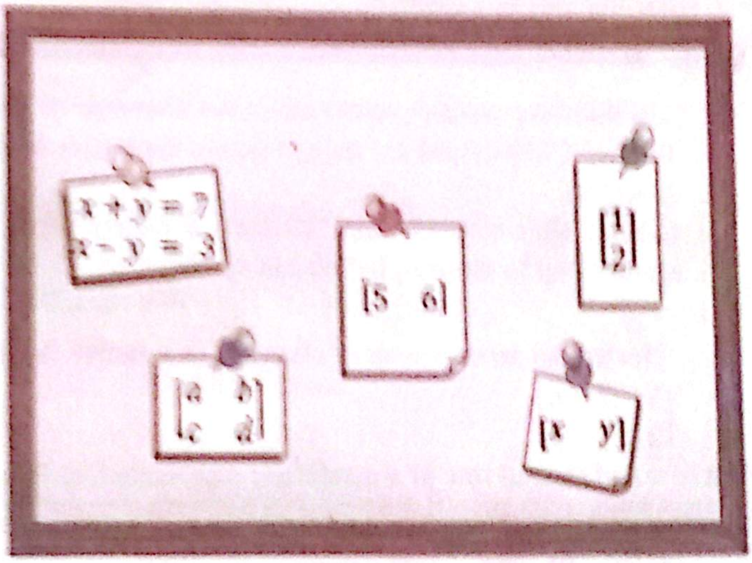
03

Matrices And Determinants

After studying this unit, students will be able to:

- Display information in the form of matrices of order 2.
- Calculate product of scalar quantity and a matrix.
- Solve situations involving sum, difference and product of two matrices.
- Evaluate the determinant and inverse of a matrix of order 2 by 2.
- Solve the simultaneous linear equations in two variables using matrix inversion method and Cramer's rule.
- Explain with examples, how mathematics plays a key role in the development of new scientific theories and technologies.
- Apply concepts of matrices to real world problems.

Matrices are used for solving system of linear equations by several methods. These methods are used in computer programs, in traffic flow, schedule air line flights, engineering, accounting, economics etc.



Introduction of Matrices

In a mathematical quiz between girls and boys, Class 8 (girls) scored 95 points, class 8 (bo scored 97 points, Class 9 (girls) scored 93 points, class 9 (boys) scored 90 points, class 10 (gi scored 96 points and class 10 (boys) scored 94 points. This information can be displayed as und

$$\begin{array}{cc} & \begin{array}{ccc} 8^{th} & 9^{th} & 10^{th} \end{array} \\ Q = & \begin{bmatrix} 95 & 93 & 96 \\ 97 & 90 & 94 \end{bmatrix} \begin{array}{l} \textit{girls} \\ \textit{boys} \end{array} \end{array}$$

This method is no doubt easy to write and manipulate. In 1850, Sylvester introduced su rectangular arrangement and he named it 'Matrix'. Later Hamilton and Cayley made furt significant contributions to the *Matrix Algebra*. Today matrices are being used in almost eve academic discipline in building models, organizing data and solving real world problems.

Matrix (plural Matrices)

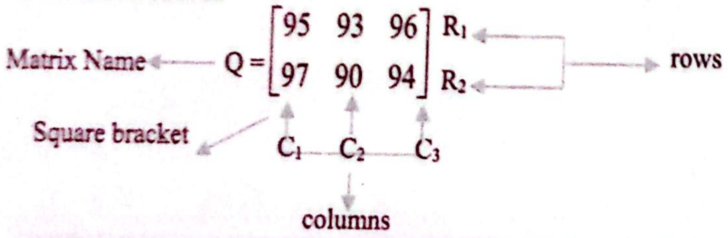
A **Matrix** is a rectangular arrangement of numbers.

e.g. $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$, $[5 \quad 6]$, $[0]$

Check Point:

If $A = [a]$ is a matrix, with only on row and only one column. Can it b called a row matrix or a colum matrix?

A Model Matrix



Key Fact:

- The numbers used in a matrix are called 'elements' or 'entries' of the matrix.
- Elements of a matrix are written within the square brackets in a definite order, in rows and columns.
- Capital alphabets are used to name a matrix while elements of a matrix are usually represented by small alphabets and numbers.

Row: Horizontal arrangement of elements in a matrix is called a row.

$$\begin{bmatrix} 5 & 2 & 6 \\ 3 & 1 & 7 \end{bmatrix}$$

First row and second row of a matrix are represented by R_1 and R_2 respectively.

Column: Vertical arrangement of elements is in a matrix called a column.

$$\begin{bmatrix} 5 & 2 & 6 \\ 3 & 1 & 7 \end{bmatrix}$$

First column, second column and third column of a matrix are represented by C_1, C_2, C respectively.

The number of rows and columns may be equal or unequal in any matrix, however the number of elements in different rows and in different columns remains the same. In [illegible] there are 2 rows and 3 columns. There are 3 elements in each row and 2 elements in each column.

Order of Matrix

If 'A' is a matrix with 'm' number of rows and 'n' number of columns, then order of the matrix is m-by-n.

Order of $\begin{bmatrix}5\\3\end{bmatrix}$ is 2-by-1 i.e. 2 rows and 1 column.

Two matrices are equivalent if and only if they have same rank.

Equal Matrices

Two matrices are said to be equal if and only if their

(i) order is same (ii) corresponding elements are same.

Example:

Check whether the following pairs of matrices are equal or not. Mention the reason in each case.

(i) $\begin{bmatrix}\sqrt{4}\\(-5)^2\end{bmatrix}=\begin{bmatrix}2\\25\end{bmatrix}$

- Matrices on both sides of equality are of order 2-by-1.
- Corresponding elements of the matrices on both sides of equality are also equal. Hence both matrices are equal.

(ii) $A=\begin{bmatrix}13\\17\end{bmatrix}, B=\begin{bmatrix}13 & 17\end{bmatrix}$

Order of matrix A is 2-by-1. Order of matrix B is 1-by-2. Order is different, we can say $A \neq B$.

(iii) $E=\begin{bmatrix}5 & 2\\3 & 6\end{bmatrix}$, $F=\begin{bmatrix}5 & 2\\3 & 3\end{bmatrix}$

Since the corresponding elements are not equal, $\therefore E \neq F$.

Can you find the value of x from $\begin{bmatrix}x & 6\\4 & 5\end{bmatrix}=\begin{bmatrix}9 & 5\\2 & 6\end{bmatrix}$? Mention the reason, if your answer is no.

Example:

Find unknowns from the following if possible.

$$\begin{bmatrix}x+2 & 5\\3 & y-4\end{bmatrix}=\begin{bmatrix}9 & 5\\3 & 7\end{bmatrix}$$

Comparing the corresponding elements of two equal matrices, we get

$$x+2=9, 5=5, 3=3, y-4=7.$$

Now from $x+2=9$, we have $x=7$ and from $y-4=7$, we have $y=11$.

Types of Matrices

Row and Column Matrices

A matrix having only one row is called a row matrix while a matrix having only one column is called a column matrix.

Row matrices: $\begin{bmatrix}5 & 3\end{bmatrix}$, $\begin{bmatrix}a+b+d\end{bmatrix}$, $\begin{bmatrix}\frac{5}{6} & \frac{4}{7}\end{bmatrix}$

Column matrices: $\begin{bmatrix}5\\6\end{bmatrix}$, $\begin{bmatrix}a+b\\b+d\end{bmatrix}$, $\begin{bmatrix}5\\4\end{bmatrix}$

Rectangular Matrix

A matrix with unequal number of rows and columns is called a rectangular matrix.

$$\begin{bmatrix} a+b & c \\ d+e & f \\ g+h & i \end{bmatrix}, \text{[illegible]} \text{ and } \begin{bmatrix} x \\ y \end{bmatrix} \text{ are all rectangular matrices.}$$

Square Matrix

A matrix with equal number of rows and columns is called a square matrix.

$$\begin{bmatrix} 5 & 3 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 9 & 5 \\ 0 & 8 & 2 \\ 2 & 5 & -1 \end{bmatrix} \text{ and } [5] \text{ are all square matrices.}$$

Null Matrix (Zero Matrix, Additive Identity Matrix)

A matrix of any order, having all the elements equal to zero, is called a null matrix.

A null matrix is represented by capital English alphabet 'O', while its elements are all zeros.

Null matrices may have any order.

$$O_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, O_1 = [0], O_{1\times 2} = [0 \;\; 0] \text{ are all null matrices.}$$

Diagonal Matrix

A square matrix in which every element except the primary (principal) diagonal elements is zero is called a diagonal matrix.

$A = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 15 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ are diagonal matrices but $D = \begin{bmatrix} 0 & 5 \\ 3 & 0 \end{bmatrix}$ is not a diagonal matrix.

Scalar Matrix

A diagonal matrix in which all the elements of primary diagonal are equal and non-zero is calle a scalar matrix.

$P = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$, is a scalar matrix of order 2, but $Q = \begin{bmatrix} 0 & 5 \\ 5 & 0 \end{bmatrix}$ is not a scalar matrix, since its primar diagonal elements are zeros and secondary diagonal elements are nonzero.

Some more examples of scalar matrices are:

$$\begin{bmatrix} 3 & 0 \\ 0 & \sqrt{9} \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 3-3 & 4 \div 2 \end{bmatrix}, \begin{bmatrix} 3+7 & 0 & 0 \\ 0 & \frac{20}{2} & 0 \\ 0 & 0 & 2\times 5 \end{bmatrix}$$

Unit Matrix (Multiplicative Identity Matrix)

A scalar matrix in which all the primary diagonal elements are equal to 1, is called a unit matri

Unit matrix is represented by I.

$$I_1 = [1], I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Write the most appropriate type of following matrices

$$A = \begin{bmatrix} 15 & 0 \\ 0 & \frac{45}{3} \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} \frac{\sqrt{9}}{3} & 5-5 \\ \frac{5}{5}-1 & \frac{15}{5\times3} \end{bmatrix}$$

Transpose of a Matrix

If rows (columns) of a matrix P are changed into columns (rows) then the resulting matrix is called transpose of the matrix P. It is represented by P^t.

Example:

If a matrix G represents number of gold medals and silver medals won by three friends, as under:

$$G = \begin{array}{cc} \text{Gold medals} & \text{Silver medals} \\ \end{array}$$

$$G = \left[\begin{array}{cc} 5 & 3 \\ 2 & 6 \\ 1 & 7 \end{array}\right] \begin{array}{l} \textit{Azka} \\ \textit{Aemen} \\ \textit{Khansa} \end{array} \quad \text{then find } G^t.$$

Key Facts:

If order of a matrix P is m-by-n, then order of P^t is n-by-m.

Solution:

$$G^t = \begin{array}{c} \begin{array}{ccc} \text{Azka} & \text{Aemen} & \text{Khansa} \end{array} \\ \left[\begin{array}{ccc} 5 & 2 & 1 \\ 3 & 6 & 7 \end{array}\right] \end{array} \begin{array}{l} \textit{Gold medals} \\ \textit{Silver medals} \end{array}$$

Symmetric Matrices and Skew Symmetric Matrices

For a square matrix A

A is Symmetric if, $A = A^t$. ⟵ ⟶ A is Skew Symmetric if, $A = -A^t$.

Example:

From the following, check for symmetric and skew symmetric matrices.

If $S = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$ then, $S^t = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$.

As $S = S^t$, therefore S is a symmetric matrix.

If $T = \begin{bmatrix} 5 & 2 \\ 6 & 11 \end{bmatrix}$ then, $T^t = \begin{bmatrix} 5 & 6 \\ 2 & 11 \end{bmatrix}$

As $T \neq T^t$, therefore T is a not a symmetric matrix.

If $V = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$

then, $V^t = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} = -V^t$

As $V = -V^t$, therefore V is a skew symmetric matrix.

Check Point:

If $\begin{bmatrix} -3 & x \\ 4 & 2 \end{bmatrix}^t = \begin{bmatrix} -3 & 4 \\ 2 & 2 \end{bmatrix}$,

then find x =?

Key Facts

- All the square matrices of order 1 are symmetric e.g. [9].
- $(A')' = A$.
- All the symmetric matrices are square but every square matrix is not necessarily symmetri

Exercise 3.1

1. Write the number of rows, columns and order of the given matrices.

   (a) $\begin{bmatrix} 1+2 & 4+5 \end{bmatrix}$ (b) $\begin{bmatrix} a+s & 0 \\ 0 & d+s \end{bmatrix}$ (c) $\begin{bmatrix} \frac{5}{7} \end{bmatrix}$ (d) $\begin{bmatrix} 4+2\times 1+5 \end{bmatrix}$

2. Check whether the following pairs of matrices are equal or not.

   (a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \sqrt{25}-\sqrt{25} & \frac{\sqrt{25}}{5} \\ \frac{\sqrt{16}}{4} & \sqrt{16-16} \end{bmatrix}$ (b) $[1 \;\; 2 \;\; 3], [1+2+3]$

   (c) $\begin{bmatrix} \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (d) $\begin{bmatrix} 1+0 & 0+0 \\ 0\times 0 & 1-0 \end{bmatrix}, \begin{bmatrix} \frac{5\times 5}{25} & \frac{3\times 0}{7} \\ 2-17+15 & 17\times\frac{1}{17} \end{bmatrix}$

3. Hyder, Hassan and Ahmed scored 7 points, 11 points and 10 points respectively in a mathematical quiz. Display the data in a row matrix (R) and a column matrix (C).

4. Write the most appropriate type of the following matrices.

   (a) $\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 \\ 0 & 3+0 \end{bmatrix}$ (c) $\begin{bmatrix} \frac{15\times 3}{9\times 5} \end{bmatrix}$

   (d) $\begin{bmatrix} \frac{\sqrt{625}}{25} & 17+15-32 \\ 18-19+1 & \frac{3}{\sqrt{9}} \end{bmatrix}$ (e) $\begin{bmatrix} 6+2 & 0 \\ -3+3 & 8 \end{bmatrix}$

5. Check for symmetric and skew symmetric matrices from the following.

   (a) $\begin{bmatrix} 5 & 2 & 3 \\ 2 & 9 & 6 \\ 3 & 6 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 9 \\ -9 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & -2 & -4 \\ 2 & 0 & -2 \\ 4 & 2 & 0 \end{bmatrix}$

   (d) $\begin{bmatrix} 0 & -7 \\ 7 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} 0 & -5 \\ -5 & 0 \end{bmatrix}$

6. Write a symmetric and a skew symmetric matrix of order 3.

Addition, Subtraction and Scalar Multiplication of Matrices

Addition and Subtraction of Matrices

Two matrices can be added or subtracted only if their order is same.

Consider $A = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$. Here addition and subtraction of A and B is impos because their order is not same.

However, addition and subtraction of A and C is possible as their order is same.

Key Facts

- In the process of addition or subtraction of matrices, only corresponding elements are added or subtracted.
- Matrices of same order are conformable for addition and subtraction.

Example : If $A = \begin{bmatrix}2-3\\65\end{bmatrix}$ and $B = \begin{bmatrix}78\\9-2\end{bmatrix}$ then find $A+B$, $A-B$, $B+A$, $B-A$.

Solution: $A+B = \begin{bmatrix}2-3\\65\end{bmatrix} + \begin{bmatrix}78\\9-2\end{bmatrix} = \begin{bmatrix}2-3+78\\65+9-2\end{bmatrix} = \begin{bmatrix}77\\72\end{bmatrix}$

$A-B = \begin{bmatrix}2-3\\65\end{bmatrix} - \begin{bmatrix}78\\9-2\end{bmatrix} = \begin{bmatrix}2-3-78\\65-9+2\end{bmatrix} = \begin{bmatrix}-79\\58\end{bmatrix}$

Similarly, $B+A = \begin{bmatrix}78+2-3\\9-2+65\end{bmatrix} = \begin{bmatrix}77\\72\end{bmatrix}$,

$B-A = \begin{bmatrix}78-2+3\\9-2-65\end{bmatrix} = \begin{bmatrix}79\\-58\end{bmatrix}$

Check Point:

Can you add $\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$ and $\begin{bmatrix}1\\2\end{bmatrix}$?

Scalar Multiplication of a Matrix

Scalar multiplication of a matrix means multiplication of a matrix with a constant.

If a matrix 'C' is multiplied by 2, then each of its element is doubled.

If $C = \begin{bmatrix}2 & 0\\6 & 0\end{bmatrix}$ then, $2C = 2 \times \begin{bmatrix}2 & 0\\6 & 0\end{bmatrix} = \begin{bmatrix}2\times 2 & 2\times 0\\2\times 6 & 2\times 0\end{bmatrix} = \begin{bmatrix}4 & 0\\12 & 0\end{bmatrix}$

Commutative Property of Matrix Addition

If A and B are two matrices of same order, then A + B = B + A, is called commutative property of matrix addition.

Example:

If $P = \begin{bmatrix}5 & 3\\2 & 9\end{bmatrix}$ and $Q = \begin{bmatrix}7 & 2\\3 & 9\end{bmatrix}$, then verify the commutative property of matrix addition.

Solution:

Commutative property of addition for above matrices is P + Q = Q + P.

L.H.S : $P + Q = \begin{bmatrix}5 & 3\\2 & 9\end{bmatrix} + \begin{bmatrix}7 & 2\\3 & 9\end{bmatrix} = \begin{bmatrix}5+7 & 3+2\\2+3 & 9+9\end{bmatrix} = \begin{bmatrix}12 & 5\\5 & 18\end{bmatrix}$

R.H.S: $Q + P = \begin{bmatrix}7 & 2\\3 & 9\end{bmatrix} + \begin{bmatrix}5 & 3\\2 & 9\end{bmatrix} = \begin{bmatrix}7+5 & 2+3\\3+2 & 9+9\end{bmatrix} = \begin{bmatrix}12 & 5\\5 & 18\end{bmatrix}$

As, P + Q = Q + P, therefore commutative property of matrix addition is verified.

Check Point:

Can you verify the commutative property of matrix addition for:
$A = [x]$ and $B = [x \;\; y]$?

Associative Property of Matrix Addition

If A, B and C are three matrices of same order, then (A + B) + C = A + (B + C), is called associative property of matrix addition.

Example: If $R=\begin{bmatrix}9&7\\5&3\\8&6\end{bmatrix}$, $S=\begin{bmatrix}10&12\\13&7\\5&2\end{bmatrix}$ and $T=\begin{bmatrix}1&5\\6&7\\5&-4\end{bmatrix}$, then verify associative property o

matrix addition.

Solution: Associative property of addition for matrices is $(R+S)+T=R+(S+T)$.

$$\text{L.H.S: } (R+S)+T=\left(\begin{bmatrix}9&7\\5&3\\8&6\end{bmatrix}+\begin{bmatrix}10&12\\13&7\\5&2\end{bmatrix}\right)+\begin{bmatrix}1&5\\6&7\\5&-4\end{bmatrix}=\begin{bmatrix}9+10&7+12\\5+13&3+7\\8+5&6+2\end{bmatrix}+\begin{bmatrix}1&5\\6&7\\5&-4\end{bmatrix}$$

$$=\begin{bmatrix}19&19\\18&10\\13&8\end{bmatrix}+\begin{bmatrix}1&5\\6&7\\5&-4\end{bmatrix}=\begin{bmatrix}19+1&19+5\\18+6&10+7\\13+5&8+(-4)\end{bmatrix}=\begin{bmatrix}20&24\\24&17\\18&4\end{bmatrix}$$

$$\text{R.H.S: } R+(S+T)=\begin{bmatrix}9&7\\5&3\\8&6\end{bmatrix}+\left(\begin{bmatrix}10&12\\13&7\\5&2\end{bmatrix}+\begin{bmatrix}1&5\\6&7\\5&-4\end{bmatrix}\right)$$

$$=\begin{bmatrix}9&7\\5&3\\8&6\end{bmatrix}+\left(\begin{bmatrix}10+1&12+5\\13+6&7+7\\5+5&2+(-4)\end{bmatrix}\right)$$

$$=\begin{bmatrix}9&7\\5&3\\8&6\end{bmatrix}+\begin{bmatrix}11&17\\19&14\\10&-2\end{bmatrix}=\begin{bmatrix}9+11&7+17\\5+19&3+14\\8+10&6-2\end{bmatrix}=\begin{bmatrix}20&24\\24&17\\18&4\end{bmatrix}$$

As, L.H.S = R.H.S, therefore associative property of matrix addition is verified.

Additive Identity in Matrices

Additive identity is such a matrix which causes no change in any matrix A while 'adding to' 'subtracting from' it. Any matrix and its additive identity matrix have the same order.

If $A=\begin{bmatrix}3\\7\\2\end{bmatrix}$, then $O=\begin{bmatrix}0\\0\\0\end{bmatrix}$ is the additive identity of A.

It is observed that: $A+O=O+A=A$

If $B=[\sqrt{7}]$, then the associated additive identity matrix will be $O_1=[0]$.

Check Point

Taking matrices A and B of same order, verify that:

(i) $k(A+B)=kA+kB$

(ii) $(h+k)A=hA+kA$

(iii) $(hk)B=h(kB)$

where h and k are constants.

Additive Inverse of a Matrix

Additive inverse of a matrix 'A' is such a matrix which when added to A, gives additive iden matrix of the same order.

Key Facts

- Matrices A and B are calle additive inverses (negative) o each other if $A+B=O$.
- Additive inverse of a matrix ha the same order as that of th matrix.

Example (a): If $X=\begin{bmatrix}5&0\\-9&4\end{bmatrix}$ then find '$-X$'.

(b): Find T from $5\times\begin{bmatrix}3&2\\1&0\end{bmatrix}-2\,T=\begin{bmatrix}5&4\\3&2\end{bmatrix}$

Solution (a): If $X=\begin{bmatrix}5&0\\-9&4\end{bmatrix}$ then $-X=\begin{bmatrix}-5&0\\9&-4\end{bmatrix}$

(b) $5\times\begin{bmatrix}3&2\\1&0\end{bmatrix}-2\,T=\begin{bmatrix}5&4\\3&2\end{bmatrix}$

$$\Rightarrow\begin{bmatrix}5\times3&5\times2\\5\times1&5\times0\end{bmatrix}-2\,T=\begin{bmatrix}5&4\\3&2\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 15 & 10 \\ 5 & 0 \end{bmatrix} - 2T = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix} \quad \Rightarrow \quad -2T = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} 15 & 10 \\ 5 & 0 \end{bmatrix}$$

$$\Rightarrow -2T = \begin{bmatrix} 5-15 & 4-10 \\ 3-5 & 2-0 \end{bmatrix} = \begin{bmatrix} -10 & -6 \\ -2 & 2 \end{bmatrix}$$

$$\Rightarrow T = \frac{-1}{2}\begin{bmatrix} -10 & -6 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} \frac{-10}{-2} & \frac{-6}{-2} \\ \frac{-2}{-2} & \frac{2}{-2} \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 1 & -1 \end{bmatrix}$$

Multiplication of Matrices

Two matrices A and B are conformable for product AB, if

number of columns in A = number of rows in B

Example:

If $A = \begin{bmatrix} 5 & 3 \\ 1 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$, then find AB and BA, if possible.

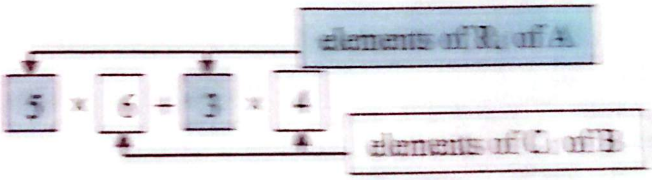
Solution:

Finding AB

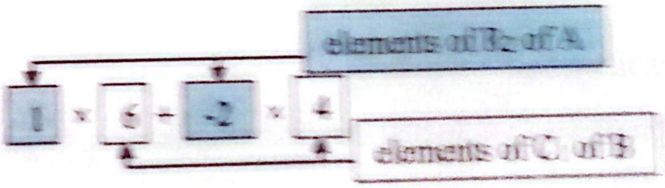
As number of columns in A = 2 = number of rows in B. So, product AB is possible.

$$AB = \begin{bmatrix} 5 & 3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 5\times 6 + 3 \times 4 \\ 1\times 6 + (-2)\times 4 \end{bmatrix} = \begin{bmatrix} 30 + 12 \\ 6 - 8 \end{bmatrix} = \begin{bmatrix} 42 \\ -2 \end{bmatrix}$$

Here



Finding the sum of products of corresponding elements of R₁ of A and C₁ of B.



Finding the sum of products of corresponding elements of R₂ of A and C₁ of B.

Finding BA

Since number of columns of B = 1, but number of rows in A = 2.

So, product BA is not possible.

Key Points

- In multiplication, the first matrix contributes its number of rows and the second matrix contributes its number of columns, in the resultant matrix. i.e. If order of matrix A is m-by-n and order of matrix B is n-by-p, then order of AB is m-by-p.
- If A and B are two matrices which are conformable for product AB then B and A are not necessarily confirmable for product BA.

Example:

Check whether the following pairs of matrices are conformable for multiplication or not? Mention order of product if possible.

(i) $A = \begin{bmatrix}5\\3\end{bmatrix}$ and $B = \begin{bmatrix}9\\3\end{bmatrix}$ (ii) $E = \begin{bmatrix}6\\3\end{bmatrix}^t$ and $F = \begin{bmatrix}6\\3\end{bmatrix}$

Solution:

(i) $A = \begin{bmatrix}5\\3\end{bmatrix}$ and $B = \begin{bmatrix}9\\3\end{bmatrix}$

Here, number of columns in B ≠ number of rows in A, so product AB is not possible.

(ii) $E = \begin{bmatrix}6\\3\end{bmatrix}^t = \begin{bmatrix}6 & 3\end{bmatrix}$ and $F = \begin{bmatrix}6\\3\end{bmatrix}$

Here, number of columns of E = number of rows of F. So, product EF is possible.

Order of E is 1-by-2 and order of F is 2-by-1, so order of EF is 1-by-1.

Example: If $A = \begin{bmatrix}5 & 3\\2 & 6\end{bmatrix}$ and $B = \begin{bmatrix}7 & 2\\4 & 9\end{bmatrix}$ then find AB and BA. Also check whether AB = BA or not. What is concluded from the result?

Solution:
$$AB = \begin{bmatrix}5 & 3\\2 & 6\end{bmatrix} \times \begin{bmatrix}7 & 2\\4 & 9\end{bmatrix}$$
$$= \begin{bmatrix}(5\times 7)+(3\times 4) & (5\times 2)+(3\times 9)\\(2\times 7)+(6\times 4) & (2\times 2)+(6\times 9)\end{bmatrix}$$
$$= \begin{bmatrix}35+12 & 10+27\\14+24 & 4+54\end{bmatrix} = \begin{bmatrix}47 & 37\\38 & 58\end{bmatrix}$$
$$BA = \begin{bmatrix}7 & 2\\4 & 9\end{bmatrix} \times \begin{bmatrix}5 & 3\\2 & 6\end{bmatrix}$$
$$= \begin{bmatrix}7\times 5+2\times 2 & 7\times 3+2\times 6\\4\times 5+9\times 2 & 4\times 3+9\times 6\end{bmatrix}$$
$$= \begin{bmatrix}35+4 & 21+12\\20+18 & 12+54\end{bmatrix} = \begin{bmatrix}39 & 33\\38 & 66\end{bmatrix}$$

Key Facts

If A and B are diagonal matrices, then C = AB is diagonal.

We see that AB ≠ BA.

From this result it is concluded that:

Commutative property does not hold in matrix multiplication, in general.

Associative Property of Matrix Multiplication

Associative property of multiplication holds in matrices. If A, B, and C are matrices and products AB and BC are possible, then:

$$(AB)\,C = A\,(BC)$$

Example: Verify Associative property of matrix multiplication for:
$$X = \begin{bmatrix}5 & 3\\2 & 6\end{bmatrix},\quad Y = \begin{bmatrix}1 & -7\\5 & 4\end{bmatrix},\quad Z = \begin{bmatrix}6\\4\end{bmatrix}$$

Solution: Associative property of matrix multiplication is: $(XY)Z = X\,(YZ)$

L.H.S: $(XY)Z = \left(\begin{bmatrix}5 & 3\\2 & 6\end{bmatrix} \times \begin{bmatrix}1 & -7\\5 & 4\end{bmatrix}\right) \times \begin{bmatrix}6\\4\end{bmatrix} = \begin{bmatrix}5\times 1+3\times 5 & 5\times(-7)+3\times 4\\2\times 1+6\times 5 & 2\times(-7)+6\times 4\end{bmatrix}\begin{bmatrix}6\\4\end{bmatrix}$

$$= \begin{bmatrix} 20 & -23 \\ 32 & 10 \end{bmatrix}\begin{bmatrix} 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 20 \times 6 - 23 \times 4 \\ 32 \times 6 + 10 \times 4 \end{bmatrix} = \begin{bmatrix} 120 - 92 \\ 192 + 40 \end{bmatrix} = \begin{bmatrix} 28 \\ 232 \end{bmatrix}$$

$$\text{R.H.S: } X(YZ) = \begin{bmatrix} 5 & 3 \\ 2 & 6 \end{bmatrix} \times \left(\begin{bmatrix} 1 & -7 \\ 5 & 4 \end{bmatrix} \times \begin{bmatrix} 6 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 5 & 3 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} 1 \times 6 + (-7) \times 4 \\ 5 \times 6 + 4 \times 4 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 3 \\ 2 & 6 \end{bmatrix} \times \begin{bmatrix} 6 - 28 \\ 30 + 16 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} -22 \\ 46 \end{bmatrix} = \begin{bmatrix} 5 \times (-22) + 3 \times 46 \\ 2 \times (-22) + 6 \times 46 \end{bmatrix} = \begin{bmatrix} 28 \\ 232 \end{bmatrix}$$

As L.H.S = R.H.S, so associative property of matrix multiplication is verified.

Distributive Property of Multiplication over Addition / Subtraction

- A (B + C) = AB + AC (left distributive property of multiplication over addition)
- A (B − C) = AB − AC (left distributive property of multiplication over subtraction)

Example (a): Verify A(B + C) = AB + AC

$$A = \begin{bmatrix} 5 & 2 \end{bmatrix}, B = \begin{bmatrix} 6 \\ 4 \end{bmatrix} \text{and } C = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

(b): Verify A(B − C) = AB − AC if $A = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$, $B = [5]$, $C = [7]$.

Solution:

(a) L.H.S: $A(B + C) = \begin{bmatrix} 5 & 2 \end{bmatrix}\left(\begin{bmatrix} 6 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 5 & 2 \end{bmatrix}\begin{bmatrix} 6 + 1 \\ 4 + 2 \end{bmatrix} = \begin{bmatrix} 5 & 2 \end{bmatrix}\begin{bmatrix} 7 \\ 6 \end{bmatrix}$

$$= \begin{bmatrix} 5 \times 7 + 2 \times 6 \end{bmatrix} = \begin{bmatrix} 35 + 12 \end{bmatrix} = \begin{bmatrix} 47 \end{bmatrix}$$

R.H.S: $AB + AC = \begin{bmatrix} 5 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ 4 \end{bmatrix} + \begin{bmatrix} 5 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \times 6 + 2 \times 4 \end{bmatrix} + \begin{bmatrix} 5 \times 1 + 2 \times 2 \end{bmatrix}$

$$= \begin{bmatrix} 30 + 8 \end{bmatrix} + \begin{bmatrix} 5 + 4 \end{bmatrix} = \begin{bmatrix} 38 + 9 \end{bmatrix} = \begin{bmatrix} 47 \end{bmatrix}$$

Comparing both the results, it is concluded that L.H.S = R.H.S

∴ Left distributive property of multiplication over addition is verified.

(b) L.H.S: $A(B - C) = \begin{bmatrix} 4 \\ 8 \end{bmatrix}([5] - [7]) = \begin{bmatrix} 4 \\ 8 \end{bmatrix}[5 - 7] = \begin{bmatrix} 4 \\ 8 \end{bmatrix}[-2]$

$$= \begin{bmatrix} 4 \times (-2) \\ 8 \times (-2) \end{bmatrix} = \begin{bmatrix} -8 \\ -16 \end{bmatrix}$$

R.H.S: $AB - AC = \begin{bmatrix} 4 \\ 8 \end{bmatrix}[5] - \begin{bmatrix} 4 \\ 8 \end{bmatrix}[7] = \begin{bmatrix} 4 \times 5 \\ 8 \times 5 \end{bmatrix} - \begin{bmatrix} 4 \times 7 \\ 8 \times 7 \end{bmatrix}$

$$= \begin{bmatrix} 20 \\ 40 \end{bmatrix} - \begin{bmatrix} 28 \\ 56 \end{bmatrix} = \begin{bmatrix} -8 \\ -16 \end{bmatrix}$$

From both the results, it is concluded that L.H.S = R.H.S

∴ Left distributive property of multiplication over subtraction is verified.

Key Facts:

(A + B) C = AC + BC (Right distributive property of multiplication over addition)

(A − B) C = AC − BC (Right distributive property of multiplication over subtraction)

Multiplicative Identity in Matrices

Multiplicative identity is such a matrix which causes no change in any matrix A when multiplied with A.

- Multiplicative identity matrices of orders 1, 2, 3, ...n are represented by $I_1, I_2, I_3, \ldots I_n$ or simply by I.
- If I is multiplicative identity matrix of square matrix 'S', then IS = SI = S.
- $\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$, $\begin{bmatrix}1\\1\end{bmatrix}$, $[1 \quad 1]$ etc. are not multiplicative identities.
- Since multiplicative identity matrix can never be a rectangular matrix, so multiplicative identity matrices are only possible for square matrices.
- Commutative property of matrix multiplication can hold in particular cases:

(i) $A \times I = I \times A = A$ (if A and I are conformable for either multiplication.)

(ii) $B \times O = O \times B = O$ (if O and B are conformable for either multiplication.)

Exercise 3.2

1. Find x, y, z, from the followings, if possible. Mention the reason if not possible.

(a) $[x \quad 9] = [2 \quad y]$ (b) $\begin{bmatrix}6 & 3\\4 & x\end{bmatrix} = \begin{bmatrix}7 & 3\\4 & 0\end{bmatrix}$ (c) $\begin{bmatrix}5x\\2y\end{bmatrix} = [-10 \quad 20]$

(d) $-\begin{bmatrix}2x & 3y\\4z & 10\end{bmatrix} = \begin{bmatrix}8 & 6\\32 & -10\end{bmatrix}$ (e) $\begin{bmatrix}x & -2y\\6 & x+y\end{bmatrix} = \begin{bmatrix}3 & -6\\6 & z\end{bmatrix}$ (f) $\begin{bmatrix}5\\6\\7\end{bmatrix} = \begin{bmatrix}5 & 0\\x & 0\\7 & 0\end{bmatrix}$

(g) $\begin{bmatrix}x+y\\x-y\end{bmatrix} = \begin{bmatrix}11\\1\end{bmatrix}$ (h) $\begin{bmatrix}5 & 10\\15 & x\end{bmatrix} + \begin{bmatrix}5 & y\\15 & 5\end{bmatrix} = \begin{bmatrix}z & 15\\30 & 7\end{bmatrix}$

(i) $5\begin{bmatrix}x\\3y\end{bmatrix} - \begin{bmatrix}36\\26\end{bmatrix} = 2\begin{bmatrix}-2x\\y\end{bmatrix}$ (j) $\begin{bmatrix}x & y & z\\-2 & -4 & 5\end{bmatrix} + 2 \times \begin{bmatrix}5 & 3 & 2\\1 & 6 & 3\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\0 & 8 & 11\end{bmatrix}$

2. Find the additive inverses of the following.

$$R = \begin{bmatrix}5 & 0 & 3\\7 & -9 & -1\\-8 & 5 & 6\end{bmatrix}, \quad S = \begin{bmatrix}-5 & 2\\3 & -6\\-9 & 4\end{bmatrix}, T = [5 \quad -6 \quad 1].$$

3. If $A = \begin{bmatrix}5 & 3\\-2 & 6\end{bmatrix}$, $B = \begin{bmatrix}10 & 8\\-8 & 6\end{bmatrix}$, $C = \begin{bmatrix}15 & 6\\0 & -12\end{bmatrix}$, then find

(i) $2A + \frac{1}{2}B - \frac{1}{3}C$ (ii) $A - \frac{1}{2}B$

4. If $A = \begin{bmatrix}1 & 2\\3 & -3\end{bmatrix}$, $B = \begin{bmatrix}1 & -2\\3 & 3\end{bmatrix}$, $C = \begin{bmatrix}2 & 0\\6 & 0\end{bmatrix}$, $D = \begin{bmatrix}2\\6\end{bmatrix}$, $E = \begin{bmatrix}2 & 2\\6 & 6\end{bmatrix}$

Then check whether: (i) $A + B = C$ (ii) $C + D = E$ (iii) $D + D = E$ (iv) $E - D = C$

5. Taking matrices A and B from Q.3, verify commutative property of matrix addition.
6. Taking matrices A, B and C from Q. 3, verify associative property of matrix addition.
7. Taking matrices A and B from Q. 4, verify that $A^t + B^t = (A + B)^t$.

8. Find the matrix 'Z' from these equations.

(i) $4\begin{bmatrix}5\\10\end{bmatrix}-5Z=\sqrt{5}\begin{bmatrix}\sqrt{45}\\\sqrt{5}\end{bmatrix}$ (ii) $Z+\begin{bmatrix}5\\-7\end{bmatrix}=3Z-\begin{bmatrix}5\\3\end{bmatrix}$

9. (a) Mention the order of the indicated products where possible.

(i) $\begin{bmatrix}1&0\\0&1\end{bmatrix}\times\begin{bmatrix}5&3\end{bmatrix}$ (ii) $\begin{bmatrix}5&3\end{bmatrix}\times\begin{bmatrix}1&0\\1&0\end{bmatrix}$ (iii) $\begin{bmatrix}1&1\\0&0\end{bmatrix}\times\begin{bmatrix}5\\3\end{bmatrix}$

(iv) $\begin{bmatrix}5\\3\end{bmatrix}\times\begin{bmatrix}0&0\\1&0\end{bmatrix}$ (v) $\begin{bmatrix}5\end{bmatrix}\times\begin{bmatrix}1&0\\1&0\end{bmatrix}$ (vi) $\begin{bmatrix}1&-2\\3&4\end{bmatrix}\times\begin{bmatrix}5\end{bmatrix}$

(vii) $\begin{bmatrix}1\\3\end{bmatrix}\times\begin{bmatrix}5\end{bmatrix}$ (viii) $\begin{bmatrix}5\end{bmatrix}\times\begin{bmatrix}1\\3\end{bmatrix}$ (ix) $\begin{bmatrix}5\end{bmatrix}\times\begin{bmatrix}9&3\end{bmatrix}$

(x) $\begin{bmatrix}9&-3\end{bmatrix}\times\begin{bmatrix}5\end{bmatrix}$ (xi) $\begin{bmatrix}5\end{bmatrix}\times\begin{bmatrix}10\end{bmatrix}$ (xii) $\begin{bmatrix}5&10\end{bmatrix}\times\begin{bmatrix}-2\\3\end{bmatrix}$

(xiii) $\begin{bmatrix}2\\3\end{bmatrix}\times\begin{bmatrix}5&10\end{bmatrix}$ (xiv) $\begin{bmatrix}5&2\\3&4\end{bmatrix}\times\begin{bmatrix}0&1\\1&0\end{bmatrix}$ (xv) $\begin{bmatrix}5&3\\2&6\end{bmatrix}\times\begin{bmatrix}5&-2\end{bmatrix}^t$

(b) Perform the indicated products in part (a) where possible.

10. If $A=\begin{bmatrix}1&0\\2&3\end{bmatrix}$, $B=\begin{bmatrix}2&1\\0&-3\end{bmatrix}$ and $C=\begin{bmatrix}3&0\\-2&1\end{bmatrix}$ then

(i) Find AB and BA, check whether AB = BA or not?
(ii) Find AC and CA, check whether AC = CA or not?
(iii) Verify A(B + C) = AB + AC
(iv) Verify (A − B) C = AC − BC
(v) Verify the associative property of matrix multiplication.
(vi) Find A^2, B^2, A + B, A − B, (A + B)(A − B), $A^2 - B^2$.
(vii) Check whether $(A + B) \times (A - B) = A^2 - B^2$ or not ?
(viii) Check whether $(A - B) \times (A + B) = A^2 - B^2$ or not?
(ix) Check whether (A + B) × (A − B) = (A − B) × (A + B) or not?
(x) Verify that $(AB)^t = B^tA^t$ and $(A^t)^t = A$

Multiplicative Inverse of a Matrix

Before we discuss inverse of a matrix, we need to examine the determinant of a matrix.

Determinant of a Matrix

When a matrix has m rows and n columns, it is called an " m-by-n" matrix and is also called dimensions of a matrix. If a matrix has the same number of rows and columns, it is called a square matrix. With every square matrix is associated a number called its determinant, defined as follows for 2-by-2 matrices:

The determinant of its matrix $\begin{bmatrix}a&c\\b&d\end{bmatrix}$ is denoted $\begin{vmatrix}a&c\\b&d\end{vmatrix}$ and is defined as follows:

$$\begin{vmatrix}a&c\\b&d\end{vmatrix}=ad-bc.$$

Example:

Find |A|, when $A = \begin{bmatrix} 5 & 6 \\ 2 & 3 \end{bmatrix}$

$$\text{Det } A = |A| = \begin{vmatrix} 5 & 6 \\ 2 & 3 \end{vmatrix}$$
$$= (5 \times 3) - (6 \times 2)$$
$$= 15 - 12 = 3$$

The theory of determinants is attribu German, Gottfried Wilhelm Leibniz. He expanded upon the earlier work of Ja mathematician Seki Kowa.

- Every square matrix has a determinant either zero or nonzero.
- Determinants can be represented by $D_1, D_2, D_3, \ldots$
- The determinant of a matrix is not a matrix rather it is a number.
- While writing determinant of a matrix, the elements of the matrix should be written the vertical bars instead of brackets used for matrices.

Example: Evaluate the determinants

$$D_1 = \begin{vmatrix} -4 & 8 \\ 1 & -2 \end{vmatrix} = (-4 \times -2) - (8 \times 1) = 8 - 8 = 0$$

$$D_2 = \begin{vmatrix} \sqrt{5} & \sqrt{4} \\ \sqrt{1} & -\sqrt{5} \end{vmatrix} = (\sqrt{5} \times -\sqrt{5}) - (\sqrt{1} \times \sqrt{4}) = -(\sqrt{5})^2 - \sqrt{4} = -5 - 2 = -7$$

Singular and Non-singular Matrices

Given that A is a square matrix, then

A is singular matrix, if |A| = 0 ⟷ A is non-singular matrix, if |A| ≠ 0

Example:

From the followings, check for the singular and non-singular matrices.

$$A = \begin{bmatrix} 5 & 10 \\ 3 & 6 \end{bmatrix}, B = \begin{bmatrix} 5 & 5 \\ 3 & \sqrt{5} \end{bmatrix}$$

Solution: $|A| = \begin{vmatrix} 5 & 10 \\ 3 & 6 \end{vmatrix} = 5 \times 6 - 3 \times 10 = 30 - 30 = 0$

As |A| = 0, therefore A is a singular matrix.

$$|B| = \begin{vmatrix} 7 & 5 \\ 3 & \sqrt{5} \end{vmatrix} = 7\sqrt{5} - 15 \neq 0.$$

As |B| ≠ 0, therefore B is non-singular matrix.

Check Point

Can you find determinant a rectangular matrix?

Adjoint of a Matrix

Adjoint of a square matrix A (of order 2) is a square matrix obtained by:

(i) interchanging the positions of primary diagonal elements of A.

(ii) changing the signs of secondary diagonal elements of A.

Adjoint of matrix A is represented by adj A.

$$\text{If } A=\begin{bmatrix}5 & -3\\ 4 & -2\end{bmatrix} \xrightarrow{\text{Interchange the position of primary diagonal elements}} \begin{bmatrix}-2 & -3\\ 4 & 5\end{bmatrix} \xrightarrow{\text{Change the signs of secondary diagonal elements}} \begin{bmatrix}-2 & 3\\ -4 & 5\end{bmatrix}=\text{adj}A$$

Finding Multiplicative Inverse of a Matrix

Any matrix $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$, will have an inverse A^{-1} if and only $\begin{vmatrix}a & c\\ b & d\end{vmatrix}\neq 0$ mean A is non singular.

Then $A^{-1}=\dfrac{1}{ad-bc}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$. We can also write: $A^{-1}=\dfrac{1}{ad-bc}$ Adjoint of A.

Example: If $A=\begin{bmatrix}5 & 4\\ 2 & 6\end{bmatrix}$, find A^{-1}.

Solution: $A=\begin{bmatrix}5 & 4\\ 2 & 6\end{bmatrix}$, Compute the value of the determinant A.

$$|A|=\begin{vmatrix}5 & 4\\ 2 & 6\end{vmatrix}=5\times6-4\times2=30-8=22,$$

Since the determinant does not equal 0, therefore A^{-1} exists.

$$A^{-1}=\frac{1}{22}\begin{bmatrix}6 & -4\\ -2 & 5\end{bmatrix},\quad \text{use formula: } A^{-1}=\frac{1}{ad-bc}\text{Adjoint of A.}$$

This can be written as: $A^{-1}=\begin{bmatrix}\frac{6}{22} & \frac{-4}{22}\\ \frac{-2}{22} & \frac{5}{22}\end{bmatrix}=\begin{bmatrix}\frac{3}{11} & \frac{-2}{11}\\ \frac{-1}{11} & \frac{5}{22}\end{bmatrix}$

Key Fact:

- Inverse of a square matrix, if exists, is unique and$(A^{-1})^{-1}=A$.
- A matrix A is invertible if its inverse exists.
- $A^{-1}A=I$, I is identity matrix.

Example: If $B=\begin{bmatrix}5 & 2\\ 6 & 3\end{bmatrix}$, find B^{-1} also verify that $BB^{-1}=B^{-1}B=I$.

Solution: If $B=\begin{bmatrix}5 & 2\\ 6 & 3\end{bmatrix}$, then $|B|=5\times3-2\times6=15-12=3\neq 0$, B^{-1} exists.

Adjoint of $B=\begin{bmatrix}3 & -2\\ -6 & 5\end{bmatrix}$, apply formula: $B^{-1}=\dfrac{1}{|B|}$ Adjoint of B

$$B^{-1}=\frac{1}{3}\begin{bmatrix}3 & -2\\ -6 & 5\end{bmatrix}=\begin{bmatrix}\frac{3}{3} & \frac{-2}{3}\\ \frac{-6}{3} & \frac{5}{3}\end{bmatrix}=\begin{bmatrix}1 & \frac{-2}{3}\\ -2 & \frac{5}{3}\end{bmatrix}$$

Check: $B^{-1}B=\frac{1}{3}\begin{bmatrix}3 & -2\\ -6 & 5\end{bmatrix}\begin{bmatrix}5 & 2\\ 6 & 3\end{bmatrix}=\frac{1}{3}\begin{bmatrix}15-12 & 6-6\\ -30+30 & -12+15\end{bmatrix}=\frac{1}{3}\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}=$

Similarly $=BB^{-1}=\frac{1}{3}\begin{bmatrix}5 & 2\\ 6 & 3\end{bmatrix}\begin{bmatrix}3 & -2\\ -6 & 5\end{bmatrix}=\frac{1}{3}\begin{bmatrix}15-12 & 6-6\\ -30+30 & -12+15\end{bmatrix}=\frac{1}{3}\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}=$

If A and B are squa matrices, then |AB|

Check Point

Verify: $(AB)^{-1}=B^{-1}A^{-1}$, by taking A and B non singular matrix.

Solution of Simultaneous Linear Equations

A large-scale use of matrices in almost every field is in the form of solution of simulta equations. While studying Economics, Statistics, Medical Sciences, Engineering etc. find the solution of linear equations in two or more variables. However, our study a confined to the solution of simultaneous linear equations only in two variables.

Conversion of Matrix Equation into System of Linear Equations

A matrix equation $\begin{bmatrix}5 & 2\\ 3 & 10\end{bmatrix}\times\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}9\\ 23\end{bmatrix}$, can be written as $\begin{bmatrix}5x+2y\\ 3x+10y\end{bmatrix}=\begin{bmatrix}9\\ 23\end{bmatrix}$

Now comparing the corresponding elements of equal matrices

$$5x+2y=9,\ 3x+10y=23$$

Which is a system of two linear equations in two variables x and y.

Conversion of System of Linear Equations into Matrix Equations

By the converse process mentioned above, the system of linear equations can be rewrit equation:

i.e. $5x+2y=9,\ 3x+10y=23$

writing in matric form, we have

$$\begin{bmatrix}5x & +2y\\ 3x & +10y\end{bmatrix}=\begin{bmatrix}9\\ 23\end{bmatrix}$$

or $\begin{bmatrix}5 & 2\\ 3 & 10\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}9\\ 23\end{bmatrix}$ or $AX=B$ is the required matrix equation.

Check P

Is $AB=I=BA$ alw

Solution of System of two Linear Equations in Two Variables

We discuss here two methods for finding solution of liner equations in two variables.

(a) Matrix inversion method (b) Cramer's rule

(a) Matrix Inversion Method

Let us find the solution of:

$$5x + 4y = 14 \quad ; \quad 3x + 7y = 13$$

This system of equations can be rewritten in matrix equation as follows:

$$\begin{bmatrix} 5 & 4 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 14 \\ 13 \end{bmatrix} \quad \ldots\ldots \text{(i)}$$

where $A = \begin{bmatrix} 5 & 4 \\ 3 & 7 \end{bmatrix}$ is a matrix of coefficients, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ is a matrix of variables and $B = \begin{bmatrix} 14 \\ 13 \end{bmatrix}$ is a matrix of constants.

Now equation (i) can also be written as:

$A X = B$

$A^{-1} A X = A^{-1} B$ (Pre multiplying with A^{-1} if it exists.)

$I X = A^{-1} B$ $(A^{-1} A = I)$

$X = A^{-1} B$ ……… (ii)

This equation indicates that the values of the variables in the matrix X are equal to the corresponding elements in the matrix A^{-1} B. To find A^{-1} B, we need A^{-1}.

As, $|A| = \begin{vmatrix} 5 & 4 \\ 3 & 7 \end{vmatrix} = 35 - 12 = 23 \neq 0$, so inverse is possible.

Also, adj $A = \begin{bmatrix} 7 & -4 \\ -3 & 5 \end{bmatrix}$

Now, $A^{-1} = \frac{1}{|A|} \times \text{adj } A = \frac{1}{23} \times \begin{bmatrix} 7 & -4 \\ -3 & 5 \end{bmatrix}$

Substituting the values in equation (ii), we get:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{23} \times \begin{bmatrix} 7 & -4 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} 14 \\ 13 \end{bmatrix} = \frac{1}{23} \times \begin{bmatrix} 98 - 52 \\ -42 + 65 \end{bmatrix} = \begin{bmatrix} \frac{1}{23} \times 46 \\ \frac{1}{23} \times 23 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Comparing the elements of both matrices, we see that $x = 2$ and $y = 1$ solution set = {(2, 1)}

Example:

Solve system of linear equation $3y = 24 - 9x$, $2y + 6x = 10$ (if possible) by matrix inversion method.

Solution: Writing the given system of linear equations in arranged form. i.e.

$9x + 3y = 24$, $6x + 2y = 10$.

Which can be written in matrix equation as:

$$\begin{bmatrix} 9 & 3 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 24 \\ 10 \end{bmatrix}$$

or $AX = B \Rightarrow X = A^{-1} B$

Now $|A| = \begin{vmatrix} 9 & 3 \\ 6 & 2 \end{vmatrix} = 9 \times 2 - 6 \times 3 = 18 - 18 = 0$

As the matrix A is a singular, so its multiplicative inverse does not exist and solution of given system of equation is not possible.

(b) Cramer's Rule

This method is named after Gabriel Cramer (1704 -1752). This rule uses determinants to find the solution of system of linear equations.

Consider a system of linear equations as $x + 2y = 6,\ 4x - 2y = 4$.

Writing it in matrix equation, we get:

$$\begin{bmatrix} 1 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix} \quad \text{or} \quad AX = B$$

The special formulae used in this method are: $x = \frac{D_x}{D}$ and $y = \frac{D_y}{D}$

Where, $D = |A| = \begin{vmatrix} 1 & 2 \\ 4 & -2 \end{vmatrix} = -2 - 8 = -10 \neq 0$. (non singular)

Original C_2 of A

ow $\quad D_x = \begin{vmatrix} 6 & 2 \\ 4 & -2 \end{vmatrix} = -12 - 8 = -20$

Contributed by B

Original C_1 of A

$D_y = \begin{vmatrix} 1 & 6 \\ 4 & 4 \end{vmatrix} = 4 - 24 = -20$

Contributed by B

Using $x = \frac{D_x}{D}$ and $y = \frac{D_y}{D}$

$x = \frac{-20}{-10}$ $\quad y = \frac{-20}{-10}$

$x = 2$ $\quad y = 2$ and solution set = $\{(2, 2)\}$

Exercise 3.3

1. (a) Mention singular and nonsingular matrices from the followings.

$A = \begin{bmatrix} 6 & 4 \\ 2 & 3 \end{bmatrix}$ $\quad B = \begin{bmatrix} -5 & 7 \\ -9 & 4 \end{bmatrix}$, $\quad C = \begin{bmatrix} 9 & 18 \\ 2 & 4 \end{bmatrix}$

$D = \begin{bmatrix} a & b + 1/a \\ a & b \end{bmatrix}$ $\quad E = \begin{bmatrix} a & b + c \\ a & d + c \end{bmatrix}$ (where $a \neq 0$ but $b = d$)

(b) If $|P| = 9$ and $P = \begin{bmatrix} 3 & 3 \\ 1 & 4k \end{bmatrix}$, then find k.

(c) If $|T| = 3$ and adj $T = \begin{bmatrix} 5 & x \\ 3 & 2 \end{bmatrix}$, then find x.

2. (a) Find the multiplicative inverses of these matrices if possible.

$$R = \begin{bmatrix} 4 & -1 \\ -6 & 2 \end{bmatrix}, \quad S = \begin{bmatrix} 5 \\ 3 \end{bmatrix}, \quad T = \begin{bmatrix} 25 & 2 \\ 50 & 4 \end{bmatrix}, \quad U = \begin{bmatrix} x & x+1 \\ y & y+1 \end{bmatrix}, \text{if } x = y.$$

(b) If $R = \begin{bmatrix} 4 & 1 \\ -6 & 2 \end{bmatrix}$ then verify that $RR^{-1} = R^{-1}R = I$.

3. If $Y = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ and $Z = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, then verify that $(YZ)^{-1} = Z^{-1}Y^{-1}$.

4. What is relation among $|A|, |A^{-1}|$ and $|\text{adj } A|$ if $A = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$?

5. Write these matrix equations into system of linear equations if possible.

(i) $\begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$

(iii) $\begin{bmatrix} 0 \\ 3 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 \\ 24 \end{bmatrix}$ (iv) $\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \end{bmatrix}$

6. (a) Write the systems of linear equations in matrix form.

(i) $x + y = 2$, $x - y = 4$ (ii) $2x + y = 90$, $5y - x = 10$ (iii) $y = 3$, $x = 4$

(iv) $\frac{5}{2}x - 3y = 1, \quad \frac{1}{2}y - 4x = 2$

(b) If matrix of coefficients of $5x - 4y = 30$, $10x - ky = 60$ is singular then find k.

7. (a) Use Matrix Inversion Method to solve the following systems of linear equations if possible.

(i) $4x + 3y = -6$, $x + 2y = 1$ (ii) $-x + 2y = 1.5$, $5x + 4y = 3$

(iii) $2y = 10 - 16x$, $24x = 15 - 3y$ (iv) $9 - x = 7y$, $14y + 2x = 18$

(b) Use Cramer's rule to solve following systems of linear equations if possible.

(i) $2x + 3y = 5$, $5x + 10y = 10$ (ii) $x = \frac{2}{3} - 2y$, $4y = 3 - 3x$

(iii) $\frac{6}{10}x + \frac{8}{10}y = 20$, $\frac{8}{10}x - \frac{6}{10}y = 10$ (iv) $16x - 10 + 2y = 0$, $15 - 3y - 24x = 0$

Applications of Matrices

The most difficult part of solving a problem in algebra is almost always translating the pro situation to mathematical language. Once an equation is translated, the rest is u straightforward. In this section, we study systems of equations and how to solve them matrices. We can use matrices to help us solve problems that involve systems of equations. U matrices often simplifies the process of solving these systems.

Example:

Anas bought 3 cream puffs and 5 pringle packs paying Rs.650. Abid bought 4 cream puffs a pringle packs paying Rs. 400. Find how much each item costs? (Hint: Either matrix inver method or Cramer's rule can be used).

Solution:

Let x be the cost of a cream puff and y be the cost of a pringle pack, then Anas's shoppin represented by $3x + 5y = 650$ and Abid's shopping is represented by equation $4x + 2y = 400$

Writing in matrix equation

$$\begin{bmatrix} 3 & 5 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 650 \\ 400 \end{bmatrix} \quad \text{or} \quad AX = B$$

Now by using Cramer's rule

$$D = |A| = \begin{vmatrix} 3 & 5 \\ 4 & 2 \end{vmatrix} = 6 - 20 = -14$$

$$D_x = \begin{vmatrix} 650 & 5 \\ 400 & 2 \end{vmatrix} = 1300 - 2000 = -700\,, \quad D_y = \begin{vmatrix} 3 & 650 \\ 4 & 400 \end{vmatrix} = 1200 - 2600 = -1400$$

$$\text{Now,} \quad x = \frac{D_x}{D} \quad \text{and} \quad y = \frac{D_y}{D}$$

$$x = \frac{-700}{-14} \quad \text{and} \quad y = \frac{-1400}{-14} \quad \Rightarrow x = 50 \quad \text{and} \quad y = 100$$

So, each cream puff costs Rs. 50 and each pringle pack costs Rs. 100.

Example:

Naveed is a chemist who is preparing an acid solution to be used as a cleaner for machine p The machine shops need several batches of 200 ml of solution at a 48% concentration. He has 60% and 40% concentration solutions. The two solutions can be combined to make the solution. How much of each solution should Naveed use to make 200 ml of solution?

Solution:

Let x represent the amount of 60% solution and let y represent the amount of 40% solution.

$x + y = 200$ The total of two amount must be 200 ml.

Now write an equation that represents the proportions of each solution needed.

$60\% + 40\% = 48\%$

$60\%\,(x) + 40\%\,(y) = 48\%\,(x + y)$ Each part contribute to the total

$0.60\,(x) + 0.40\,(y) = 0.48\,(x + y)$ Multiply 100 to remove the decimals

$12\,x - 8y = 0$ Write the equation in standard form

Step 1: Write a system of equations. Then write the system as a matrix equation.

$x+y=200,\ \ 12x-8y=0$ *in matrix form*: $AX=B$, and $X=A^{-1}B$

$$\begin{bmatrix}1 & 1\\12 & -8\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}200\\0\end{bmatrix}, \text{ where } A=\begin{bmatrix}1 & 1\\12 & -8\end{bmatrix}, X=\begin{bmatrix}x\\y\end{bmatrix} \text{ and } B=\begin{bmatrix}200\\0\end{bmatrix}$$

Step 2: To solve the matrix equation, first find the inverse of the matrix A.

$$A^{-1}=\frac{1}{-20}\begin{bmatrix}-8 & -1\\-12 & 1\end{bmatrix}, \text{ where } |A|=-8-12=-20 \text{ and } \textit{Adjoint of } A=\begin{bmatrix}-8 & -1\\-12 & 1\end{bmatrix}$$

$$X=\frac{1}{-20}\begin{bmatrix}-8 & -1\\-12 & 1\end{bmatrix}\begin{bmatrix}200\\0\end{bmatrix}, \text{ where } X=A^{-1}B$$

$$\begin{bmatrix}x\\y\end{bmatrix}=\frac{1}{-20}\begin{bmatrix}-1600\\-2400\end{bmatrix}=\begin{bmatrix}80\\120\end{bmatrix}$$

We have $x=80$ and $y=120$.

This means that 80 ml of the 60% solution is added to 120 ml of the 40% solution to make 200 ml of the 48% solution.

Exercise 3.4

1. From the two acute angles of a right triangle, one angle is 10 (degrees) less than the other. Find both the acute angles by using Cramer's rule.
2. In a quiz, Abdullah correctly replied 15 out of 20 questions from literature and science, getting Rs.125. If every correct literature reply is credited Rs. 5 and every correct science reply is credited Rs. 10, then find the number of correct replies of literature and science separately.
3. If Asad bought 3 notebooks and 2 books paying Rs. 600 and Iman bought 4 notebooks and 1 book paying Rs. 550, find the cost of each notebook and book? (Use Cramer's rule).
4. Sum of ages of two sisters Sundas and Zenab is 22 years. If Sundas is 1 year older than twice the age of Zenab. Find the ages of both the sisters by using matrix inversion method.
5. Two trucks have capacities of 10 tons and 12 tons. They made a total of 20 round trips to haul 226 tons of sand to the community park. How many round trips did each truck make?
6. To make 20 kg of aluminum alloy with 70% aluminum, a metallurgist wants to use two metals with 55% and 80% aluminum content. How much of each metal should he use.
7. Soybean meal contains 16% protein and corn meal contains 9% protein. How many pounds of each should be mixed together in order to get a 350-lb mixture that is 12% protein?
8. The perimeter of a lot is 190 m. The width is one fourth of the length. Find the dimensions.

I have Learnt

- Displaying information in the form of matrices of order 2.
- Calculating product of scalar quantity and a matrix.
- Solving situations involving sum, difference and product of two matrices.
- Evaluating the determinant and inverse of a matrix of order 2 by 2.
- Solving the simultaneous linear equations in two variables using matrix inversion method and Cramer's rule.
- Explaining with examples, how mathematics plays a key role in the development of new scientific theories and technologies.
- Applying concepts of matrices to real world problems

MISCELLANEOUS EXERCISE-3

1. Encircle the correct option in the following.

i. If $\begin{bmatrix} 5 & 3 \\ 2 & 9 \end{bmatrix}^t = \begin{bmatrix} 5 & \frac{x}{2} \\ 3 & 9 \end{bmatrix}$ then $x =$

(a) 6 (b) 4 (c) -6 (d) -4

ii. If $I_3 = \begin{bmatrix} y & 0 & x \\ 0 & z & 0 \\ x & 0 & 1 \end{bmatrix}$ then

(a) $y = x = 1$ (b) $x = z = 0$ (c) $x = z = 1$ (d) $y = z = 1, x = 0$

iii. Additive inverse of unit matrix of order 2, is

(a) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

iv. Multiplicative inverse of a null matrix of order 2, is

(a) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 0 \end{bmatrix}$ (d) impossible

v. A is a symmetric matrix if

(a) $A^t \neq A$ (b) $(A^t)^t \neq -A$ (c) $(A^t)^t = -A$ (d) $(A^t)^t = A^t$

vi. $\begin{bmatrix} \frac{1}{2} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} =$

(a) $\begin{bmatrix} \frac{1}{2} \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (c) $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 \end{bmatrix}$ (d) impossible

vii. Order of matrix A is 1-by-2 and order of matrix B is 2-by-3 then order of AB is

(a) 1-by-3 (b) 3-by-1 (c) 2-by-2 (d) 3-by-2

viii. If $AB = B$, then $A = \ldots\ldots$

(a) I (b) A^{-1} (c) B (d) B^{-1}

ix. $\begin{bmatrix} 15 \\ 25 \end{bmatrix} \times \begin{bmatrix} 3 & 2 \end{bmatrix} =$

(a) $[95]$ (b) $\begin{bmatrix} 45 \\ 50 \end{bmatrix}$ (c) $\begin{bmatrix} 45 & 50 \end{bmatrix}$ (d) $\begin{bmatrix} 45 & 30 \\ 75 & 50 \end{bmatrix}$

x. If $|T| = -1$ then $T^{-1} =$

(a) $-T$ (b) adj T (c) $-$adj T (d) T

xi. Matrix equation for $y + x = 144 + y$ and $x + 2y = x + 13$ is

(a) $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 144 \\ 13 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 144 \\ 13 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 144 \\ 13 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \times \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} 144 \\ 13 \end{bmatrix}$

xii. The matrix of coefficients for $x - y = 3$, is

(a) $[3]$ (b) $[1 \quad -1]$ (c) $\begin{bmatrix} x \\ y \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

xiii. $[1 \quad 2]\,[3] =$

(a) $[3 \quad 6]$ (b) $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$ (c) $[5]$ (d) impossible

xiv. If A and B are two matrices, conformable for the product AB then $(AB)^t =$

(a) A^tB^t (b) $(BA)^t$ (c) B^tA^t (d) AB

xv. If $\begin{bmatrix} -3 & 5 \\ -3 & x-1 \end{bmatrix}$ is singular matrix, then $x =$

(a) 4 (b) 6 (c) -6 (d) -4

xvi. If adj $A = \begin{bmatrix} 5 & 6 \\ 2 & 3 \end{bmatrix}$ then $|A| =$

(a) $|\text{adj } A|$ (b) $|A^t|$ (c) $|-A|$ (d) all a, b, c

2. If $P = \begin{bmatrix} 5 & -1 \\ 2 & -4 \end{bmatrix}$, then show that $PP^{-1} = P^{-1}P = I$.

3. Solve: $5x - 4 = y$; $2y + 8 = 10x$, by Matrix Inversion Method if possible.

4. Use Cramer's rule for solution of $5x + 2y = 19$ and $10x + 4y = 38$, if possible.

5. The daily sum of labour and the material costs of Haani's and Massab's Home Industries are Rs.20,000 and Rs.40,000 respectively. If labour and the material rate, for both the Home Industries is same but Haani's Home Industry has 6 workers who make 80 items and Massab's Home Industry has 12 workers who make 160 items, find the labour and the material cost.

04

Linear and Quadratic Inequalities

After studying this unit students will be able to:

- Solve linear inequalities in one variable.
- Solve quadratic inequalities in one variable.
- Solve two linear inequalities with two unknowns simultaneously.
- Interpret and identify the region in plane bounded by two linear inequalities in two unknowns.

You are checking a bag at an airport. Bags can weigh no more than 60 pounds. Your bag weighs 26.5 pounds. Find the possible weights '*w*' (in pounds) that you can add to the bag.

Write a model:

Weight of bag + Weight you can add ≤ Weight limit

$26.5 + w \leq 60$ (writing inequalities)

$26.5 + w - 26.5 \leq 60 - 26.5$ (subtracting 26.5 from both sides)

$w \leq 33.5$ (simplifying)

You can add less than 33.5 pounds.

Inequalities in One Variable

Furqan and Qasim are the hockey players in the school hockey team. If we compare their scoring for the season, only one of the following statements will be true:

Furqan scored fewer goals than Qasim.

Furqan scored same number of goals as Qasim.

Furqan scored more goals than Qasim.

Let 'a' represents the number of goals of Furqan and 'b' represents the number of goals of Qasim. We can compare the score using an inequality or an equation.

$a < b, \qquad a > b, \qquad a = b$

This is an example of trichotomy property.

Trichotomy Property

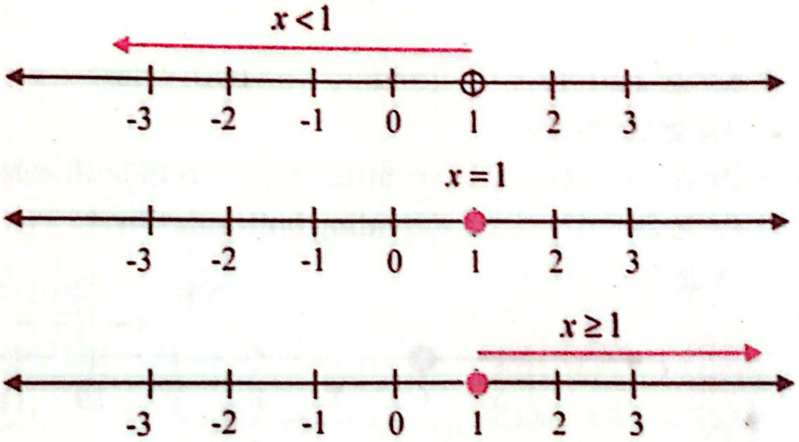
For any two real numbers a and b, exactly one of the following statements will be true:

$a < b, \qquad a > b, \qquad a = b$

Graph of Inequality

On a number line, the graph of an inequality in one variable is the set of points that represents all solutions of the inequality. To graph an inequality in one variable, we use open circle for the symbols '<' or '>' and a closed circle for the symbols '≤' or '≥'.

The graph of $x < 1, x = 1$ and $x \geq 1$ respectively are:



Example: Solve $9x+6<8x-2$. Graph the solution set.

Solution:

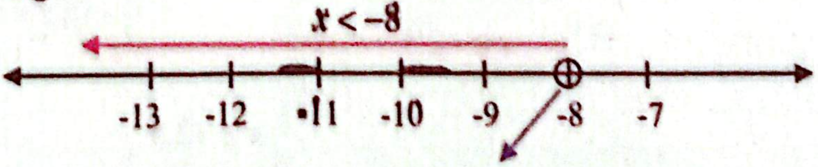
$$9x+6<8x-2$$

$$9x+6+(-8x)<8x+(-8x)-2 \quad \text{(Add} -8x \text{ to both sides)}$$

$$x+6<-2$$

$$x+6+(-6)<-2+(-6) \quad \text{(Add} -6 \text{ to both sides)}$$

$$x<-8$$



Circle indicates that -8 is not included.

Any real number less than – 8 is solution.

Check: Substitute −8 for x in $9x+16<8x-2$, the inequalities should not be true. Then

number less than −8, the inequality should be true.

Key Fact

(i) Addition and subtraction properties of inequalities.

For any real number a, b and c:

- If $a>b$, then $a+c>b+c$ and $a-c>b-c$
- If $a<b$, then $a+c<b+c$ and $a-c<b-c$

(ii) Multiplication and division properties of inequalities.

- If c is positive and $a<b$, then $ac<bc$ and $\frac{a}{c}<\frac{b}{c}$
- If c is positive and $a>b$, then $ac>bc$ and $\frac{a}{c}>\frac{b}{c}$
- If c is negative and $a<b$, then $ac>bc$ and $\frac{a}{c}>\frac{b}{c}$
- If c is negative and $a>b$, then $ac<bc$ and $\frac{a}{c}<\frac{b}{c}$

Example: Solve $6x-7\geq 2x+17$. Graph the solution.

Solution: $6x-7\geq 2x+17.$

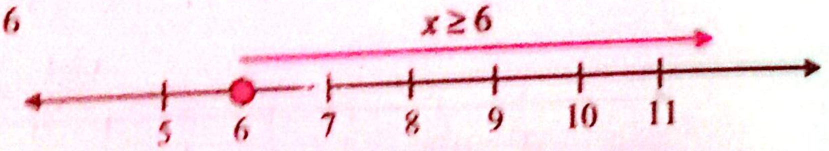
$6x-7+7\geq 2x+17+7$ (adding 7 on both sides)

$6x\geq 2x+24$

$6x-2x\geq 2x-2x+24$ (subtracting $2x$ from both sides)

$4x\geq 24$ (dividing both sides by 4)

$x\geq 6$



Example: Solve and graph the solution

$10.7x-6+5.5x\geq 3(5x-3)$

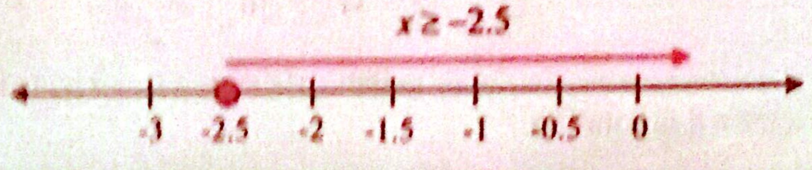
Solution: $10.7x-6+5.5x\geq 3(5x-3)$

$16.2\ x-6\geq 15x-9$

$16.2x-15x\geq -9+6$

$1.2x\geq -3$

$x\geq -2.5$



Check Point:

Describe and correct the error.

a. $17-3x\geq 56$

$-3x\geq 39\ \Rightarrow\ x\geq -13$

b. $-4(2x-3)<28$

$-8x-12<28$

$-8x<-40\ \Rightarrow\ x>5$

Solving Quadratic Inequalities

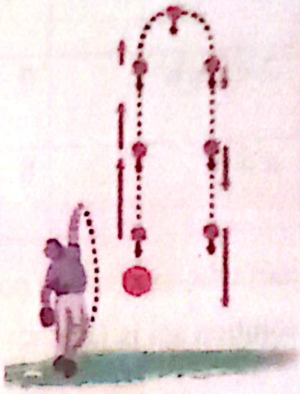
We will discuss the solution of quadratic inequalities in one variable. Quadratic inequalities can be derived from quadratic equations. The word 'Quadratic' comes from the word 'Quadratum' which means square in Latin. From this, we can define quadratic inequalities on second degree equation. The general form of a quadratic equation is $ax^2+bx+c=0$. Further if the quadratic polynomial ax^2+bx+c is not equal to zero, then they are either $ax^2+bx+c>0$ or $ax^2+bx+c<0$ and are called quadratic inequalities. We explain quadratic inequalities with daily life example.

Consider a ball is thrown upward with an initial velocity of 64 feet per second. The function $h(t) = 64t - 16t^2$ gives the height of the ball after t seconds. During what time span is the ball in flight? We can find the times that the ball is in flight by solving the inequality $64t - 16t^2 > 0$.

$$64t - 16t^2 > 0$$

$$-16t^2 + 64t > 0$$

$$-16t(t-4) > 0$$



This inequality shows that the product of two numbers is positive if both of the numbers $(-16t, t-4)$ are positive or both are negative.

Case I:	Case II:
When both numbers are positive	When both numbers are negative
$-16t>0$ and $t-4>0$	$-16t<0$ and $t-4<0$
$t<0$ and $t>4$	$t>0$ and $t<4$
Here, t < 0 and at the same time t > 4 is never true.	Means: $0<t$ and $t<4$ $0 < t < 4$

The solution set is $\{0 < t < 4\}$. So, the ball is in flight between 0 and 4 seconds after it is thrown.

Example: Solve $x^2-4x-12>0$

Solution: $x^2-4x-12>0$

$$x^2-6x+2x-12>0 \quad \Rightarrow \quad x(x-6)+2(x-6)>0$$

$$\Rightarrow \quad (x-6)(x+2)>0$$

Another way that you can solve quadratic inequalities is by using three test points. First solve the equation $(x-6)(x+2)=0$

$$x-6=0 \text{ or } x+2=0$$

$$\Rightarrow \quad x=6,\ x=-2$$

These solutions are called critical points. The points -2 and 6 separate the x-axis into three regions $x<-2,\ -2<x<6,\ x>6$.

Solving Quadratic Inequalities

We will discuss the solution of quadratic inequalities in one variable. Quadratic inequalities can be derived from quadratic equations. The word 'Quadratic' comes from the word 'Quadratum' which means square in Latin. From this, we can define quadratic inequalities on second degree equation. The general form of a quadratic equation is $ax^2+bx+c=0$. Further if the quadratic polynomial ax^2+bx+c is not equal to zero, then they are either $ax^2+bx+c>0$ or $ax^2+bx+c<0$ and are called quadratic inequalities. We explain quadratic inequalities with daily life example.

Consider a ball is thrown upward with an initial velocity of 64 feet per second. The function $h(t)=64t-16t^2$ gives the height of the ball after t seconds. During what time span is the ball in flight? We can find the times that the ball is in flight by solving the inequality $64t-16t^2>0$.

$$64t-16t^2>0$$

$$-16t^2+64t>0$$

$$-16t(t-4)>0$$

This inequality shows that the product of two numbers is positive if both of the numbers $(-16t, t-4)$ are positive or both are negative.

Case I:

When both numbers are positive

$-16t>0$ and $t-4>0$

$t<0$ and $t>4$

Here, $t<0$ and at the same time $t>4$ is never true.

Case II:

When both numbers are negative

$-16t<0$ and $t-4<0$

$t>0$ and $t<4$

Means: $0<t$ and $t<4$

$0<t<4$

The solution set is $\{0<t<4\}$. So, the ball is in flight between 0 and 4 seconds after it is thrown.

Example: Solve $x^2-4x-12>0$

Solution: $x^2-4x-12>0$

$x^2-6x+2x-12>0 \quad \Rightarrow \quad x(x-6)+2(x-6)>0$

$\Rightarrow \quad (x-6)(x+2)>0$

Another way that you can solve quadratic inequalities is by using three test points.

First solve the equation $(x-6)(x+2)=0$

$x-6=0$ or $x+2=0$

$\Rightarrow \quad x=6, x=-2$

These solutions are called critical points. The points -2 and 6 separate the x-axis into three regions $x<-2,\ -2<x<6,\ x>6$.

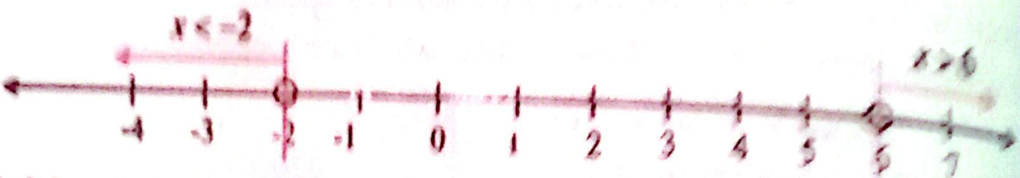
Choose a value from each of the parts and substitute it into $(x-6)(x+2) > 0$.

Construct a table to organize your results.

Part of x-axis	Point x	$(x-6)(x+2)$	Is $(x-6)(x+2) > 0$?
$x < -2$	-3	$(-3-6)(-3+2) = 9$	Yes
$-2 < x < 6$	0	$(0-6)(0+2) = -12$	No
$x > 6$	8	$(8-6)(8+2) = 20$	Yes

The part of x-axis where points are solution of $(x-6)(x+2) > 0$, belongs to the solution se

The solution set is $\{x | x < -2 \text{ or } x > 6\}$.



Example: Find the solution set of $x^2 + 6x - 27 \leq 0$.

Solution: $x^2 + 6x - 27 \leq 0 \quad \Rightarrow \quad x^2 + 9x - 3x - 27 \leq 0$

$x(x+9) - 3(x+9) \leq 0 \quad \Rightarrow \quad (x+9)(x-3) \leq 0$

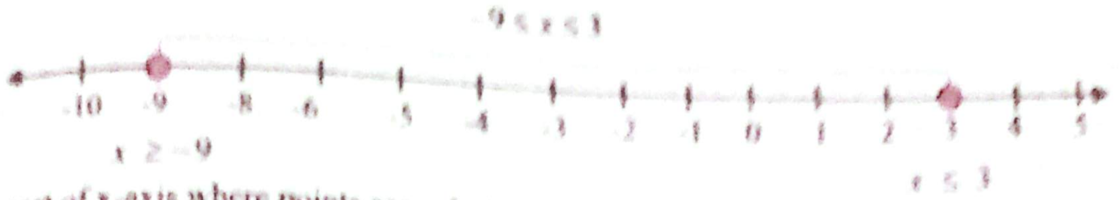
First solve the equation $(x-3)(x+9) = 0$

$x - 3 = 0$ or $x + 9 = 0 \quad \Rightarrow \quad x = 3, x = -9$

The point 3 and −9 are separate, its x-axis in three parts.

Make a table to organize results.

Part of x-axis	Point x	$(x-3)(x+9)$	Is $(x-3)(x+9) \leq 0$?
$x \geq 3$	4	$(4-3)(4+9) = 13$	No
$-9 \leq x \leq 3$	0	$(0-3)(0+9) = -27$	Yes
$x \leq 9$	-10	$(-10-3)(-10+9) = 12$	No



The part of x-axis where points are solution to $(x-3)(x+9) \le 0$ belong to the solution set. The solution set is $\{x \mid -9 \le x \le 3\}$.

Exercise 4.1

Solve the inequality and graph the solution.

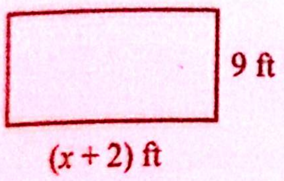
1. $5x-12 \le 3x-4$
2. $1-8x \le -4(2x-1)$
3. $\frac{-2}{3}x-2<\frac{1}{3}x+8$
4. $8-\frac{4}{5}x>-14+2x$
5. $-0.6(x-5) \le 15$
6. $\frac{-1}{4}(x-12)>-2$

Translate the phrase into an inequality. Then solve the inequality and graph the solution.

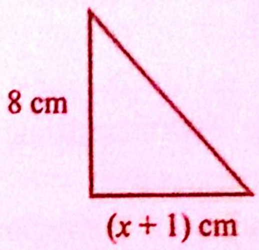
7. Four more than the product of 3 and x is less than 40.
8. Twice the sum of x and 8 appears less than or equal to -2.

Write and solve an inequality to find the possible values of

9. Area > 81 square feet



10. Area ≤ 44 square centimeters



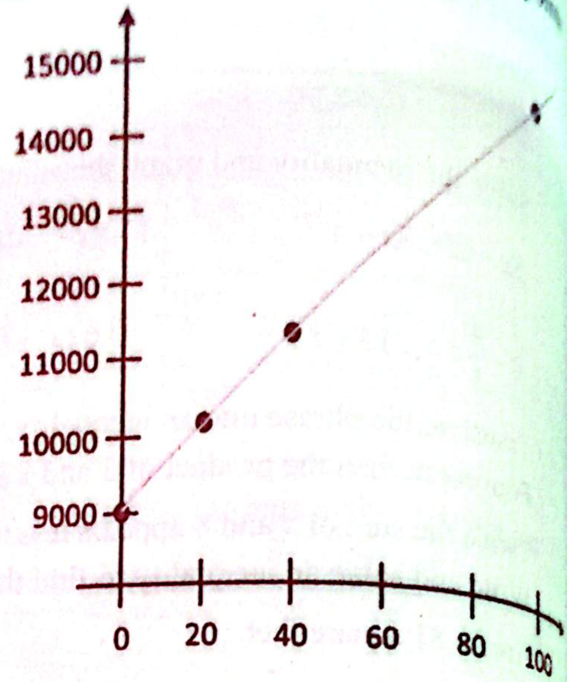
Solve each inequality

11. $(x+1)(x-3)>0$
12. $(x-9)(x+1)<0$
13. $x^2-x-90>0$
14. $x^2+4x-21<0$
15. $x^2+8x+16 \ge 0$
16. $9x^2-6x+1 \le 0$
17. Mr. Khalid has a field and want to make a rectangular garden with perimeter of 68ft. He would like the area of a garden to be at least 240 square feet. What would the width of the garden be?
18. Are $-2, -1, 0, 1, 2$ solution of the inequality $2x^2+3x+1 \le 0$.
19. The stopping distance $d(x)$ (in meters) of a car traveling at x km/h is modeled by $d(x)=0.05x^2+0.2x$. If the maximum stopping distance allowed in a school zone is 35 meters, find the speed range that ensures the stopping distance does not exceed this limit.
20. A rectangular garden is to be enclosed with a fence. The total length of the fence is at most 60 meters and the width x of the garden is twice its length. Find the possible dimensions of the garden.

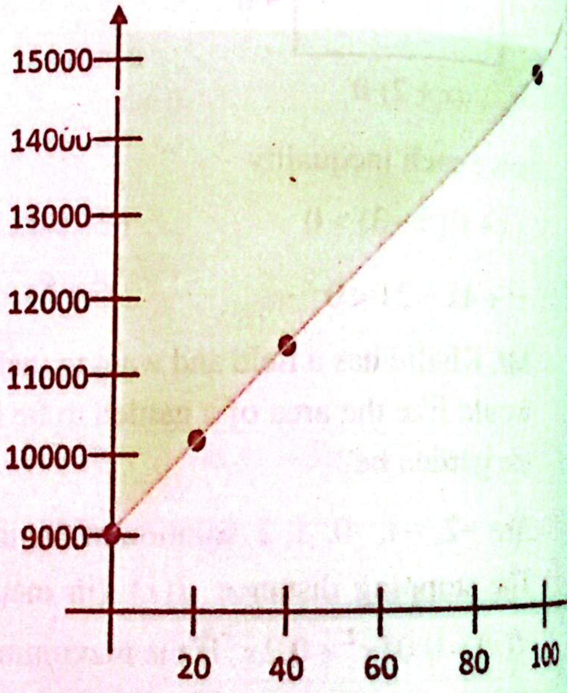
Solving System of Linear Inequalities

Mr. Waleed wants to rent a car for a business trip. A car rental company advertises that rental rate is Rs.9000 plus Rs.60 per km. Mr. Waleed would like to compare this rate with rates offered by other car rental agencies. First, he determines the equation containing the that represent the relationship between the numbers of km (x) and the total cost of the rental (y). The initial cost of the car is Rs. 9000. Since this is the point where zero km are driven, it would be the y-intercept of the graph.
The slope would be the rate of change in the total cost. In this case, the rate is Rs.60 per km.
Since the slope is 60. Thus, an equation of the line is $y=60x+9000$.

km (x)	Cost (y)
0	9000
20	10200
40	11400
100	15000



The graph of $y = 60x + 9000$ separates the coordinate plane into two regions.
The line is called boundary of the region. To graph an inequality, first you graph the boundary and then determine which region to shade. The graph $y > 60x + 9000$ contains the points that are located above the boundary.



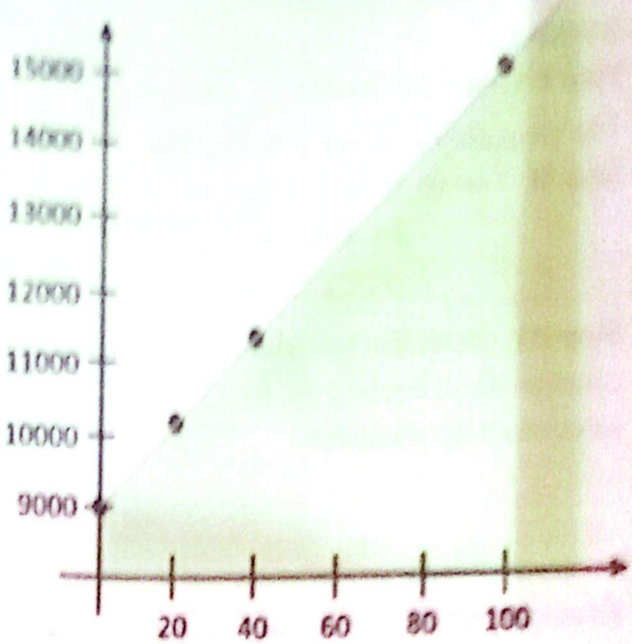
In that region, the value of the dependent variable y is greater than the value of $60x+9000$. This graph represents car rental costs that are greater than those offered by car rental company A. For example, another company B charges 14000 for a car rental with 50km.

The point (50, 14000) lies above the boundary.

The graph of $y > 60x + 9000$ contains points that are located below the boundary. In that region, the value of y is less than the value of $60x + 9000$

The graph represents car rental costs that are less than those offered by company A.

For example, a company C charges 14000 For a car rental with 100km. the point (100, 14000) lies below the boundary.



- When graphing an inequality, the boundary you draw may be solid or dashed.
- If the inequality uses the symbol ≤ or ≥ which include equality then the boundary will be solid. Otherwise, it will be dashed.
- After graphing the boundary, you must determine which region is to be shaded. Test a point on one side of the line. If the ordered pair satisfies the inequality, that region contains solution of the inequality. If the ordered pair does not satisfy the inequality, the other region is the solution.

Key Facts

A linear inequality in two variables: such as $x - 3y < 6$, is the result of replacing the "=" in a linear equation with <,≤,> or ≥. A solution of an inequality in two variables x and y is an ordered pair (x, y) that produces a true statement when the values of x and y are satisfied into the inequality.

Example: Which ordered pair is not a solution of $x - 3y \leq 6$?

a. (0, 0) b. (6, –1) c. (10, 3) d. (–1, 2)

Solution:

Test (0, 0): $x - 3y \leq 6$ (Write inequality)

$0 - 3(0) \leq 6$ (Substitute 0 for x and 0 for y.)

$0 \leq 6$ (True)

Test (6, –1): $x - 3y \leq 6$ (Write inequality)

$6 - 3(-1) \leq 6$ (Substitute 6 for x and –1 for y.)

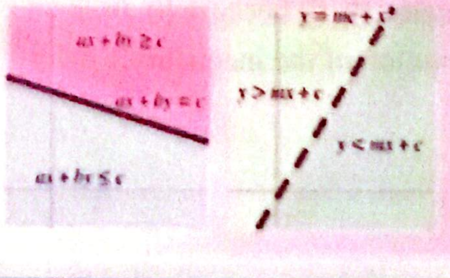
$9 \leq 6$ (False)

Similarly (10, 3) and (–1, 2) also true but (6, –1) is not a solution.

Feasible region

Inequalities represent regions graphically. Below are the rules in drawing feasible region:

> or <	≥ or ≤
Dotted line	Solid line



Example: Graph the inequality $y > 5x - 4$.

Solution:

Step I: Graph the equation $y = 5x - 4$.

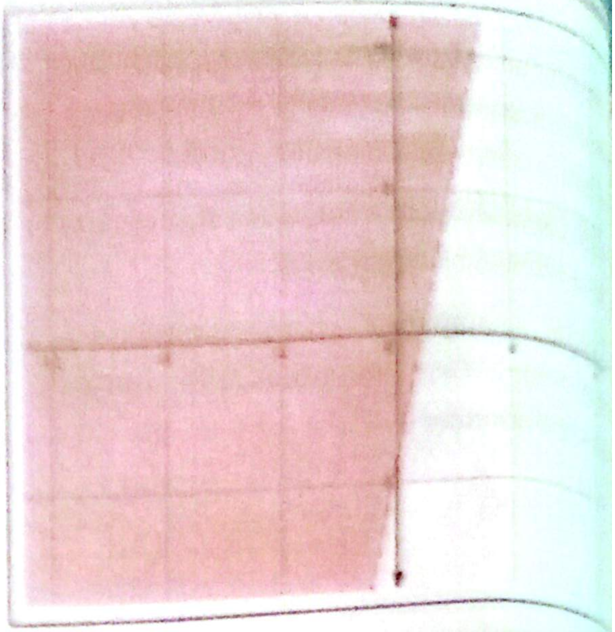
The inequality is >, use a dashed line.

Step II: Test (0, 0) in $y > 5x - 4$.

$$0 > 4(0) - 4$$

$$0 > -3 \text{ (True)}$$

Step III: Shade the half plane that contains (0, 0) because (0, 0) is a solution of the inequality.



Example: Graph the in equality $x + 2y \leq 0$.

Solution:

Step I: Graph the equation $x + 2y = 0$.

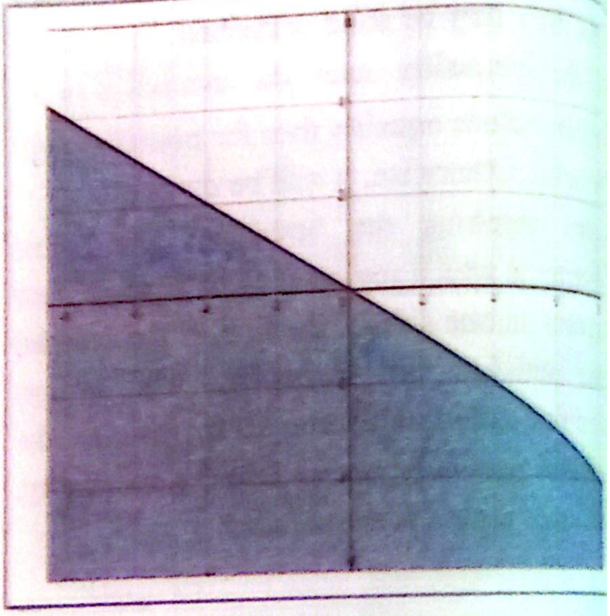
The inequality is ≤, use a solid line.

Step II: Test (1, 0) in $x + 2y \leq 0$.

$$1 + 2(0) \leq 0$$

$$1 \leq 0 \text{ (False)}$$

Step III: Shade the half plane that does not contain (1, 0) because (1, 0) is not a solution of the inequality.



Example: Graph the in equality $y \geq -2$.

Solution:

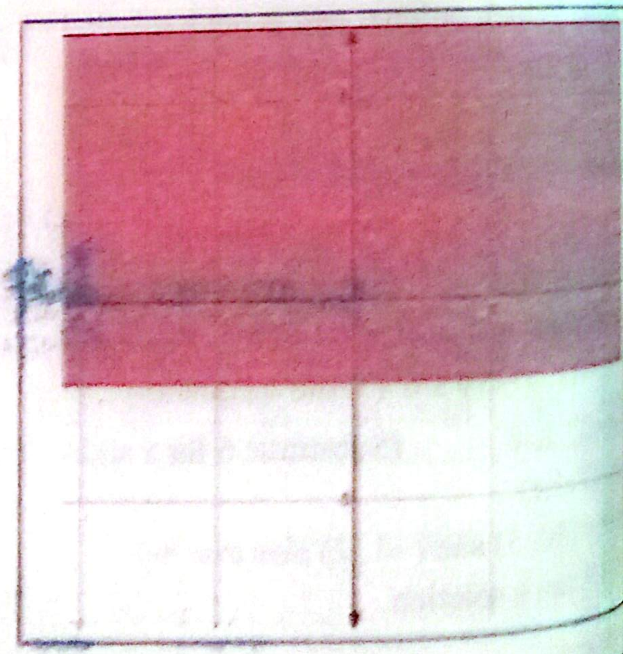
Step I: Graph the equation $y = -2$.

The inequality is ≥, use a solid line.

Step II: Test (0, 0) in $y \geq -2$

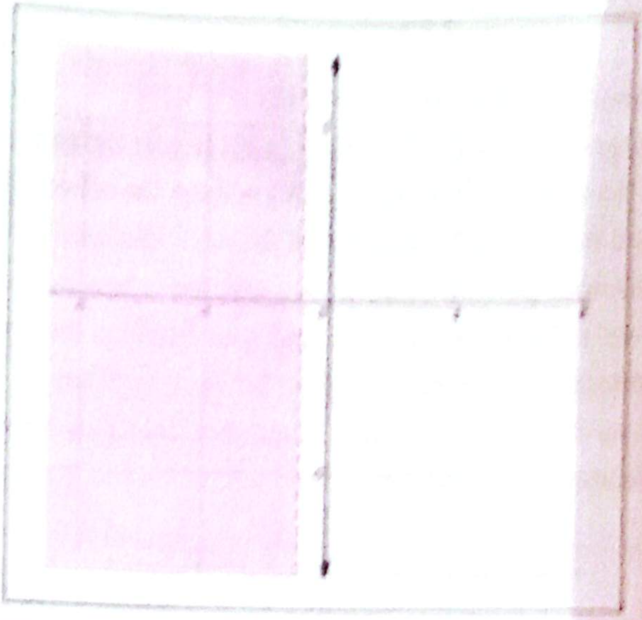
$$0 \geq -2 \text{ (True)}$$

Step III: Shade the half plane that contains (0, 0) because (0, 0) is a solution of the inequality.



Example: Graph the in equality $x < -1$.

Solution:



Step I: Graph the equation $x = -1$.

The inequality is $\geq$, use a solid line.

Step II: Test (0, 0) in $x < -1$

$$0 < -1 \text{ (False)}$$

Step III: Shade the half plane that does not contain (0, 0) because (0, 0) is not a solution of the inequality.

Exercise 4.2

Tell whether the ordered pair is a solution of the inequality.

1. $(0,0);\ x+y<-4$
2. $(-2,-3);\ y-x>-2$
3. $(5,2);\ 2x+3y\geq 14$
4. $(-1,5);\ 4x-7y>28$
5. $(-9,-7);\ y\leq 8$
6. $(-4,0);\ x\geq -3$

Graph the inequality.

7. $x-y<-3$
8. $-3y-2x<12$
9. $x-y\geq 2$
10. $2x+y\geq 8$
11. $x-y\leq -11$
12. $y<-5$
13. $x\geq 4$
14. $\frac{1}{2}(x+2)+3y<8$
15. Can we use (0, 0) as a test point when graphing $x+y>0$? Explain with reason.

Write the verbal sentence as an inequality. Then graph the inequality:

16. Three less than x is greater than or equal to y.
17. The product of -2 and y is less than or equal to the sum of x and 6.
18. The sum of x and the product of 4 and y is less than -2.

Absolute Valued Equations and Inequalities

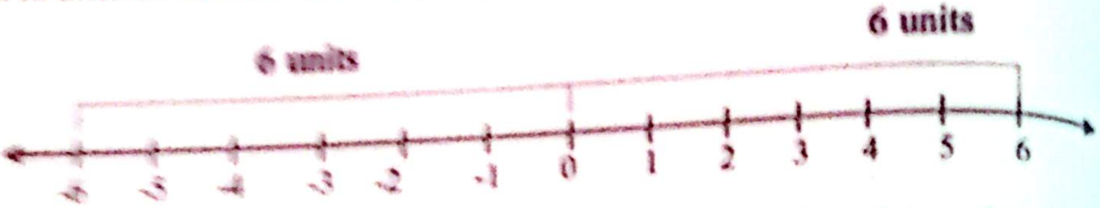
Equations with Absolute Value

Suppose you and your friend, each live in the same street as your school, but on opposite sid of the school. You each live 6 km from the school. What can you say about your trips to scho each day? Do you each travel the same distance?

Consider placing both houses and the school on a number line with the school at the orig

Your home is located at 6, and your friend is located at -6.

Certainly, -6 and 6 are quite different but they do have something in common. They have t same distance from 0 on the number line. This means that you and your friend travel the sam distance, but in different direction, when you go to school.



We say that -6 and 6 have the same absolute value. The absolute value of a number is the number of units it is from 0 in the number line.

We use the symbol $|x|$ to represent the absolute value of a number x.

The absolute value of 6 is $|6| = 6$, and absolute value of -6. is $|-6| = 6$.

Absolute value →

For any real number x

- If $x \geq 0$, then $|x| = x$
- If $x > 0$, then $|x| = -x$
- Or $|x| = \begin{cases} x \text{ when } x \geq 0 \\ -x \text{ when } x > 0 \end{cases}$

In words: The definition states that the absolute value of a non-negative is the number itself an the absolute value of a negative number is the opposite of the number.

Since distance is always non-negative, we can think of a number's absolute value as its distanc from zero on number line.

The absolute value of a number is always positive or zero.

Example: Find the absolute value of 7 and -11.

Solution: $|7| = 7$ $\qquad |-11| = -(-11) = 11$

Example: Find the absolute value of $x - 5$.

Solution: Let's make a list of possible cases.

If x is 5 or greater, then $|x-5| \geq 0$,

$$|x-5| = x-5$$

If x is less than 5, then $|x-5| < 0$,

$$|x-5| = -(x-5) = 5-x$$

Example: Find the solution set.

a. $|2x+5| = 11$

b. $|3-2x| = -5$

c. $|2x-3| = |x+6|$

d. $2|x+3|+2 = 16$

Solution:

a. $|2x+5| = 11$

$2x+5 = 11$ or $2x+5 = -11$

$2x = 11-5$ $\quad$ $2x = -11-5$

$2x = 6$ $\quad$ $2x = -16$

$x = 3$ $\quad$ $x = -8$

S.S $= \{3, -8\}$

b. $|3-2x| = -5$

The absolute value principle reminds us that absolute value is always non-negative. The equation $|3-2x| = -5$ has no solution. The solution set is ϕ.

c. $|2x-3| = |x+6|$

$2x-3 = x+6$ or $2x-3 = -(x+6)$

$2x-x = 6+3$ $\quad$ $2x-3 = -x-6$

$x = 9$ $\quad$ $2x+x = -6+3$

$x = -1$

S.S $= \{-1, 9\}$

d. $2|x+3|+2 = 16$

$2|x+3|+2 = 16$

$2|x+3| = 14-2$

$|x+3| = 7$

$x+3 = 7$ or $x+3 = -7$

$x = 4$ $\quad$ $x = -10$

S.S $= \{4, -10\}$

Student can verify the solution.

Example: A machine that fills the boxes of sugar is to fill the boxes with 14 kg of sugar. After the boxes are filled, another machine weighs the boxes. If the box is more than 0.3 kg above or below the desired weight, the box is rejected. What is the weight of the heaviest and the lightest box that the machine will pass?

Solution: Let w = the weight of the box, then:

$|w-14| = 0.3$ (as per condition)

$w - 14 = 0.3$ or $w - 14 = -0.3$

$w = 14.3$ or $w = 13.7$

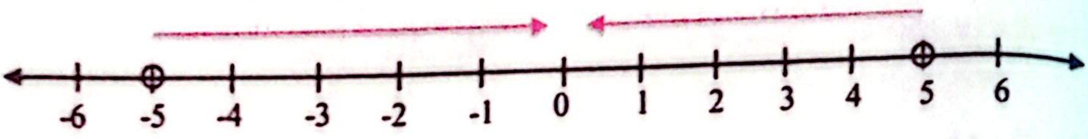
The heaviest weight allowed to pass is 14.3kg. The lightest weight allowed to pass is 13.7kg.

Inequalities With Absolute Value

The absolute value of a number is its distance from 0 on the number line. We can use this id to solve inequalities involving absolute value.

Example: Solve $|x| < 5$.

Solution: $|x| < 5$ means that the distance between x and 0 is less than five units. To make $|x|<$ true, we must substitute the values of x that are less than five units from 0.

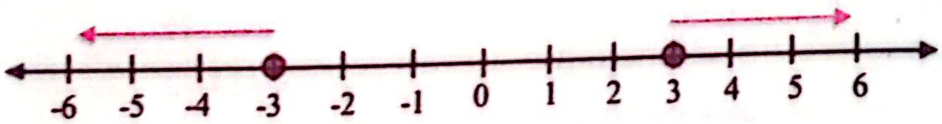


All the number between – 5 and 5 are less than 5 units from zero.

The absolute solution set $= \{|x|: -5 < x < 5\}$.

Example: Solve $|x| \geq 3$. Graph the solution.

Solution: The solution of $|x| \geq 3$. are those numbers whose distance from 0 is greater than o equal to 3. In other words, those numbers x such that $x \leq -3$ or $x \geq 3$. The solution set i $\{|x|: x \leq -3 \text{ or } x \geq 3\}$.



Example: Solve and graph the solution.

a. $|2x-7| \geq 11$

b. $|3x-2| < 4$

Solution:

a. $|2x-7| \geq 11$ (Apply $|x| \geq a: x \geq a$ or $x \leq -a$)

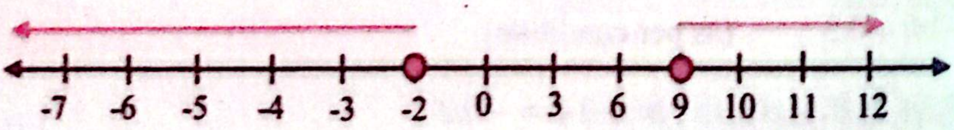
$2x - 7 \geq 11$ or $2x - 7 \leq -11$

$2x \geq 11 + 7$ or $2x \leq -11 + 7$

$2x \geq 18$ or $2x \leq -4$

$x \geq 9$ or $x \leq -2$

Solution set $= \{x | x \geq 9$ or $x \leq -2\}$



b. $|3x-2| < 4$ (Apply: $|x| < a$: $-a < x < a$)

$-4 < 3x - 2 < 4$

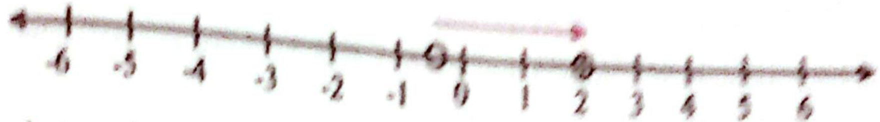
$-4 < 3x-2$ and $3x-2 < 4$

$-4+2 < 3x$ and $3x < 4+2$

$-2 < 3x$ and $3x < 6$

$\frac{-2}{3} < x$ and $x < 2$

Solution set $= \left\{x : \frac{-2}{3} < x < 2\right\}$



Example: Solve $|-4x-5|+3<9$. Graph the solution.

Solution: $|-4x-5|+3<9 \Rightarrow |-4x-5|<6$

$-6 < -4x-5 < 6$

$-6 < -4x-5$ and $-4x-5 < 6$

$-1 < -4x$ and $-4x < 11$

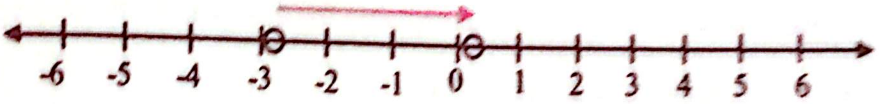
$\frac{1}{4} > x$ and $x > \frac{-11}{4}$

We can write as: $-\frac{11}{4} < x < \frac{1}{4}$

Solution set $= \left\{x : \frac{-11}{4} < x < \frac{1}{4}\right\}$

Key Facts:

- $|x| \le a$ implies that $-a \le x \le a$
- $|x| \ge a$ implies that $x \le -a$ or $x \ge a$



Example:

You are buying a new calculator and find 10 models in a shop. The prices are Rs. 980, Rs. 570, Rs. 470, Rs. 820, Rs. 560, Rs. 770, Rs. 580, Rs. 390, Rs. 680 and Rs. 550.

a. Find the mean of the calculator price.

b. You are willing to pay the mean price with an absolute deviation of at most Rs. 100. How many of the calculator prices meet your condition?

Solution:

a. Find the mean first:

$$\text{Mean} = \frac{980+570+470+820+560+770+580+390+690+550}{10}$$

$$= \frac{6380}{10} = 638$$

b. Write and solve an inequality.

An absolute deviation of at most Rs.100 from the mean 638, is given by the inequality

$$|x-638| \le 100$$

$$-100 \le x - 638 \le 100$$

$$-100 \le x - 638 \text{ and } x - 638 \le 100$$

$$538 \le x \text{ and } x \le 738$$

We write as: $538 \le x \le 738$

The price of calculator must be at least 538 and at most 738. Five prices meet your condition for is Rs. 570, Rs. 560, Rs. 580, Rs. 690 and Rs. 550.

Exercise 4.3

Solve the absolute valued equations (1-6).

1. $8|x-3| = 88$
2. $|2x+9| = 30$
3. $|x-3| = -26$
4. $3|x+6| = 9x-6$
5. $|x-3| = 2x$
6. $\left|\frac{4-5x}{6}\right| = 7$
7. Explain, why the equation $|3x-6|+7=4$ has no solution.
8. Before the start of a professional basketball game, a basketball must be inflated to an air pressure of 8 pounds per square inch (psi) (Absolute error is the absolute deviation). Find the minimum and maximum air pressure acceptable for the basketball.

Solve the inequality and graph the solution (9-14).

9. $|9x-1| \le 10$
10. $|2x-7| < 1$
11. $5\left|\frac{1}{2}x+3\right| > 5$
12. $3|14-x| > 6$
13. $\left|\frac{3x-2}{5}\right| \le 2$
14. $\left|\frac{2-5x}{4}\right| \ge \frac{2}{3}$
15. An essay contest requires that essay entries consists of 500 words with an absolute deviation of at most 30 words. What are the possible number of words that the essay can have?

Solving Systems of Linear Inequalities

A system of linear inequalities in two variables or simply a system of [illegible] two or more linear inequalities in the same variable. As:

$$\left.\begin{array}{l} x - y > 9 \quad \text{Inequality I} \\ 2x + y < 9 \quad \text{Inequality II} \end{array}\right\} A$$

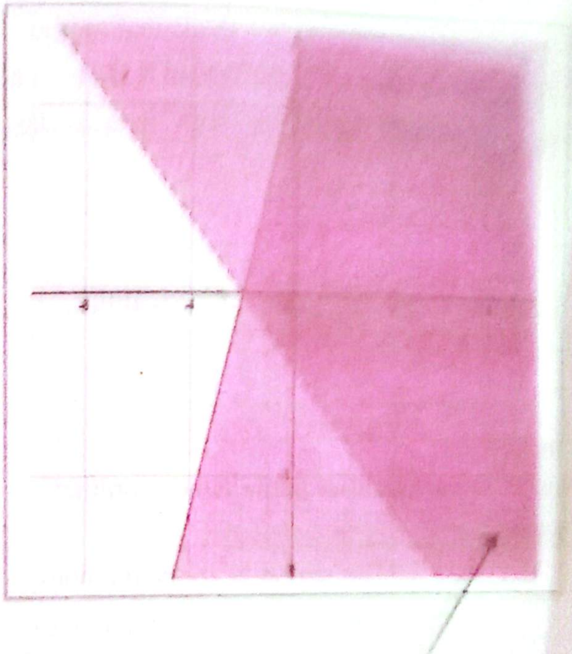
A solution of a system of linear inequalities is an ordered pair that [illegible] inequality in the system. For example, (7, − 6) is a solution of the system A.

The graph of a system of linear inequalities is the graph of all solutions of the system.

Example: Graph the system of inequalities:

$x + y > -2$

$3x - y \geq -6$



Solution:

Step I: Graph both inequalities in the same coordinate plane.

Step II: Find the solution of each inequality. Preferably taking (0, 0) as test point to decide the solution region.

Step III: The graph of the system in the intersection of the two half planes. Which is shaded red.

This dark shaded region is a solution of these inequalities.

Check: Choose a point in the region such as (0, 1). To check this solution, substitute 0 for *x* and 1 for *y* into each inequality.

Inequality I:	Inequality II:
$x + y > -2$	$3x - y \geq -6$
$0 + 1 > -2$	$3(0) - 1 \geq -6$
$1 > -2$ True	$-1 \geq -6$ True

- Graph each inequality in the [illegible]
- Find the intersection of the half planes
- The graph of system is the [illegible]

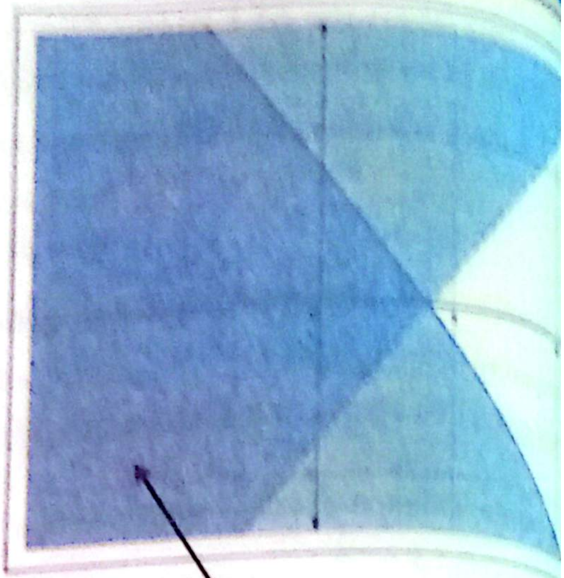
Example: Graph the system:

$$x+y\leq 4$$
$$x-y<4$$

Solution:

To graph the inequality $x+y\leq 4$, we graph $x+y=4$ using a solid line. We then consider (0, 0) as a test point and find that it is a solution, we shade all points in that region.

Next, we graph $x-y<4$. We graph $x-y=4$ as a dashed line and consider (0, 0) as a test point. Again (0, 0) is a solution, we shade that region. Finally, the solution set of the system is the region that is commonly shaded in both inequalities.



This dark shaded region is a solution of these inequalities.

Example: Graph the system of inequalities.

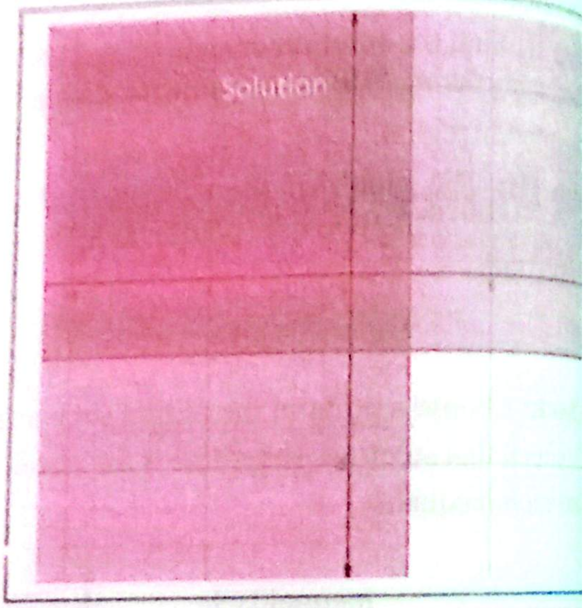
$$x\geq -2$$
$$x\leq 2$$

Solution:

Graph two inequalities in the same coordinate system.

$y=-2$ (The region is above the line)

$x=2$ (The region is left of the line)



Exercise 4.4

1. Describe the steps you would take to graph the system of inequalities shown.

   $x-y<6$......(Inequality I)

   $y\geq 3$......(Inequality II)

2. Tell whether the ordered pair is a solution of the system of inequalities.
   - a. $(3,1),(0,0),(-3,1),(4,1);\ x+y<1, x-y<2$
   - b. $(1,-1),(4,1),(2,0),(3,2);\ 2x-y\leq 5, x+2y>2$

3. Graph the system of inequalities.

a. $\begin{cases} -x + y < 0 \\ x + y > 3 \end{cases}$ b. $\begin{cases} x - y < 0 \\ x + y < 1 \end{cases}$

c. $\begin{cases} y - 2x \geq 3 \\ y - 2x \leq 3 \end{cases}$ d. $\begin{cases} y \leq -2 \\ x \geq 2 \end{cases}$

e. $\begin{cases} x > -2 \\ 2x + y < 3 \end{cases}$ f. $\begin{cases} x + y \leq 3 \\ x - y \leq 4 \end{cases}$

g. $\begin{cases} y \geq 2x - 1 \\ y < -x + 4 \end{cases}$ h. $\begin{cases} x < 8 \\ x - 4y \leq -8 \end{cases}$

4. Does the system of inequalities have any solution?

$$x + y > 5$$
$$x + y < 1$$

5. Open-Ended Question: Describe a real-world situation that can be modeled by a system of linear inequalities. Then write and graph the system of inequalities.

I have Learnt

- On a number line, the graph of an inequality in one variable is the set of points that represent all solutions of the inequality.
- To graph an inequality in one variable are an open circle for < or > and a closed circle for ≤ or ≥.
- Dividing / multiplying each side of an inequality by a positive number produces an equivalent inequality.
- Dividing / multiplying each side of an inequality by a negative number and reversing the direction of the inequality symbol produces an equivalent inequality.
- A linear inequality in two variables, such as $ax + by < c$ is the result of replacing the "=" sign in a linear equation with $<, \leq, >$ or $\geq$.
- A solution of an inequality in two variables x and y is an ordered pair (x, y) that produces a true statement when the values of x and y are substituted into the inequality.
- The boundary line for an inequality in one variable is either vertical or horizontal.
- If an inequality has only the variable x, substitute the x-coordinate of the test point into the inequality.
- If an inequality has only the variable y, substitute the y-coordinate of the test point into the inequality.
- A solution of a system of linear inequalities is an ordered pair that is a solution of each inequality in the system.
- The graph of a system of linear inequalities is the graph of all solutions of the system.

MISCELLANEOUS EXERCISE 4

1. Tick the correct option.

i. The solution of the inequality $6x - 7 \ge 2x + 17$ is:

a. $x < 6$ b. $x \le 6$ c. $x > 6$ d.

ii. The solution of the inequality $x < 3$ for $x \in W$ is:

a. $\{0, 1, 2\}$ b. $\{-2, -1, 0, 1, 2\}$ c. $\{1, 2\}$ d.

iii. In general, we use a test point when graphing the inequality:

a. $(-1, -1)$ b. $(0, 0)$ c. $(2, 2)$ d.

iv. The solution of the $\frac{x}{-2} < 3 - x$ is:

a. $x < 2$ b. $x < 6$ c. $x > 2$ d.

v. Which ordered pair is a solution of the inequality $4x - y \ge 3$?

a. $(0, 0)$ b. $(-1, 2)$ c. $(1, 1)$ d.

vi. Which ordered pair is a solution of the system $2x - y \le 5$ and $x + 2y > 2$?

a. $(1, -1)$ b. $(4, 1)$ c. $(2, 0)$ d.

vii. The solution for $-5x \ge -80$ is:

a. $\{x \mid x < 16\}$ b. $\{x \mid x \le 16\}$ c. $\{x \mid x > 16\}$ d.

viii. $(7, -2)$ is a solution of the inequality:

a. $x - y < -4$ b. $2x + y < 10$ c. $x + 10y < 1$ d.

Solve the inequalities, if possible, graph the solution.

2. $\frac{2}{3}x - 4 \ge 1$

3. $1 - 3x \le -14 + 2x$

4. $7(x - 1) > -8 + 7x$

5. $-3(2x - 1) \ge 1 - 8x$

Graph on a plane.

6. $y > 2x$

7. $y < 3x$

8. $y \ge -5$

9. $x \le 6$

Graph the system of inequalities:

10. $3x + 4y \ge 12$
$5x + 6y \le 30$

11. $y - x \ge 1$
$y - 4 \le 4$

12. $8x + 5y \le 40$
$x \ge 0$

13. $2x + y \le 12$
$y \ge 0$

Write a system of inequalities for the shaded region described.

14. The shaded region is a rectangle with vertices (2, 1), (2, 4), (6, 4) and (6, 1).

15. The shaded region is a triangle with vertices at (3, 0), (3, 2) and (0, −2).

05

Algebraic Fractions

After studying this unit students will be able to:

- Describe rational expression.
- Factorize and simplify rational expression.
- Demonstrate manipulation of algebraic fractions.
- Performs operations on rational expressions.
- Solve rational equations.
- Apply the concept of rational equations to real world problems.

X-ray

Use of letters to represent an unknown quantity was introduced by "Rene Descartes", a French Mathematician, in 1637. Today 'x' is used by most nations as the standard letter for a single unknown. Infect x-rays were so named because the scientists who discovered them did not know what they were and thus labeled them the 'unknown rays' or x-rays. Algebra is basically an extension of Arithmetic.

A factory worker wants to work overtime to fulfill his basic necessities of life. The factory owner has decided to give him Rs. 60 per hour for working extra. But sometimes he does not have enough time to work overtime daily. So he cannot find his accurate monthly income for working extra. If we represent overtime of the worker in hours for a month by an unknown quantity say 't', then we can say that his salary for working overtime will be '60t'.

Algebraic Expressions

A statement in which variables or constants or both are connected by arithmetic operatio (i.e. +, −, ×, ÷) is called an algebraic expression.

For example,

$\frac{-5x^2 + 4}{4}$, $3(a+b)-4$, 0 , -5

$z-\sqrt{2}\,t$, $\frac{1}{x}$, $\frac{x}{y+1}$ $\sqrt{b^2-4ac}$ etc.

History

Muhammad Bin Musa Al–Khawarizimi wa first Muslim mathematician who introdu Algebra and wrote a book entitled Hisab-Al-J Wal Muqabala in 820 A.D. He was known 'Father of Algebra'.

Key Facts

An algebraic expression may contain:

- Numbers
- Signs of operations (+ , − , ×, ÷)
- Variables (*a, b, c,..., x, y, z*)
- Grouping symbols −, ( ) , { } , [ ]

Kinds of Algebraic Expressions

Algebraic expressions are of three kinds.

- Polynomial Expressions
- Rational Expressions
- Irrational Expressions

(a) Polynomial Expressions (Polynomials)

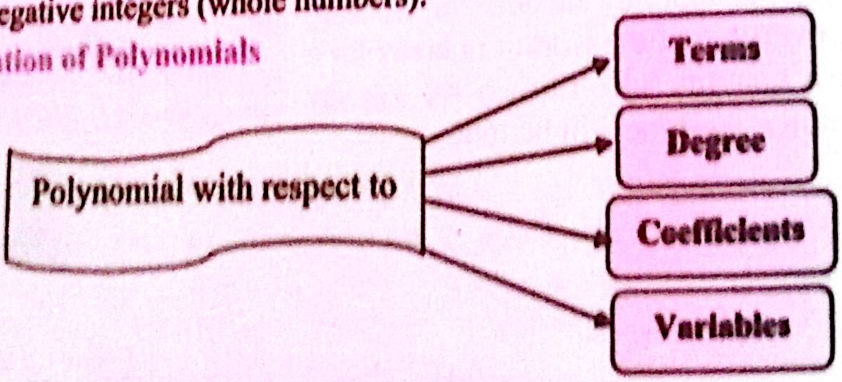
Polynomials are algebraic expressions consisting of one or more terms in which exponent the variables involved are non-negative (either zero or positive) integers.

For example:

$0,\ -2,\ \frac{3}{4}x+\frac{3}{4}y^2z,\ -\sqrt{\frac{3}{9}}\,y^3,\ \sqrt{2}x^4-\pi x^2-\sqrt{10}$ etc.

The expressions x^{-3}, $y^2+\frac{1}{y^2}$, $\sqrt[3]{y^4}$, $2y^{\frac{1}{2}}$ are not polynomials because their exponents a not non negative integers (whole numbers).

Classification of Polynomials



(i) Types of Polynomials w.r.t. Terms

- A polynomial having one term is called a monomial. e.g., 3, x, 0 etc.
- A polynomial having two terms is called a binomial.

  e.g. $a + b,\ 2x + y,\ 7ab - \frac{1}{3},\ x^{3} + 2\frac{1}{4}$ etc.
- A polynomial having three terms is called a trinomial.

  e.g. $x^2 + 2xy + y^2,\ \frac{1}{8}x^3 + \frac{3}{4}x^2 - 9$ etc.

All other polynomials having more than three non- zero terms are called polynomials of four terms, five terms, six terms and so on.

When a polynomial is written in descending order, then the first coefficient is called the leading coefficient. e.g., in $p(x) = 3x^4 - 2x^2 + 1$, the leading coefficient is 3. Here $p(x)$ means that the variable in the polynomial is 'x'.

(ii) Types of Polynomials w.r.t. Degree

➤ **Zero polynomial or no degree polynomial:**

'0' is called a polynomial of no degree. Also, $0x^3 + 0x$ is a no degree polynomial, because coefficients are always zero in zero polynomial.

➤ **Constant polynomial:**

A polynomial having degree zero is called a constant polynomial. e.g. $2, -5, \frac{1}{2}, \sqrt{5}$ are all constant polynomials.

➤ **Linear polynomial:**

A polynomial having degree one is called a linear polynomial. e.g., x, $2x - y$, $-7xy^0$ etc.

➤ **Quadratic polynomial:** A polynomial having degree two is called a quadratic polynomial. e.g., $2x^2 + 7$, $ax + 2xy + 3$, $-\frac{3}{4}xyz^0$ etc.

➤ **Cubic polynomial:**

A polynomial having degree three is called a cubic polynomial. e.g. $9x^3 - 7x + 5$, $9xzy$, $3x^2y - \frac{3}{4}z$ etc.

All other polynomials have no specific name w.r.t. degree but simply, we call them polynomials of degree four, degree five and so on.

(iii) Types of Polynomials w.r.t. Coefficients

Polynomials can also be classified according to the coefficients of the variables involved in each term of them. e.g., $-3x^2 + 4x - 1$ is a polynomial with integers co-efficients because $-3, 4, -1 \in Z$.

Similarly, $\frac{1}{2}y^3 - \pi x^2 y + \frac{22}{7}$ is a polynomial with real numbers because $\frac{1}{2}, -\pi, \frac{22}{7} \in R$

Just like this we can also check the coefficients in any polynomial and classify it accordingly.

(iv) Types of Polynomials w.r.t. Variables

Classification of polynomials is also possible with respect to the variables involved in them. e.g., $x^2y - 7$ is a polynomial in two variables x , y. We can express it as $P(x, y) = x^2y - 7$.

Similarly, $6x^2yz + 12xy^3z^3 + 9z^4$ is a polynomial in three variables x , y, z. We may write it as $Q(x, y, z) = 6x^2yz + 12xy^3z^3 + 9z^4$.

Recall

The highest exponent of the variable involved in a polynomial is called its degree. If more than one variable are being multiplied in terms of a polynomial. Then the degree of the polynomial is the sum of exponent of all the variables present in a term.

(b) Rational Expression

An algebraic expression of the form $\frac{P(x)}{Q(x)}$,where $P(x)$and $Q(x)$ are polynomials and $Q(x) \neq$

(i.e. it is not a zero polynomial) is called a rational expression.

For example: $\frac{1}{2}$, $\frac{3}{4x^2}$, $\frac{2x-1}{x^2+3}$, $\frac{2x+4}{x^2+5x+6}$, $\frac{x^2+x}{x^3+x}$, 5 etc.

Memory Pl

Rational expression the expression in whic exponent of variable integers.

Reduction / Simplification of Rational Expressions

In algebraic rational expressions, we can multiply or divide the numerator and denominator by a same non–zero expression.

e.g. $$\frac{x^2+x}{x^3+x} = \frac{(x^2+x)\div x}{(x^3+x)\div x} = \frac{x+1}{x^2+1}$$

Hence, $\frac{x+1}{x^2+1}$ is the lowest or reduced form of $\frac{x^2+x}{x^3+x}$.

Reduction of a rational expression is done on the same rules as in reduction of a rational number. To reduce a rational expression to its lowest form, we have to

- Factorize the numerator and denominator completely.
- Divide the numerator and denominator by their common factors.
- Rewrite the remaining terms as a fraction, computing all products. This will be the reduced form.

Example: Reduce each rational expression to its lowest form.

(a) $\frac{x^2y^3}{x^4y^3}$ (b) $\frac{b-a}{a^3-b^3}$

Solution: (i) By using laws of exponents, we have

$$\frac{x^2y^3}{x^4y^3}=\frac{y^{3-3}}{x^{4-2}}$$

$$=\frac{y^0}{x^2}=\frac{1}{x^2}$$

(ii) Since: $(a^3-b^3)=(a-b)(a^2+ab+b^2)$

So: $$\frac{b-a}{a^3-b^3}=\frac{(b-a)}{(a-b)(a^2+ab+b^2)}$$

$$=\frac{-\cancel{(a-b)}}{\cancel{(a-b)}(a^2+ab+b^2)}$$

$$=\frac{-1}{a^2+ab+b^2}$$

[illegible]

What will be the degree of polynomial $\frac{x^3y^2-\sqrt{3}y^2z^3x}{xy^2}$ after simplification?

(c) Irrational Expression

An algebraic expression which cannot be expressed in the form $\frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials but Q is not a zero polynomial, is called an irrational expression.

For example: $\frac{x}{\sqrt{y}}$, $x^{\frac{1}{2}}y-7$, $\sqrt{x}-\frac{1}{\sqrt[3]{y}}$, $\sqrt[3]{x^2+y^2}$, $24x^{\frac{3}{2}}y^{-2}+\frac{9}{y^2}-7$ etc.

Evaluation of Algebraic Expressions

The process of replacing the variables by numbers in an algebraic expression is called 'substitution'. The process of getting a numerical answer by substituting numbers for each variable is called 'evaluation'.

Example: Find the value of $\frac{3}{4}x^2-\frac{2}{3}x+1$ at $x=\frac{-1}{2}$.

Solution: $\frac{3}{4}x^2-\frac{2}{3}x+1$

By substituting the value of x, we have

$$=\frac{3}{4}\left(\frac{-1}{2}\right)^2-\frac{2}{3}\left(\frac{-1}{2}\right)+1$$

$$=\frac{3}{4}\left(\frac{1}{4}\right)-\frac{2}{3}\left(\frac{-1}{2}\right)+1$$

Memory Plus

Irrational expressions are those expression in which exponent of variable are non-integers.

$$= \frac{3}{16} + \frac{1}{3} + 1$$

$$= \frac{9+16+48}{48} = \frac{73}{48} = 1\frac{25}{48}$$

Example: If $p = -5$ and $q = 2$, find the value of $p^2q + pq^2 + 2p \,.\, pq$

Solution: $p^2q + pq^2 + 2p \,.\, pq$

$= \underbrace{p^2q + pq^2 + 2p^2q}$ $\leftarrow$ simplify the given expression

$= 3p^2q + pq^2$ $\leftarrow$ combine like terms to reduce number of terms

$= 3\,(-5)^2\,(2) + (-5)\,(2)^2$ $\leftarrow$ substitute the given values

$= 3\,(25)\,(2) + (-5)(4)$ $\leftarrow$ simplify exponents

$= 150 - 20 = 130$

Example: Find the value of the expression

$3.1\,x^4 - 0.2x^3y^2 + 0.3x^2y^2 + 1.1\,xy^3 - y^4$ at $x = 1$, $y = -1$.

Solution: $3.1x^4 - 0.2x^3y^2 + 0.3x^2y^2 + 1.1\,xy^3 - y^4$

$= 3.1\,(1)^4 - 0.2\,(1)^3\,(-1)^2 + 0.3\,(1)^2\,(-1)^2 + 1.1\,(1)\,(-1)^3 - (-1)^4$

$= 3.1 - 0.2\,(1)\,(+1) + 0.3\,(1)\,(+1) + 1.1\,(1)\,(-1) - (+1)$

$= 3.1 - 0.2 + 0.3 - 1.1 - 1$

$= 3.4 - 2.3 = 1.1$

Exercise 5.1

1. Reduced the following expressions to their lowest form.

(i) $\frac{15ax^3y^2}{25a^2.xy^6}$ (ii) $\frac{38k^2p^3m^4}{57k^3pm^2}$ (iii) $\frac{mn^4pq}{m^2n^3p^4}$

(iv) $\frac{3abc}{15a^2b^2c}$ (v) $\frac{46l^3m^4n^5}{69l^2m^3n^4}$ (vi) $\frac{x-3}{3-x}$

(vii) $\frac{x^2-81}{x+9}$ (viii) $\frac{(r+3)(r+4)}{r^2-16}$

2. Evaluate the following expressions for the given value of each variable.

(i) $3\,(r^2 - s^2)$, if $r = 2$, $s = -1$ (ii) $\frac{1}{2}mv^2$ at $m = 18.75$ and $v = 5.6$

(iii) $\sqrt{2gs}$ when $g = 32.2$ and $s = 144.9$

(iv) $3x - y + \frac{1}{z}$ if $x = \frac{-1}{2}, y = 3, z = \frac{-1}{3}$

(v) $0.1d^3 + 0.01d + 1$ if $d = -0.2$ (vi) $\frac{4}{7}b^3 - 3\frac{1}{2}b^2 + b - 3$ if $b = \frac{1}{2}$

3. If nth triangular number is represented by $T(n) = \frac{n(n+1)}{2}$, then find 100th triangular number.

4. If $P(x) = x^2 + 2x - 15$, $D(x) = x - 3$ and $Q(x) = x + 5$, show that $\frac{P(2)}{Q(2)} = D(2)$.

5. If $g(x) = \frac{1}{2x^3} + \frac{x}{2} + 2$, find $g\left(-\frac{1}{3}\right)$.

6. The volume of a basketball (sphere) is approximately 38808 cm^3. The radius 'r' of the ball is given by

$r = \sqrt[3]{\frac{3v}{4\pi}}$, where v is its volume.

Determine the radius of that ball. (take $\pi = \frac{22}{7}$)

Basic Operations on Algebraic Fractions

Operations with rational numbers and rational algebraic expressions are very similar. A rational number can be expressed as the quotient of two integers. A rational algebraic expression can be expressed as the quotient of two polynomials. In either case, the denominator can never be 0.

Rational Numbers: $\frac{3}{11}, \frac{27}{100}, \frac{-7}{25}, \frac{61}{-89}$

Rational Algebraic Expressions: $\frac{5}{x}, \frac{y}{10}, \frac{2x}{x-5}, \frac{y^2-36}{y+6}, \frac{x^2-6x+5}{\frac{5x}{x+1}}$

If P and Q are any two algebraic expressions then $\frac{P}{Q}$ is called an algebraic fraction, where Q ≠ 0 e.g., $\frac{a-b}{a^2+2ab+b^2}$

To write a fraction in simplest form, we divide both the numerator and denominator by their greatest common factor (GCD).

Addition of Algebraic Expressions

Algebraic expressions can be added as explained in the following examples.

Example: Add the following expressions.

(i) $\frac{x^2-1}{2}+\frac{4(2x^2-x+1)}{3}$ (ii) $4x+7+\frac{(x+3)(x-1)}{x-2}$

Solution:

(i) $\frac{x^2-1}{2}+\frac{4(2x^2-x+1)}{3}$

$$=\frac{3(x^2-1)+8(2x^2-x+1)}{6}$$

$$=\frac{3x^2-3+16x^2-8x+8}{6}$$

$$=\frac{19x^2-8x+5}{6}$$

(ii) $4x+7+\frac{(x+3)(x-1)}{x-2}$

$$=\frac{(4x+7)(x-2)+(x+3)(x-1)}{x-2}$$

$$=\frac{4x^2+7x-8x-14+x^2+3x-x-3}{x-2}$$

$$=\frac{5x^2+x-17}{x-2}$$

Key Fact

For all rational expressions

$\frac{a}{b}$ and $\frac{c}{d}$, $b\neq 0, d\neq 0$:

$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$, $\frac{a}{b}-\frac{c}{d}=\frac{ad-bc}{bd}$

Subtraction of Algebraic Expressions

In subtraction of algebraic expressions, we have to change the signs of the expression which is being subtracted and all other steps are the same as in addition of algebraic expressions.

Example: Subtract $\frac{3x^2-7x+4}{x^2-4}$ from $\frac{3x-1}{x+2}$.

Solution: $\frac{3x-1}{x+2}-\frac{3x^2-7x+4}{x^2-4}$

$$=\frac{3x-1}{x+2}-\frac{3x^2-7x+4}{(x+2)(x-2)} \longleftarrow \text{By using the formula } a^2-b^2=(a+b)(a-b)$$

$$=\frac{(3x-1)(x-2)-(3x^2-7x+4)}{(x+2)(x-2)}$$

$$=\frac{3x^2-7x+2-3x^2+7x-4}{(x+2)(x-2)}=\frac{-2}{(x+2)(x-2)}=\frac{-2}{x^2-4}$$

Multiplication of Algebraic Expressions

Multiplication of algebraic expression can be explained with the help of following examples.

Example: Find the product of $\frac{x^2-9}{x^3+2x}$ and $\frac{x}{x+3}$.

Solution: $\frac{x^2-9}{x^2+2x} \times \frac{x}{x+3}$

$$=\frac{(x+3)(x-3)}{x(x+2)} \times \frac{x}{x+3} = \frac{x-3}{x+2}$$

Example: Multiply x^2-x-20 by $\frac{3}{x^3+4x^2}$.

Solution: $(x^2-x-20) \times \frac{3}{x^3+4x^2}$

$$=(x^2-5x+4x-20) \times \frac{3}{x^2(x+4)}$$

$$=(x-5)(x+4) \times \frac{3}{x^2(x+4)} = \frac{3x-15}{x^2}$$

Key Facts

For all rational expressions

$\frac{a}{b}$ and $\frac{c}{d}$, $b \neq 0$, $d \neq 0$:

$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$, $\quad \frac{a}{b} + \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$

Division of Algebraic Expressions

The process of dividing any two algebraic expressions is explained with the help of following examples.

Example: Divide $\frac{2x^2-11x-21}{x^2-49}$ by $\frac{2x+3}{x+7}$.

Solution: $\frac{2x^2-11x-21}{x^2-49} \div \frac{2x+3}{x+7}$

$$=\frac{2x^2-14x-3x-21}{(x)^2-(7)^2} \times \frac{x+7}{2x+3}$$

$$=\frac{2x(x-7)+3(x-7)}{(x+7)(x-7)} \times \frac{x+7}{2x+3}$$

$$=\frac{(x-7)(2x+3)}{(x+7)(x-7)} \times \frac{x+7}{2x+3} = 1$$

Complex Algebraic Expressions

For simplifying the complex algebraic expression involving addition, subtraction, multiplication and division, we apply the BODMAS rule similar to arithmetic expression.

Example: Simplify.

$$\frac{3x+2}{x+2}+\frac{x-2}{2x+10}\div\frac{2x^2-4}{x^2+2x-15}$$

Solution: $\frac{3x+2}{x+2}+\frac{x-2}{2x+10}\times\frac{x^2+2x-15}{2x^2-8}$

$$=\frac{3x+2}{x+2}+\frac{x-2}{2(x+5)}\times\frac{x^2+5x-3x-15}{2(x^2-4)}$$

$$=\frac{3x+2}{x+2}+\frac{x-2}{2(x+5)}\times\frac{x(x+5)-3(x+5)}{2(x+2)(x-2)}$$

$$=\frac{3x+2}{x+2}+\frac{x-2}{2(x+5)}\times\frac{(x+5)(x-3)}{2(x+2)(x-2)}$$

$$=\frac{3x+2}{x+2}+\frac{x-3}{4(x+2)}$$

$$=\frac{4(3x+2)+(x-3)}{4(x+2)}=\frac{12x+8+x-3}{4(x+2)}=\frac{13x+5}{4x+8}$$

Example: Simplify the following.

$$\left[\frac{a}{(a+b)^2-2ab}\div\frac{(a+b)^3-3ab(a+b)}{a^4-b^4}\right]+\frac{(a+b)^2-4ab}{(a+b)^2-3ab}$$

Solution:

$$\left[\frac{a}{(a+b)^2-2ab}\div\frac{(a+b)^3-3ab(a+b)}{a^4-b^4}\right]+\frac{(a+b)^2-4ab}{(a+b)^2-3ab}$$

$$=\left[\frac{a}{(a+b)^2-2ab}\times\frac{a^4-b^4}{(a+b)^3-3ab(a+b)}\right]+\frac{(a+b)^2-4ab}{(a+b)^2-3ab}$$

$$=\left[\frac{a}{a^2+b^2+2ab-2ab}\times\frac{(a^2-b^2)(a^2+b^2)}{a^3+b^3+3ab(a+b)-3ab(a+b)}\right]+\frac{(a+b)^2-4ab}{(a+b)^2-3ab}$$

$$=\left[\frac{a}{a^2+b^2}\times\frac{(a^2-b^2)(a^2+b^2)}{a^3+b^3}\right]+\frac{(a+b)^2-4ab}{(a+b)^2-3ab}$$

$$=\left[\frac{a}{a^2+b^2}\times\frac{(a+b)(a-b)(a^2+b^2)}{(a+b)(a^2-ab+b^2)}\right]+\frac{(a+b)^2-4ab}{(a+b)^2-3ab}$$

$$=\left[\frac{a(a-b)}{a^2-ab+b^2}+\frac{(a+b)^2-4ab}{(a+b)^2-3ab}\right]$$

$$= \left[\frac{a(a-b)}{a^2-ab+b^2} \times \frac{(a+b)^2-3ab}{(a+b)^2-4ab}\right]$$

$$= \left[\frac{a(a-b)}{a^2-ab+b^2} \times \frac{a^2+2ab+b^2-3ab}{a^2+2ab+b^2-4ab}\right]$$

$$= \left[\frac{a(a-b)}{a^2-ab+b^2} \times \frac{a^2+b^2-ab}{a^2+b^2-2ab}\right]$$

$$= \left[\frac{a(a-b)}{a^2-2ab+b^2}\right]$$

$$= \left[\frac{a(a-b)}{(a-b)^2}\right]$$

$$= \frac{a}{a-b}$$

Another method for simplifying a complex rational expression involves first adding or subtracting, as necessary, to get one rational expression in the numerator and one rational expression in the denominator. The problem is thereby simplified to one involving the division of two expressions.

Example: Simplify. $\dfrac{\frac{3}{x}-\frac{2}{x^2}}{\frac{3}{x-2}+\frac{1}{x^2}}$

Solution: $\dfrac{\frac{3}{x}-\frac{2}{x^2}}{\frac{3}{x-2}+\frac{1}{x^2}} = \dfrac{3\ \frac{x-2}{x^2}}{\frac{3x^2+x-2}{x^2(x-2)}}$, taking LCM

$$= \frac{3x-2}{x^2} \times \frac{x^2(x-2)}{3x^2+x-2}$$ to divide, multiply, by the reciprocal of the divisor.

$$= \frac{3x-2}{x^2} \times \frac{x^2(x-2)}{3x^2+3x-2x-2} = \frac{(3x-2)}{x^2} \times \frac{x^2(x-2)}{(3x-2)(x+1)}$$ factoring

$$= \frac{x-2}{x+1}$$ removing a common factor from numerator and denominator.

Exercise 5.2

1. Add the following expressions.

(i) $\frac{x}{2}, \frac{x}{5}$ (ii) $\frac{x-2}{2}, \frac{x+10}{9}$ (iii) $\frac{4+x}{4}, \frac{x-1}{7}, \frac{5x}{2}$

(iv) $\frac{3x}{x+5}, \frac{10}{5x+25}$ (v) $\frac{24x}{6x-18}, \frac{3(1+x)}{x-3}$

2. Subtract.

(i) $\frac{23-x}{5}$ from 7 (ii) $\frac{6(x-8)}{7}$ from $\frac{5(x-7)}{3}$

(iii) $2x^2-2x+1$ from $\frac{x+1}{x}$.

3. Divide the first expression by the second.

(i) x^4-10x^2+9, x^2-2x-3 (ii) $x^3-3x^2y+3xy^2-y^3$, $x-y$

(iii) $\frac{4x^2-16}{5x}, \frac{2x+4}{15}$ (iv) $\frac{x^2+5x}{x-3}, \frac{x^2-25}{x-3}$

4. Simplify the following.

(i) $\frac{x}{2}+\frac{x}{3}-\frac{x}{4}+\frac{x}{5}$ (ii) $\frac{1}{2}\left(4-\frac{x}{3}\right)-\frac{5}{6}+\frac{1}{3}\left(11-\frac{x}{2}\right)$

(iii) $\frac{2}{x+1}+\frac{x}{x-1}-\frac{x+2}{x-1}$ (iv) $\frac{x^2-25}{5}-\frac{x}{4}+\frac{3x}{20}$

(v) $\frac{45a^2b^3c^4}{27x^4y^3z}\times\frac{243xy^2z^3}{180a^2bc^3}$ (vi) $\frac{m^2}{8n}\times\frac{36p^3q^2}{81mn}\div\frac{15mpx^5}{270n^2x^3y}$

(vii) $3x+\frac{3x^2-27}{x+3}+\frac{1}{x-3}$ (viii) $\frac{5x+5}{3(2x-1)}+\frac{6-2x}{2(1-2x)}$

(ix) $\frac{2a}{2a-3}-\frac{5}{6a+9}-\frac{4(3a+2)}{3(4a^2-9)}$ (x) $\frac{5}{5+x-18x^2}-\frac{2}{2+5x+2x^2}$

(xi) $\frac{m+3}{24m}-\frac{m+1}{24m}+\frac{3m-1}{6m^2+18m}+\frac{12m-4}{m+3}$

(xii) $\frac{1-p^2}{1+q}\times\frac{1-q^2}{p+p^2}\times\left(1+\frac{p}{1-p}\right)$ (xiii) $\left(1-\frac{x}{1+\frac{x}{1-x}}\right)+(1+x^3)$

Solving Rational Equations

An equation that contains one or more rational expressions is called a rational equation. Here are some examples:

$$\frac{2}{5}-\frac{x}{4}=\frac{1}{x}, \qquad \frac{x-4}{x+1}-\frac{6}{x-2}=\frac{7}{x+1}, \qquad x^2+\frac{5}{x}=9$$

Equations of this type occur frequently in applications. It is easiest to solve a rational equation if the fractions are eliminated. This can be done by multiplying each side of the equation by the least common denominator (LCD). Remember that when you multiply each side by the LCD, each term on each side must be multiplied by the LCD.

Example: Solve $\frac{3}{x-7}+1=\frac{8}{x^2-9x+14}$. Verify the solution.

Solution: $\frac{3}{x-7}+1=\frac{8}{x^2-9x+14}$

$(x-2)(x-7).\frac{3}{x-7}+(x-2)(x-7).1=(x-2)(x-7).\frac{8}{(x-2)(x-7)}$ Multiply by LCD $(x-2)(x-7)$.

$3(x-2)+(x^2-9x+14) = 8$ Simplify.

$x^2-6x+8 = 8$ Combine the like terms.

$x^2-6x = 0$ Simplify.

$x(x-6)=0$

$x=0$, or $x=6$. The solutions are 0 and 6.

Check : if $x=0$, $\frac{3}{0-7}+1=\frac{8}{0^2-9.0+14}$, we have: $\frac{4}{7}=\frac{4}{7}$, True.

if $x=6$, $\frac{3}{6-7}+1=\frac{8}{6^2-9.6+14}$, we have: $-2=-2$, True.

S . S = {0 , 6}

Note: Remember that a rational expression is undefined when the value for variables results in a denominator of zero. When solving rational equations, we check that for solution denominator should be non-zero. If denominator is zero in any value of x, these values be excluded from list of solutions.

Example: Solve $\frac{x}{x-2}+\frac{1}{5}=\frac{2}{x-2}$.

Solution: $\frac{x}{x-2}+\frac{1}{5}=\frac{2}{x-2}$.

$5(x-2).\frac{x}{x-2}+5(x-2).\frac{1}{5}=5(x-2).\frac{2}{x-2}$ Multiply by LCD $5(x-2)$.

$5x+x-2=10$ Simplify.

$6x-2=10$ Combine the like terms.

$x=2$

The solution appears to be 2, but the expression $\frac{x}{x-2}+\frac{1}{5}=\frac{2}{x-2}$ are undefined when $x=2$.

We see that 2 is extraneous solution.
There is no solution.

Example: If a certain number is added 5 times the reciprocal of 2 more than that number, th result is 4.

Solution: Suppose a number is x. According to condition, added to 5 times the reciprocal o 2 more than that number, the result is 4:

$x+5(\frac{1}{x+2})=4$ As per statement, written in mathematical form

$(x+2).x+(x+2).5(\frac{1}{x+2})=(x+2).4$ Multiply by LCD on both sides.

$x^2+2x+5=4x+8$ Simplify

$x^2-2x-3=0$ Collecting all like terms on one side

$(x-3)(x+1)=0$ Factoring

$x=3$ or $x=-1$, the solution can be checked.

The solution can be verified:

$x+5(\frac{1}{x+2})=4$ (As per statement, written in mathematical form)

$(x+2).x+(x+2).5(\frac{1}{x+2})=(x+2).4$ (Multiply by LCD on both sides.)

$x^2+2x+5=4x+8$ (Simplify)

$x^2-2x-3=0$ (Collecting all like terms on one side)

$(x-3)(x+1)=0$ (Factoring)

$x=3$ or $x=-1$, the solution can be checked.

Result: The required number is –1 or 3.

Example: A car travels 300 km in the same time that a freight train travels 200 km. T speed of the car is 20 km/h more than the speed of the train. Find the speed of the car and t speed of the train.

Solution: We know that distances both vehicles traveled. We also observe that both vehic traveled the same amount of time and that the speed of the car is 20 km/h faster than that the train.

Suppose the distance is S and speed is V, we know that S=VT, where T is time. If V be t speed of the train, then car moves 20 km/h faster than the speed of the train. So, the speed the car is V+20. We can equate the formula for the two vehicles in terms of T. We have:

$$T = \frac{S}{V}$$

$$\text{car's time} = \text{train's time}$$

$$\frac{S}{V} = \frac{300}{V+20} = \frac{200}{V}$$

$$\frac{300}{V+20} = \frac{200}{V} \quad \text{(Solve for V)}$$

$$(V+20)\,V.\frac{300}{V+20} = V(V+20).\frac{200}{V} \quad \text{(Multiply each side by LCD)}$$

$$300V = 200(V+20) \quad \text{(Simplify)}$$

$$100V = 4000$$

$$V = 40$$

The speed of the train is 40 km/h and the speed of the car is 60 km/h.

Exercise 5.3

Solve and check your solutions (1-7).

1. $\frac{6x}{x-11}+1=\frac{3}{x-11}$
2. $\frac{2y}{y+3}=\frac{-4}{y-7}$
3. $\frac{x+7}{x+4}-1=\frac{x+10}{2x+8}$
4. $\frac{3y}{y+1}=\frac{12}{y^2-1}+\frac{y+4}{y+1}$
5. $x+\frac{5}{x}=-6$
6. $\frac{y+2}{y^2+6y-7}=\frac{8}{y^2+3y-4}$
7. $\frac{5}{y+1}+\frac{3y+5}{y^2+4y+3}=\frac{2}{y+3}$
8. **Kaleem can mow a lawn in 4 hours. Moiz can mow the same lawn in 5 hours. How long would it take both of them, working together, to mow the lawn.**
9. **You have an 8-pint mixture of paint that is made up of equal amounts of yellow paint and blue paint. To create a certain shade of green, you need a paint mixture that is 80 % yellow. How many pints of yellow paint do you need to add to the mixture?**
10. **Waqar takes 9 hours longer to build a wall than it takes Wasi. If they work together, they can build the wall in 20 hours. How long would it take each, working alone, to build the wall?**

I have Learnt

- Describing rational expression.
- Factorizing and simplifying rational expression.
- Demonstrating manipulation of algebraic fractions.
- Performing operations on rational expressions.
- Solving rational equations.
- Applying the concept of rational equations to real world problems.

MISCELLANEOUS EXERCISE

1. **Tick the correct option.**

i. An expression which is the ratio of two polynomials but the polynomial in denominator is non-zero is called

(a) polynomial (b) rational Expression

(c) compound Expression (d) irrational Expression

ii. The degree of $x^2y^3 - \frac{xy^2z^3}{y} - \sqrt{25}z^5$ is

(a) 5 (b) 6 (c) 7 (d) none

iii. Constant polynomial is also called

(a) linear polynomial (b) no degree polynomial

(c) expression (d) zero degree polynomial

iv. Ali is 2 years younger than his sister Ayesha. If Ayesha's present age is x years, then the age of Ali after 5 years will be

(a) $(x+7)$ years (b) $(x-2)$ years

(c) $(x+3)$ years (d) $(x-7)$ years

v. The value of $2\{x^3-(x^2\overline{-3-2}x^2)\}$ at $x=2$ is

(a) 2 (b) 14 (c) -2 (d) 6

vi. Reduced form of the expression $\frac{x^2y^3 - y^2x^3 + x^2y^2z}{x-y-z}$ is

(a) x^2y^2 (b) not possible (c) $\frac{x^2y^2(x-y-z)}{y-x+z}$ (d) $-x^2y^2$

vii. If $y=2-\frac{1}{y}$, then the value of $y^2+\frac{1}{y^2}$ is

(a) 4 (b) zero (c) not possible (d) 2

viii. Simplified from of $\frac{(a+b)^2-(a-b)^2}{8ab}$ is

(a) $\frac{2(a^2+b^2)}{8ab}$ (b) 2 (c) $\frac{a^2+b^2}{4ab}$ (d) $\frac{1}{2}$

ix. Difference of the sum of a and b from the product of a and b is

(a) $ab-a-b$ (b) $a+b-ab$ (c) $2ab-b$ (d) none

x. $\frac{x^3y^3+y^3z^3+z^3x^3}{x^3y^3z^3}=$

(a) $x^3+y^3+z^3$ (b) $x^6+y^6+z^6$

(c) $\frac{1}{x^3}+\frac{1}{y^3}+\frac{1}{z^3}$ (d) $\frac{1}{x^6}+\frac{1}{y^6}+\frac{1}{z^6}$

xi. Leading coefficient in $\frac{x^2}{2}-\frac{1}{8}-\frac{x^4}{4}+\frac{x^3}{7}$ is

(a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) $\frac{1}{7}$ (d) $-\frac{1}{4}$

xii. Coefficients in the polynomial $\sqrt{16}x^2y-\frac{1}{2}y^3+\frac{22}{7}z$ are the elements of the set of:

(a) Integers (b) Irrational numbers

(c) Odd numbers (d) Rational numbers

xiii. The degree of the quotient in $(x-y)^3\div(x-y)^2$ will be

(a) 3 (b) 2 (c) 1 (d) [illegible]

2. Find the value of the circumference of a circle whose radius is 12.5cm. (Take $\pi=\frac{22}{7}$)

3. The surface S of a sphere of radius 'r' is given by the formula $S=4\times\frac{22}{7}r^2$, find

(i) The surface of a sphere whose radius is 1.4 inches;

(ii) The radius of a sphere whose surface is $38\frac{1}{2}$ square feet.

4. In a right-angled triangle, if a and b denote the lengths of the sides containing the rig angle and c denotes the length of the hypotenuse, it is known that $c^2 = a^2 + b^2$. By substitution find which of the following sets of numbers represent the sides of right-angle triangle.

(i) 7 , 24 , 25 (ii) 1.6 , 6.3 , 6.5

(iii) 12 , 35 , 36 (iv) 3 , 4 , 5

5. If $a = 3$, $b = 4$ and $c = 1$, find the value of : $\sqrt{2ab+4ac}+\sqrt{9b}+\frac{2abc}{3}$

6. Subtract the sum of:

$2x^3-3x+4$ and $-3x^2+2x-7$ from $4x^3-3x^2+x-6-\{2x^3-(x-6)\}$

7. Divide the product of $x-2$, $x+3$ and $2x-7$ by the sum of $3(x^2-2x-2)$ and

$5x-x^2-15$.

8. Simplify the following.

(i) $\frac{x}{x+2}-\frac{5x+3}{x-2}+\frac{1}{2}$ (ii) $\frac{x}{x^2-y^2}\times\frac{x^2+2xy+y^2}{x+y}\div\frac{3x}{x-y}$

9. Solve.

(i) $\frac{12}{x^2-16}-\frac{24}{x-4}=3$ (ii) $\frac{y}{2y-6}-\frac{3}{x^2-6x+9}=\frac{y-2}{3y-9}$

10. When checking a possible solution of a rational equation, is it necessary to check that the solution does not make any denominator equal 0? Why or why not?

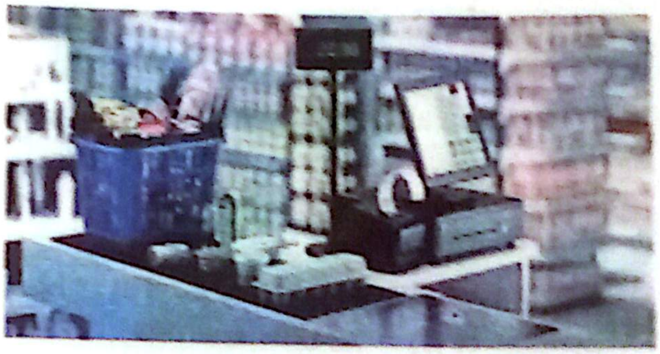
06

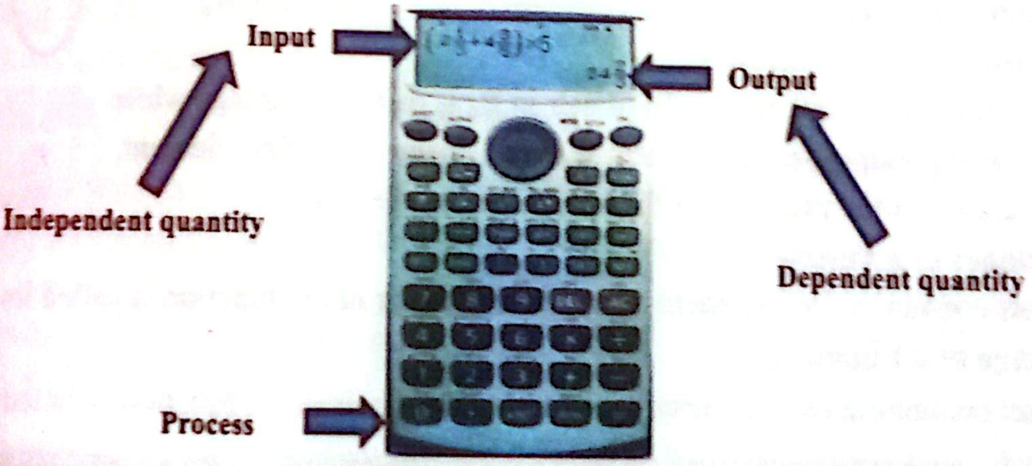
Functions And Graphs

After studying this unit students will be able to:

- Recognize function as a rule of correspondence and find domain and range of a function.
- Identify types of functions (into, onto, one-to-one, injective, surjective and bijective) by using Venn diagrams.
- Determine value of a function and perform operations on functions.
- Find inverse of a function.
- Formulate composite functions as: $fg(x) = f(g(x))$.
- Identify, sketch and interpret graphs of constant function, identity function, linear and non-liner function, absolute valued function and function of the type $y = x^n$ where n is a + ve or − ve integer and and $x > 0$
- Determine gradient of the curve through tangent line.
- Solve a system of one linear and one quadratic equation graphically.
- Solve real life problems related to functions.

Every machine needs some input to perform its functions. When we go shopping, we pay our bill at the counter. The cashier inputs the prices of all the items purchased into the calculating machine. The machine performs its function and processes out a bill. This bill is the output produced by the machine. It means every machine needs some input to give some output, i.e; every output is related to some input.





Our calculator is also a machine, we input some values in it and it gives us the result after performing some process.

Cartesian product of any two non-empty sets A and B denoted by A × B (read as A cros set containing all ordered pairs (x, y) such that $x \in A$ and $y \in B$. We can write it as:

$$A \times B = \{(x, y) \mid x \in A \wedge y \in B\}$$

For example, if A = {a, b} and B = {2}, then

$$A \times B = \{(a, 2), (b, 2)\}$$

- If n(A) = p and n(then n(A×B) = p
- Number of subs A×B = $2^{n(A \times B)}$ =

All the subsets of A × B are:

$R_1 = \{\}$, $R_2 = \{(a, 2)\}$, $R_3 = \{(b, 2)\}$ and $R_4 = \{(a, 2), (b, 2)\}$

Any subset of the cartesian product A×B is called a binary relation(or simply a relation) f A to set B. Basically, a binary relation relates an element of one set to an element of the o

All the above four subsets are binary relations from set A to set B.

Domain of any relation R denoted by dom(R) is a set containing first element of each orde in the relation. Dom(R_4) = {a, b}

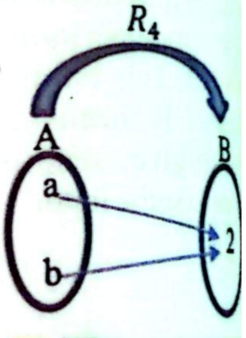
Range of a relation R denoted by ran(R) is a set containing second element of each ordered the relation. Ran(R_4) = {2}

In R_4, 2 is image of a and b but a and b are preimages of 2

Function

A function f from set A to set B is a relation, rule or mapping which maps each element of to a unique element of set B such that domain of f = A and there is no repetition in the first e of ordered pairs in f.

Mathematically, it is written as f : A → B and is read as f is a function from A to B.



In the above example in which A = {a, b} and B = {2},

Consider R_4 = {(a, 2), (b, 2)},

which is shown in arrow diagram at right.

Domain of R_4 = {a, b} = A

There is no repetition in the first element of ordered pairs in R_4.

Therefore, R_4 is a function.

The first element of each ordered pair in R_4 is called pre-image while its corresponding second element is called image of the first element.

In (a, 2), a is the pre-image of 2 and 2 is the image of a.

Check P

R_1, R_2 and R_3 a functions. Justif

Domain of a Function

A set containing first element of each ordered pair in the function is called its domain.

Range of a Function

A set containing second element of each ordered pair in the function is called its range.

Key Facts

If f is a function from set A to set B, i.e., f : A→B, then B is called co-domain of f and ra of f is a subset of co-domain.

Example: If A = Set of first three natural numbers

B = Set of prime factors of 6

Is f = {(1, 2), (2, 3),(3, 2)} a function from A to B?

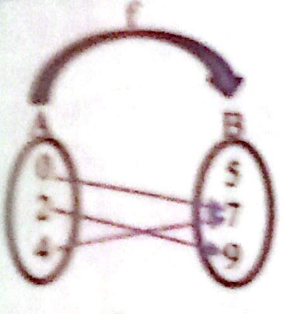
Solution: A = {1, 2, 3} , B = {2, 3}

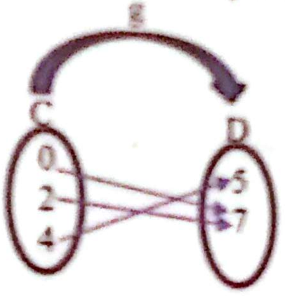
Here f is a subset of A × B and dom f = {1, 2, 3} = A.

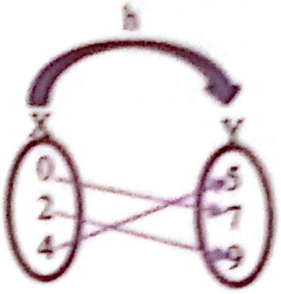
Also, there is no repitition in the first element of the ordered pairs in f, therefore f is a function.

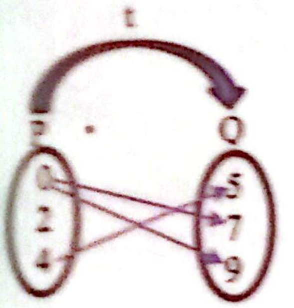
Thinking Corner!
Every function in a relation but its converse is not true.
Do you agree?

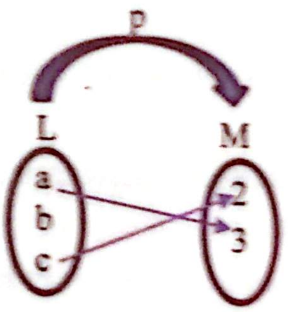
Example: Which of the following relations indicated by the given figures are functions?

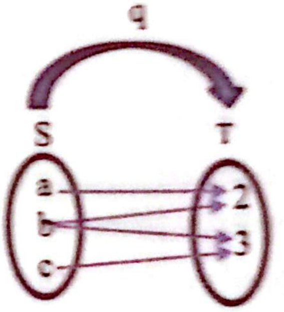












Solution: Here f, g and h are functions as the domain of each of them is equal to their corresponding first set and no first element is repeated in their respective mapping.

But in mapping t, dom(t) ≠ P(∵ 2 has no image in set Q) and there is repetition of 0 (∵ it is mapped to two elements 7 and 9 of Q), therefore t is not a function.

In mapping p, dom(p) ≠ L (∵ b has no image in set M), so p is not a function.

In mapping q, although dom(q) = S but b is repeated (∵ it is the pre-image of two elements 2 and 3 of T), therefore q is not a function.

Types of Functions

Range of a function is required for identifying the type of a function.

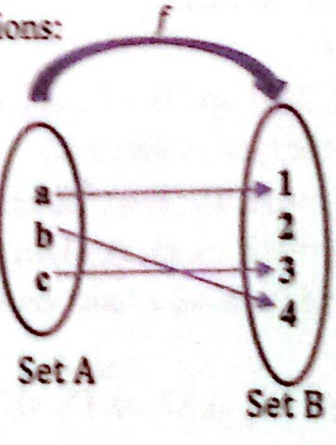
For any two non-empty sets A and B, these are the five types of functions:

Into Function

A function f : A→B will be into if there is atleast one element in set B which is not an image of any element of set A.

i.e. range(f) ⊂ B.

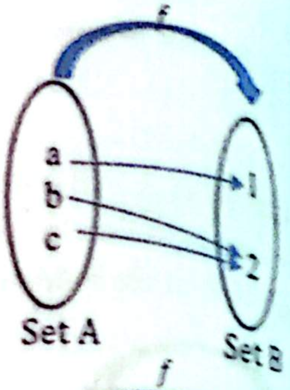
In the adjoining arrow diagram, f = {(a, 1), (b, 4), (c, 3)} is an into function.



Key Fact:
In into function, range f ≠ B.

Onto Function

A function f : A→B will be onto if very element of set B is an image of atleast one element of set A. i.e. range(f) = B.



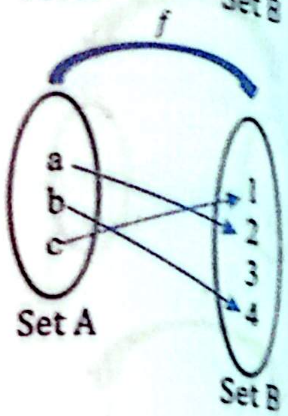
In the adjoining arrow diagram, f = {(a, 1), (b, 2), (c, 2)}is an onto function.

An onto function is also called a surjective function.

One-to-One Function

A function f : A→B will be one-to-one if distinct elements of set A have distinct images in set B.

In the adjoining arrow diagram, f = {(a, 2), (b, 4), (c, 1)}is a one-to-one function.



Points To Ponder!

- We can write one-to-one function as one-one or simply 1-1 function.
- There should be no repetition in the second element of the ordered pairs in a 1-1 function.
- Range of a one-one function may not be equal to set B.

Injective Function

A function which is into as well as 1-1 is called an injective function.

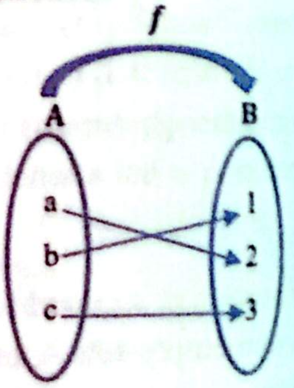
In the above arrow diagram, f = {(a, 2), (b, 4), (c, 1)}is an injective function.

Bijective Function

A function which is both onto and 1-1 is called a bijective function.

In the arrow diagram at right,

f = {(a, 2), (b, 1), (c, 3)} is a bijective function.



Bijective function shows one-one correspondence between the elements of two sets.

Example:

If A = {2, 3, 5} and B = {a, e, i, o, u}, which of the following relations from A to B are functio

Also tell the type of function.

(i) R_1= {(2, i), (2, o), (5, u), (3, e)} **(ii)** R_2= {(3, i), (2, o), (5, i)}

(iii) R_3= {(2, u), (5, i), (3, o)}

Is it possible to have a function from A onto B?

Solution:

(i) R_1= {(2, i), (2, o), (5, u), (3, e)}

$Dom(R_1) = \{2, 3, 5\} = A$, but there is repitition in the first element of first two ordered pairs, therefore R_1 is not a function.

(ii) $R_2 = \{(3, i), (2, o), (5, i)\}$

$Dom(R_2) = \{2, 3, 5\} = A$ and there is no repitition in the first element of the ordered pairs in R_2, therefore R_2 is a function. $Range(R_2) = \{i, o\} \subset B$ which implies that R_2 is an into function.

(iii) $R_3 = \{(2, u), (5, i), (3, o)\}$

$Dom(R_3) = \{2, 3, 5\} = A$ and there is no repitition in the first element of the ordered pairs in R_3, therefore R_3 is a function.

$Range(R_3) = \{i, o, u\} \subset B$ which implies that R_3 is an into function. Also, there is no repitition in the second element of the ordered pairs in R_3, so R_3 is one-one function.

Hence, R_3 is an injective function.

Also number of elements in set B is more than number of elements in set A, therefore A onto B function is not possible.

Evaluating a Function

A function is usually written as $y = f(x)$, in which the values of y (dependent variable) depends upon the values of x (independent variable) and is read as 'y is a function of x' or simply y is equal to f of x'.

Evaluation of a function is a process of finding the value of dependent variable by substituting any specific value of the independent variable.

Example: Find the value of $f(x) = -2x^3 + x^2$ at:

(i) $f(-1)$ (ii) $f(t)$ (iii) $f(\frac{-1}{2})$ (iv) $f(x+2)$

Solution: (i) Substituting $x = -1$ in $f(x) = -2x^3 + x^2$, we have

$$f(-1) = -2(-1)^3 + (-1)^2 = -2(-1) + 1 = 3$$

(ii) Replacing x by t in $f(x)$, we have

$$f(t) = -2(t)^3 + (t)^2 = -2t^3 + t^2$$

(iii) Substituting $x = \frac{-1}{2}$ in $f(x)$, we have

$$f(\frac{-1}{2}) = -2\left(\frac{-1}{2}\right)^3 + \left(\frac{-1}{2}\right)^2 = -2\left(\frac{-1}{8}\right) + \frac{1}{4} = \frac{1}{2}$$

(iv) Replacing x by $x + 2$ in $f(x)$, we have

$$f(x+2) = 2(x+2)^3 + (x+2)^2 = -2x^3 - 16 - 12x^2 - 24x + x^2 + 4x + 4$$

$$= -2(x^3 + 8 + 6x^2 + 12x) + (x^2 + 4x + 4)$$

$$= -2x^3 - 11x^2 - 20x - 12$$

Example: The value of a photocopy machine t years after its purchase is given by $V(t) = 9500 - 860t$ dollars., Find:

(i) V(5) and tell what does V(5) mean?

(ii) Find t when V(t) = 6920 and explain what it represents.

Solution:

(i) $V(t) = 9500 - 860t$

$V(5) = 9500 - 860 \times 5 = 5200$ dollars

It means the value of machine after 5 years will be 5200 dollars.

(ii) When V(t) = 6920, then $6920 = 9500 - 860t$

$$\Rightarrow 860t = 9500 - 6920 = 2580$$

$$\Rightarrow t = \frac{2580}{860} = 3$$

Hence, t = 3, which tells us that the value of photocopy machine will be 6920 dollars after 3 ye

Exercise 6.1

1. If A = {2, 4, 6, 8} and B = {1, 3, 5}, which of the following relations are functions from

to B?

(i) $R_1 = \{(2, 3), (6, 5), (8, 3), (4, 1)\}$

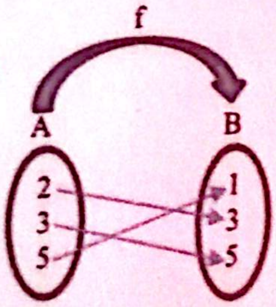
(ii) $R_2 = \{(2, 3), (6, 1), (8, 3), (6, 5)\}$

(iii) $R_3 = \{(2, 3), (6, 5), (8, 3)\}$

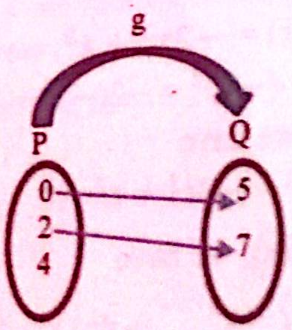
(iv) $R_4 = \{(2, 3), (6, 3), (8, 3), (4, 1)\}$

2. Identify the functions from the following diagrams. Also write their type.

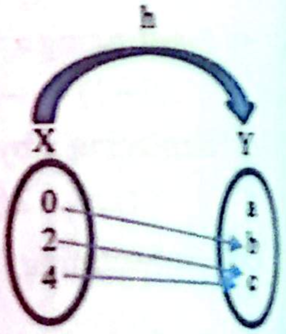
(i)



(ii)



(iii)



3. If $f(x) = x^2 - \frac{1}{2}x + 3$, evaluate the following.

(i) f(2) (ii) f(−1) (iii) $f(\frac{2}{3})$ (iv) f(t + 1)

4. If X = Set of prime factors of 6 and Y = Set of first three non-negative integers, che whether the function defined by $f : X \rightarrow Y = \{(x, y) \mid x - y = 1\}$ is injective or bijective.

A function f is defined by $f(x) = t - ax$, find the values of a and t if 0 and 2 are the images of 1 and -1 respectively.

A function f is defined by $f(x) = mx + c$, is f bijective? Justify your answer.

Operations on Functions

Operations on functions are the ways of combining functions to create new functions. Some common operations on any two functions $f(x)$ and $g(x)$ are:

(i) **Addition of functions:** $(f+g)(x) = f(x) + g(x)$.
For example, if $f(x) = x^2$ and $g(x) = 2x$, then
$(f+g)(x) = f(x) + g(x) = x^2 + 2x$

(ii) **Subtraction of functions:** $(f-g)(x) = f(x) - g(x)$.
For example, if $f(x) = 2x^2 + 3x$ and $g(x) = x^2 - 2x$, then,

$$\begin{aligned}(f-g)(x) &= f(x) - g(x) = (2x^2+3x)-(x^2-2x)\\ &= 2x^2 + 3x - x^2 + 2x = x^2 + 5x\end{aligned}$$

(iii) **Multiplication of functions:** $(f \times g)(x) = f(x) \times g(x)$.
For example, if $f(x) = x^2$ and $g(x) = -2x$, then $(f \times g)(x) = f(x) \times g(x)$

$$= x^2 \times (-2x) = -2x^3$$

(iv) **Division of functions:** $(f \div g)(x) = f(x) \div g(x)$, where $g(x) \neq 0$
For example, if $f(x) = x^2$ and $g(x) = 2x$, then $(f \div g)(x) = f(x) \div g(x)$

$$= x^2 \div (2x) = \frac{x^2}{2x} = \frac{x}{2}$$

(v) **Composition of functions:** The composition of $f(x)$ and $g(x)$ is a function denoted by fog and is defined as:

$$(fog)(x) = f(g(x))$$

For example, if $f(x) = x^2$ and $g(x) = 2x - 1$, then:

$$(fog)(x) = f(g(x)) = f(2x-1) = (2x-1)^2 = 4x^2 - 4x + 1$$

In other words, the output of inner function $g(x)$ will become the input of the outer function $f(x)$.

In this example, we first applied the function g to x, which resulted in $2x - 1$, then applied the function f to the result which gave us $(2x-1)^2$.

Furthermore, for finding $(gof)(x)$, $f(x)$ will be the input of the outer function $g(x)$.

So, $(gof)(x) = g(f(x)) = g(x^2) = 2x^2 - 1$

Key Facts:

- $(fog)(x)$ and $(gof)(x)$ can simply be written as $fg(x)$ and $gf(x)$ respectively.
- $f(f(x)) = f^2(x)$ and $f(f^2(x)) = f^3(x)$

Example: Aahil wants to have a new bike which costs 24000 rupees but he has only a saving of 12000 rupees. If his parents have agreed to give him additional 1000 rupees per week for doing chores. However, he also spends 500 rupees per week on snacks.How much will Aahil have as his

saving after 10 weeks. Also, tell in how many weeks he will be able to save the money for
a new bike?

Solution: Aahil's saving function: $S(x) = 12000 + 1000x$, where x represents number of

His spending function: $T(x) = 500x$

Total money with Aahil after x weeks $= S(x) - T(x)$

$$= 12000 + 1000x - 500x = 12000 + 500x$$

After 10 weeks, saving with Aahil $= 12000 + 500(10) = 17000$ rupees

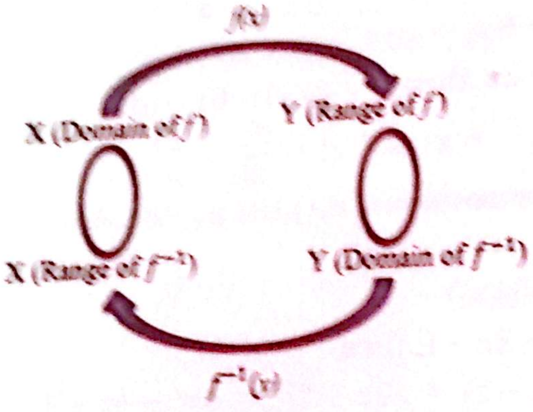
Now, $24000 = 12000 + 500x \Rightarrow 24000 - 12000 = 500x \Rightarrow x = \frac{12000}{500} = 24$

Hence, he can save required amount in 24 weeks.

Inverse of a Function

The inverse of any function f(x) is a function denoted by $f^{-1}(x)$ which reverses the effect of
and it undoes what $f(x)$ does. If $f: X \to Y$, then $f^{-1}: Y \to X$, therefore inverse of only that
exists which is bijective(both one-one and onto).

Thus, if $y = f(x)\ \forall\ x \in X$, then $x = f^{-1}(y)\ \forall\ y \in Y$. It can be illustrated in the following
as :



Key Fact:

Mathematically,
$f(f^{-1}(x)) = f^{-1}(f(x)) =$
and $(f^{-1})^{-1} = f$

It is clear from above diagram that domain of f = range of f^{-1} and range of f = domain of

Finding the Inverse of an Algebraic Function

If $f(x)$ is any algebraic function, then its inverse $f^{-1}(x)$ involves the following steps:

I. Write $y = f(x)$

II. Find the value of x in terms of y from the equation of step I, means write x in terms

III. Replace x by $f^{-1}(y)$ in the resulting equation of step II.

IV. Replace each y by x in the resulting equation of step III to get $f^{-1}(x)$.

The result obtained in step IV can be verified by showing $f^{-1}(f(x)) = x$.

Example: If $f: R \to R$ is defined by $f(x) = 3x + \frac{1}{2}$, find $f^{-1}(x)$.

Solution: Writing $f(x) = y$, we have,

$$y = 3x + \frac{1}{2}$$

$$3x = y - \frac{1}{2} = \frac{2y-1}{2} \quad \Rightarrow \quad x = \frac{2y-1}{6} \quad \text{(Isolating } x\text{)}$$

So, $f^{-1}(y)=\frac{2y-1}{6}$

Hence, $f^{-1}(x)=\frac{2x-1}{6}$ **(Replacing each y by x)**

Exercise 6.2

1. If $f(x)=4(x-1)$ and $g(x)=x^2-2x+1$, find
   (i) $(f+g)(x)$ (ii) $(f-g)(x)$ (iii) $(f\times g)(x)$ (iv) $(f\div g)(x)$
2. If $f(x)=4x$ and $g(x)=x+1$, find,
   (i) $(fog)(x)$ (ii) $(gof)((x)$ (iii) $(fof)(x)$ (iv) $(gog)(x)$
3. Find $(fog)(x)$ and $(gof)((x)$, if
   (i) $f(x)=3-2x$ and $g(x)=x+1$
   (ii) $f(x)=\frac{2}{x}$ and $g(x)=\frac{2x}{x-1}$
   (iii) $f(x)=3x$ and $g(x)=\frac{2}{\sqrt{x-1}}$
   (iv) $f(x)=x^2-1$ and $g(x)=\sqrt{x-1}$
4. If $f(x)=x^2$ and $g(x)=2x+1$, find the value of x if:
   (i) $(fog)(x)=(gof)((x)$ (ii) $f(x)=g(x)$ (iii) $(gof)(x)=9$
5. Find the inverse of the following functions.
   (i) $f(x)=2x-1$ (ii) $g(x)=\frac{2}{x-3}$, where $x\neq 3$
   (iii) $f(x)=\sqrt{x+5}$, where $x\geq -5$ (iv) $g(x)=(x-3)^2$, where $x\geq 3$
6. If $f(x)=\frac{3}{x-5}$ and $g(x)=x+\frac{3}{2}$, find the value of $f^{-1}(1)$ and $g^{-1}\left(\frac{1}{2}\right)$.

Graphs of Some Algebraic Functions

Polynomial Functions

A function $f: R\rightarrow R$ defined as $f(x)=a_0+a_1x+a_2x^2+a_3x^3+\cdots++a_{n-1}x^{n-1}+a_nx^n$, where $a_0, a_1, a_2, a_3, \dots, a_{n-1}, a_n$ are all real numbers and n (called the degree of polynomial) is a non-negative integer is called a polynomial function.

For example, $f(x)=2x^3-x^2+x$ is a polynomial function of degree 3 having leading coefficient 2.

Key Facts

- The highest exponent of the variable involved in a polynomial is called its degree.
- The coefficient of highest degree term in a polynomial is called the leading coefficient.

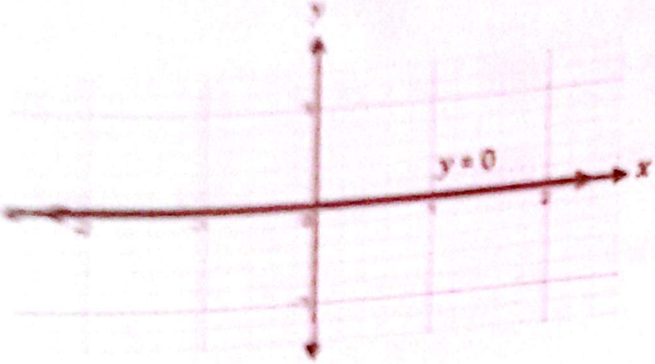
(i) Zero Polynomial Function

A polynomial function of no degree is called a zero polynomial function. This function is of the form:

$y = f(x) = 0$ in which $a_0, a_1, a_2, a_3, \dots, a_{n-1}, a_n$ are all zero.

The graph of $y = 0$ is a straight line that lies on the x-axis indicating that the value of y is
zero regardless of the value of x. It passes through origin and extends infinitely in both
and negative x-directions.

The graph of $y = 0$ is shown in the given coordinate plane.



Do You Know?

A vertical line intersects the graph of a function only at one point. If it intersects the graph at more than a point, then that graph will not be the graph of a function. This is called vertical line test.

(ii) Constant Function

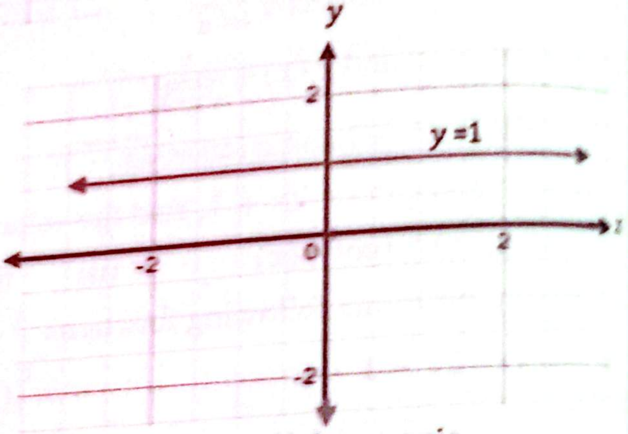
A polynomial function of degree zero is called a constant function.

This function is of the form,

$$y = f(x) = a = ax^0,$$

where a is any non-zero constant.

For example, $y = 3$, $y = \frac{-1}{2}$ etc.



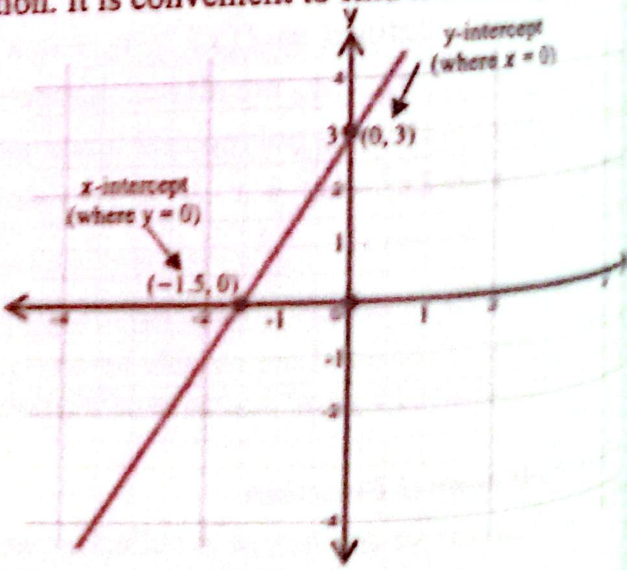
The graph of $y = 1$ (shown in the given graph) is a straight line parallel to x-axis.

(iii) Linear Function

A polynomial function of degree one is called a linear function. This function is of the form, $y = f(x) = ax + b$, where $a, b \in R$ and $a \neq 0$. For example, $y = 2x + 3$, $y = 2x$ etc.

The graph of a linear equation is a straight line. For sketching the graph of a linear function,
first find two points which satisfy the given equation. It is convenient to find x-intercept and y-intercept for getting two points.

To calculate x-intercept, substitute $y = 0$ in $y = 2x + 3$ and to calculate y-intercept, substitute $x = 0$ in the equation.



x-intercept: Put $y = 0$, $0 = 2x + 3$

$$\Rightarrow x = -\frac{3}{2} = -1.5$$

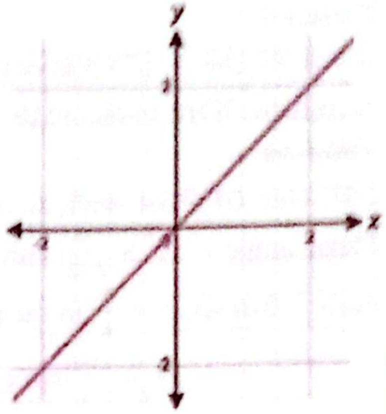
So, the point where the graph intersects x-axis is $(-1.5, 0)$.

y-intercept: Put $x = 0$, $y = 3$

The point where the graph intersects y-axis is $(0, 3)$.

From the graph, it is clear that this line does not pass through origin. We can find the coordinates of any point on the graph just by taking its distance from the axis.

(iv) Identity Function



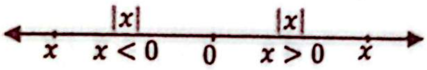
An identity function can be represented as:
$y = f(x) = x$, means it has the same value as the input.

For example, $f(2) = 2$, $f(-3) = -3$ etc.

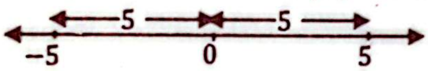
It is a linear function and its graph is a straight line bisecting first and third qudrants passing through origin.

(v) Absolute Valued Function

The absolute value of a real number is its distance from 0 on the number line.

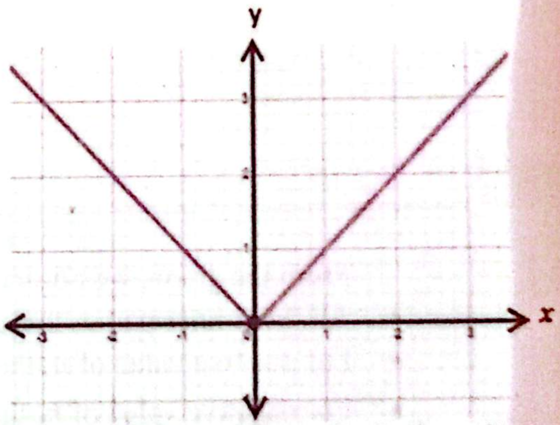


For example, the absolute value of 5 written as $|5| = 5$ and of -5 written as $|-5| = 5$.



The absolute value of a real number x denoted by $|x|$ is defined as:

$$|x| = \begin{cases} x, x \geq 0 \\ -x, x < 0 \end{cases}$$



The function defined by $f(x) = |x|$, is called the absolute valued function or the modulus function.

Its graph is V-shaped with vertex at origin.
It bisects first quadrant for $x > 0$ and bisects second quadrant for $x < 0$.
Its domain is set of all real numbers but range is set of non-negative real numbers.

Absolute valued functions are widely used in many real life problems including distance, displacement, magnitude and to calculate energy wave etc.

Example:

The profit (in rupees) of a company is given by the function $P(x) = |2x - 100|$, where x is the number of units sold. If 120 units are sold, what is the profit?

Solution:

Profit function, $P(x) = |2x - 100|$

Substituting $x = 120$, $P(120) = |2 \times 120 - 100| = 140$

Thus, the company makes a profit of 140 rupees.

Example:

A wave's energy (E) is given by the function $E(x) = |3\sin(x)|$, where x is the wave's phase
in radians. If the phase angle is 30°, what is the wave's energy ?

Solution:

First convert phase angle to radians as under:

Phase angle $= 30^\circ \times \frac{\pi}{180^\circ}$ radians $= \frac{\pi}{6}$ radians

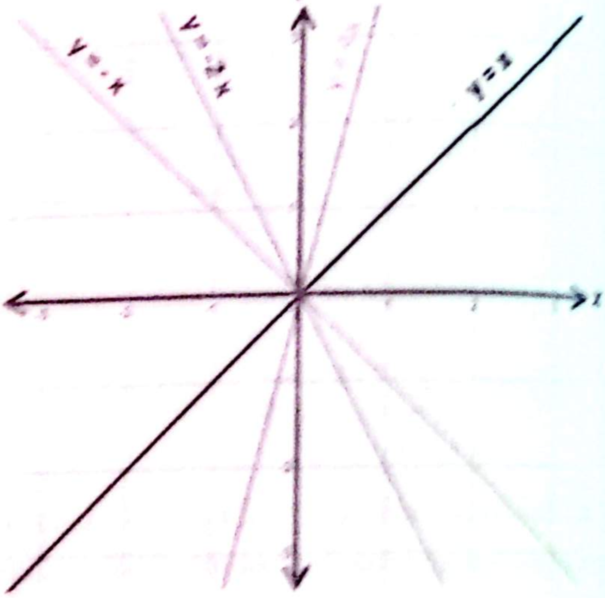
Now substitute $x = \frac{\pi}{6}$ in the given function,

$$E\left(\frac{\pi}{6}\right) = \left|3\sin\left(\frac{\pi}{6}\right)\right| = \left|3 \times \frac{1}{2}\right| = \frac{3}{2}$$

Hence, the wave's energy is $\frac{3}{2}$ units.

(vi) Graphs of the functions of the form y = nx, where n is an Integer.

For the graph of the function $y = nx$;
n is a + ve or − ve integer and and $x > 0$,
we take different integral values of n.
The graph for $n = -2, -1, 1, 4$ is drawn
at right.

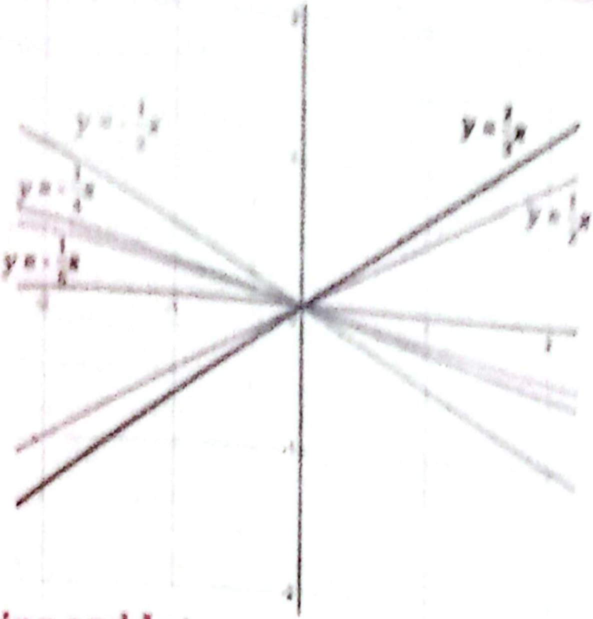


Points To Ponder!

From the graph, we observe that:

- All these lines passes through origin.
- For positive value of n, the line passes through first and third quadrant.
- For negative value of n, the line passes through second and fourth quadrant.

Similarly, we can sketch the linear graphs for fractional values of n as drawn here.

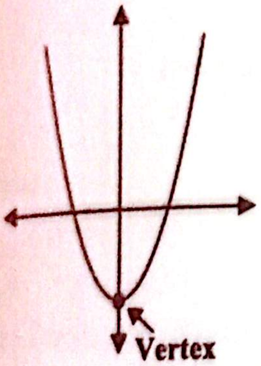


Identifying, Sketching and Interpreting Graphs of Some Non-Linear Functions

(i) Quadratic Function

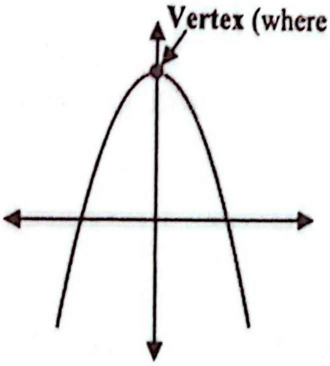
A polynomial function of degree two is called a quadratic function. This function is of the form,
$f(x) = ax^2 + bx + c \quad ; a, b, c \in R$ and $a \neq 0$.
For example, $y = x^2 - 2x - 3$, $y = -4x^2 + 1$ are quadratic functions.
The graph of a quadratic function is a U-shaped curve called parabola that opens either upward (if $a > 0$) or downward (if $a < 0$).

Opens Upward



(where y attains minimum value)

Opens Downward



(where y attains maximum value)

The vertical line passing through the vertex of parabola is called 'axis of symmetry' as the graph is symmetrical about y-axis.
In order to sketch the graph of such quadratic functions, follow the following steps.
Step-1: Identify the shape of the graph whether it opens upward or downward.
Step-2: Find x-intercept and y-intercept.
Step-3: Find the vertex of the parabola.
For finding vertex of the parabola, we proceed as follows:

$$y = ax^2 + bx + c$$

$$= a\left(x^2 + \frac{b}{a}x\right) + c \qquad \text{(Isolate the constant term and take coefficient of } x^2 \text{ as common from first two terms)}$$

$$= a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right) + c$$

(Adding and subtracting square of t coefficient of x to complete the s

$$= a\left(x + \frac{b}{2a}\right)^2 - a\left(\frac{b}{2a}\right)^2 + c$$

$$= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$$

$$= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} \quad \dashrightarrow (i)$$

Enlighten Yoursel

If the graph of a function interse x-axis at point(a, 0), then x-coor 'a' of the point is called x-interc y-coordinate of that point will alw be zero. Similarly y-coordinate 'b' o (0, b) is called y-intercept in whi x-coordinate is always zero.

Comparing (i) with

$$y = a\,(x - h)^2 + k \quad \dashrightarrow (ii) \text{ (general form of the curve)}$$

We have,

$$h = \frac{-b}{2a} \quad \text{and} \quad k = c - \frac{b^2}{4a}$$

So, the graph of the curve, y $= ax^2 + bx + c$ is a parabola with vertex at (h, k). The po is the turning point of the graph. The curve is symmetrical about the line $x = h$. To sketch the graph of parabola, we draw its vertex , x- and y-intercepts and some other p get a smooth curve.

Example:

Sketch the graph of $y = x^2 - 2x - 3$.

Solution:

$y = x^2 - 2\,x - 3$ $\dashrightarrow$ (i)

Comparing (i) with y $= ax^2 + bx + c$, we have $a = 1, b = -2$ and $c = -3$

Here, $a = 1 > 0$, so the curve will open upward.

$h = \frac{-b}{2a} = \frac{-(-2)}{2(1)} = 1$ and

$k = c - \frac{b^2}{4a} = -3 - \frac{(-2)^2}{4(1)} = -4$

So, the **vertex** of parabola is $(h, k) = (1, -4)$.

For y-intercept, put $x = 0, y = -3$

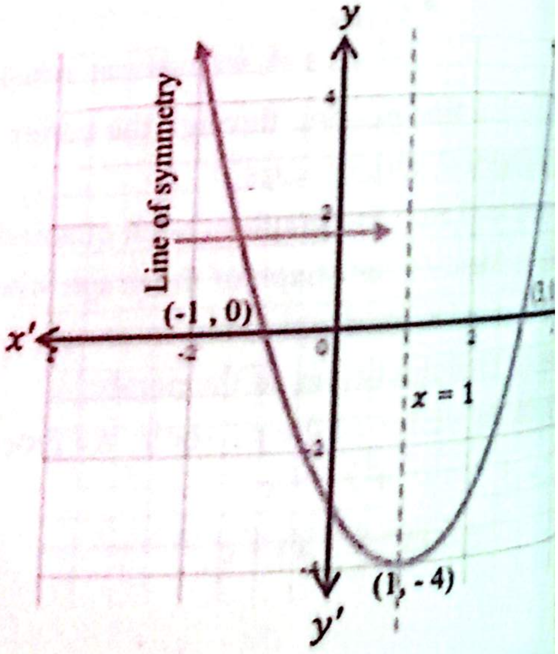
For x -intercept, put $y = 0$.

$$0 = x^2 - 2\,x - 3$$
$$0 = x^2 - 3x + x - 3$$
$$0 = x(x - 3) + 1(x - 3)$$
$$0 = (x - 3)(x + 1)$$

This implies, either $x - 3 = 0$ or $x + 1 = 0$

$$\Rightarrow x = 3 \text{ or } x = -1$$

So, the graph will intersect x-axis at (3, 0) and (−1, 0). The parabola is symmetrical about the vertical line $x = 1$ through the turning point (1, - 4).



(ii) Cubic Function

A polynomial function of degree three is called a cubic function. This function is of the form:

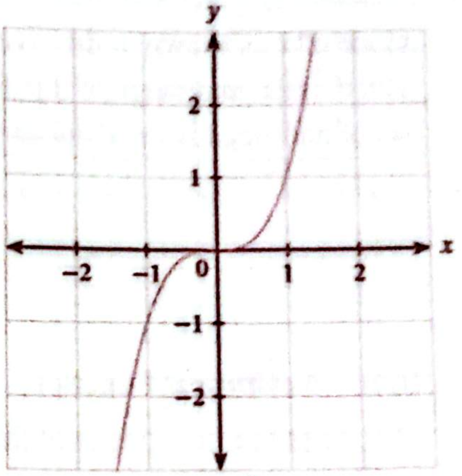
$f(x) = ax^3 + bx^2 + cx + d \quad ; a, b, c, d \in R$ and $a \neq 0$.

We will discuss here the graph of the functions $y = ax^3$.

To sketch the graph of $y = x^3$(when $a = 1$) , we need the shape, x and y intercepts and turning point.

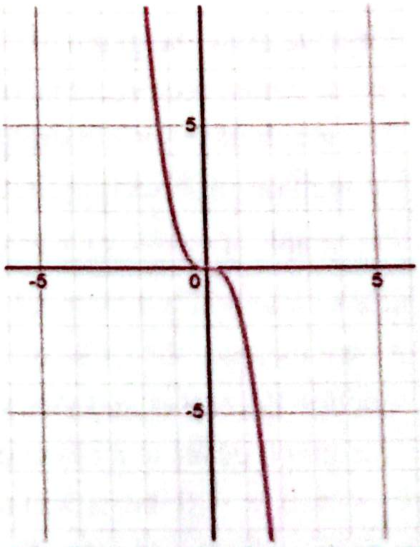
Here leading coefficient is 1 which is positive which shows that the graph will concave up in the first quadrant and concave down in the third quadrant. For each point (x, y) on the graph, the point $(-x, -y)$ is also on the graph. At $x = 0$, $y = 0$, so this graph intersects both the axis at origin and is symmetric about origin as shown in the figure. Also origin is the turning point of this graph. The domain and range of this function is the set of all real numbers.

From bottom left to top right



$y = x^3$

From top left to bottom right



Example:

Sketch the graph of $y = -2x^3$.

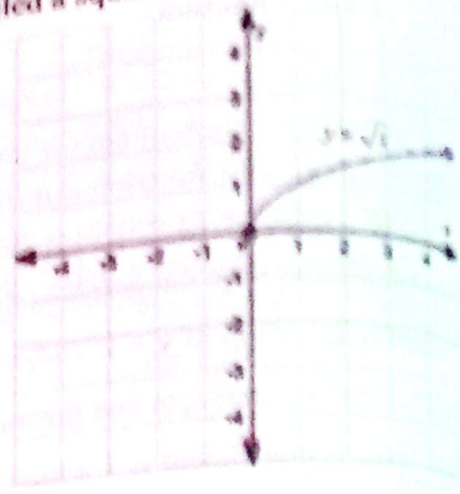
Solution: $y = -2x^3 \quad \text{--------}\!\!\rightarrow$ (i)

Here, the leading coefficient is -2 (negative), which shows that the graph will concave up in the second quadrant and concave down in the fourth quadrant.

At $x = 0$, $y = 0$, so the graph passes through origin which is the turning point of the graph.For each point $(-x, y)$ on the graph, the point $(x, -y)$ is also on the graph, so this graph is symmetric about origin. The domain and range of this function is the set of all real numbers.

(iii) Square Root Function

The function defined by $f(x) = \sqrt{x}$, where $x \geq 0$ is called a square root function.



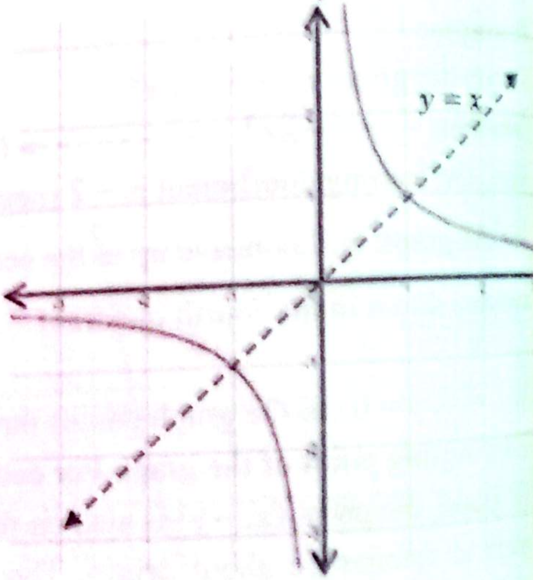
The graph of this function will remain in first quadrant and increases with increase in the the value of x as shown in the given graph. The domain and range of this function is the set of non-negative real numbers.

(iv) Reciprocal Function

The reciprocal function is of the form $f(x) = \frac{k}{x}$, where $x \neq 0$ and k(a constant)$\neq 0$. We will discuss here simple example of reciprocal fu where $f(x) = \frac{1}{x}$. The graph of reciprocal function shows how the function's output decreases as the increases illustrating inverse relationships.

Enlighten Yourself!

- The reciprocal of a very large number is a very small number.
- The reciprocal of a very small number is a very large number.
- The reciprocal of zero is undefined.



First we draw the graph of the original function $y = x$ whose reciprocal is to be drawn.
The graph of $y = x$ is a straight line passing through origin bisecting first and third quadrants.
The graph of $y = \frac{1}{x}$ is a curve which is undefined at $x = 0$ and is symmetrical about the line $y = x$.
As we increase the value of x , the value of the function decreases and vice versa.
This graph never touches coordinate axis.
$y \to \infty$ as $x \to 0$, but it will never touch y-axis and
$x \to \infty$ as $y \to 0$, but it will never touch x-axis.
The domain of this curve is all real numbers except zero.

Have You Noticed?

- The graph of $f(x) = \frac{1}{x}$ has two branches.
- The graph of $y = \frac{1}{x}$ exists in the first and third quadrants only.
- As $x \to \infty$, $f(x) \to 0$ from above.
  As $x \to -\infty$, $f(x) \to 0$ from below.
  As $x \to 0$ from the right, $f(x) \to \infty$,
  As $x \to 0$ from the left, $f(x) \to -\infty$.

Enlighten Yourself!

The symbol ∞ was first used by John Wallis in 1655 which means unbouded.

Example:

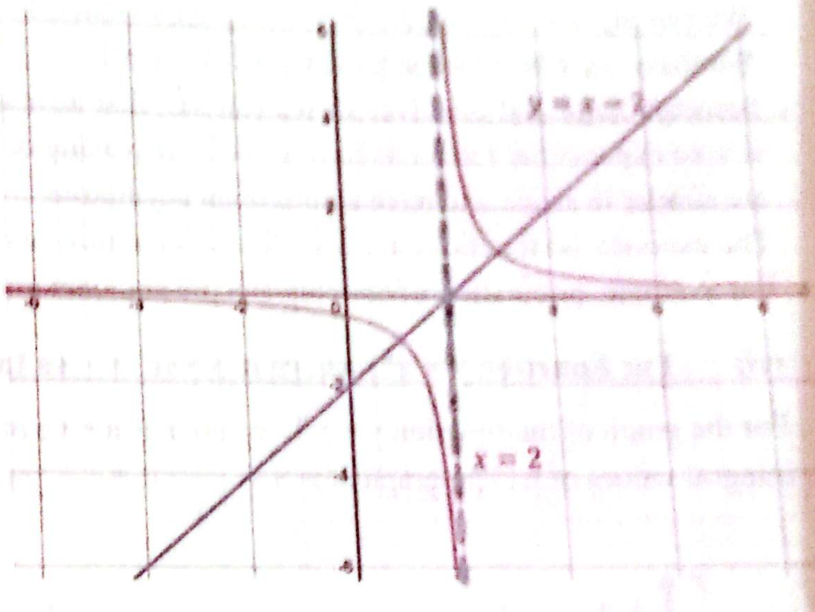
Sketch the graph of $y = \frac{1}{x-2}$.

Solution:

y-intercept: $y = \frac{-1}{2} = -0.5$

As $x \to \infty$, $y \to 0$

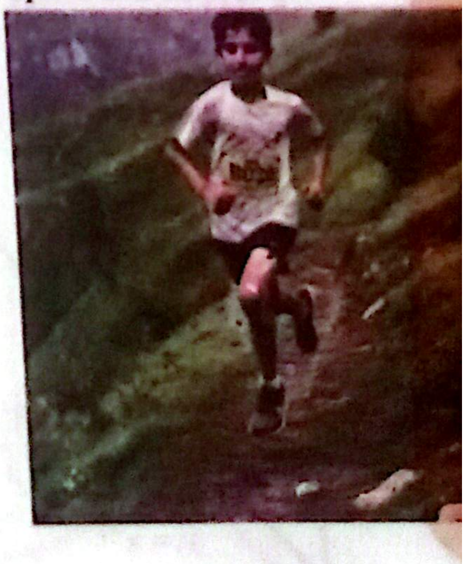
As $x \to -\infty, y \to 0$



Example:

Saim can jog 4 miles downhill in the same time which he takes to jog 2 miles uphill. He tak downhill 3mph faster than he jogs uphill. Find his jogging rate uphill and downhill.

Solution:



Let his uphill rate = x

Then, his downhill rate = $x + 3$

Uphill time = $\frac{2}{x}$

Downhill time = $\frac{4}{x+3}$

According to the given condition,

downhill time = uphill time

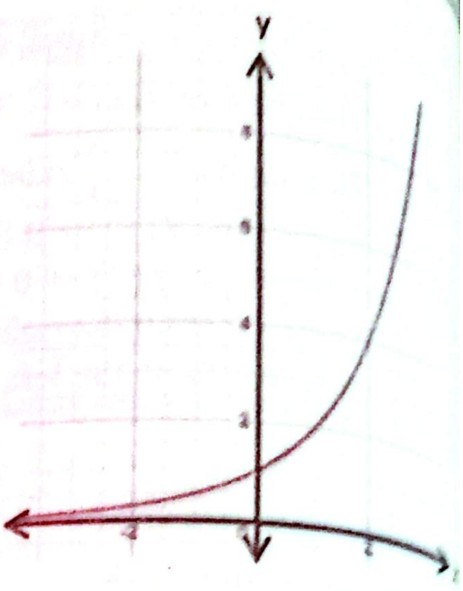
$$\Rightarrow \frac{4}{x+3} = \frac{2}{x}$$

$$\Rightarrow 4x = 2(x+3)$$

$$\Rightarrow 4x = 2x + 6 \Rightarrow 4x - 2x = 6 \Rightarrow x = 3$$

Thus, his uphill rate = 3mph and his downhill rate = 3 + 3 = 6mph

(v) Exponential Function



The most simple exponential function is of the form $f(x) = a^x$, where $a > 0$ and $a \neq 1$.

For example, $y = 2^x$ is an exponential function.

For sketching its graph, we find:

y-intercept: $y = 1$ by puting $x = 0$.

Asymptote: Horizontal asymptote, $y = 0$ (x-axis).

As x becomes large and negative, the graph of $y = 2^x$ becomes closer to x -axis but never quite reaches it.

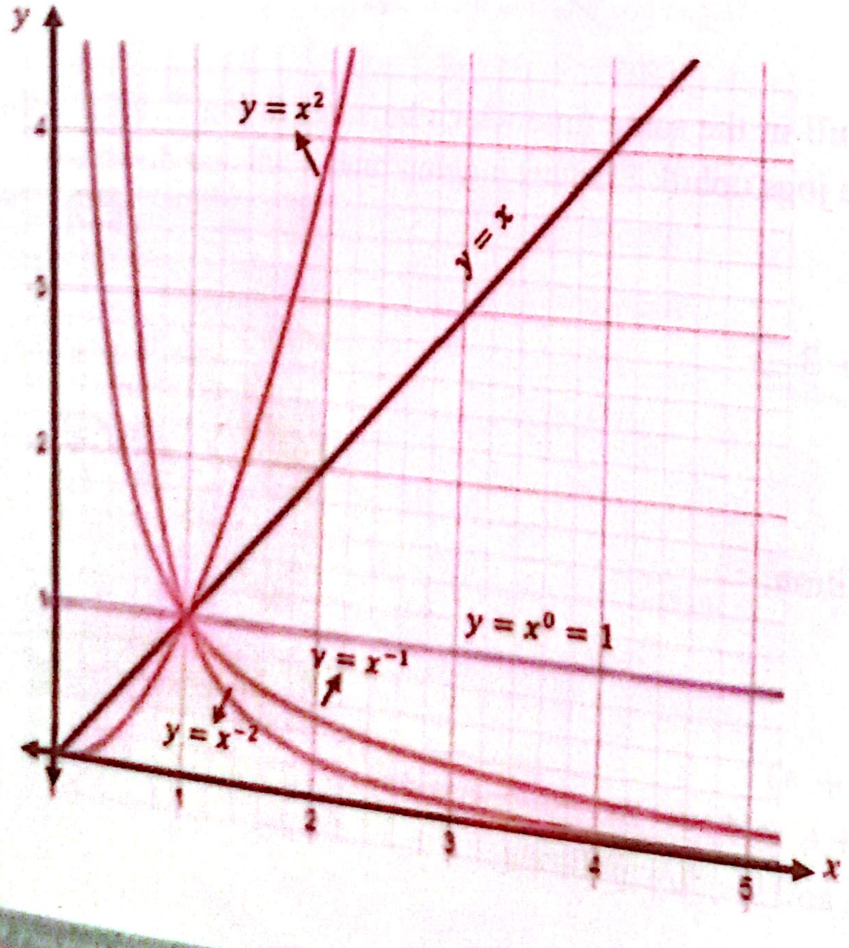
We can say that as x approaches $-\infty$, y approaches 0.

Similarly, as x becomes large and positive, y also becomes large and positive, so we can say that as x approaches ∞, y also approaches simple exponential functions have smooth increasing or decreasing graphs. All exponentia are similar in shape and have a horizontal asymptote.

The exponential functions are used for the quantities which increase or decrease exponentia For example, population of animals, people and bacteria usually grow in an exponential w

(vi) The Function $y = x^n$, where n is +ve or −ve Integer and $x > 0$

For the graph of the function $y = x^n$, where n is a + ve or − ve integer and $x > 0$, we take integral values of n. The graphs for $n = -2, -1, 0, 1, 2$ are shown in the given graph.



Enlighten Yourself

The graph of $y = x^n$ (w
n is +ve or −ve intege
$x > 0$):

- Is a horizontal line a n = 0.
- Bisects first quadra at n = 1.
- Will remain in first quadrant.
- Passes through (1, 1)

Solution of a Linear and a Quadratic Equation Graphically

As we know that the graph of a linear equation is a straight line and of a quadratic equation is a curve. The point of intersection of these two graphs is a point where both the graphs intersect each other.

Example:

Solve $f(x) = 2x + 1$ and $g(x) = x^2 - x + 3$ graphically.

Solution:

Table of values for $f(x) = 2x + 1$

x	-2	-1	0	1	3
$f(x)$	-3	-1	1	3	7

For the graph of $g(x) = x^2 - x + 3$,we compare it with$y = ax^2 + bx + c$, we have:

$a = 1, b = -1$ and $c = 3$.

Here, $a = 1 > 0$, so the curve will open upward.

Also $h = \frac{-b}{2a} = \frac{-(-1)}{2(1)} = \frac{1}{2} = 0.5$

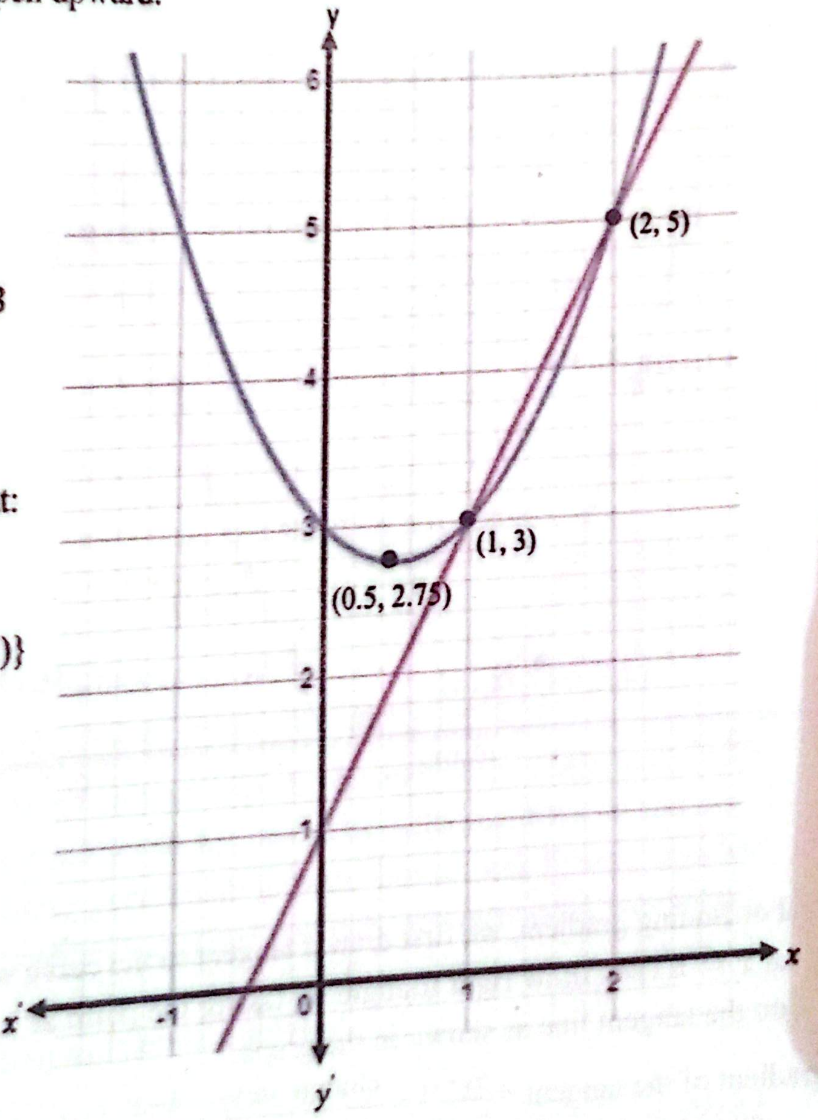
and $k = c - \frac{b^2}{4a} = 3 - \frac{(-1)^2}{4(1)} = 2.75$

So, the vertex of parabola is:

$(h, k) = (0.5, 2.75)$.

Table of values for $g(x) = x^2 - x + 3$

x	-1	0	0.5	1	2	3
$g(x)$	5	3	2.75	3	7	9



Both the graphs intersect each other at:

(1, 3) and (2, 5).

Hence, solution set = $\{(1, 3)$ and $(2, 5)\}$

Determining the Gradient of Curves through Tangents

The gradient of a curve at a given point is the slope of the tangent line drawn to the curve at that point. It is obtained by drawing a tangent to the curve at that point. The tangent to a curve is a line which touches the curve onle at one point externally. The slope of tangent to the curve is its gradient. The slope of the tangent = $\frac{\text{change in } y}{\text{change in } x}$ $(\frac{rise}{run})$.

So, the gradient of the curve = $\frac{\text{change in } y}{\text{change in } x} = \frac{y_2-y_1}{x_2-x_1}$

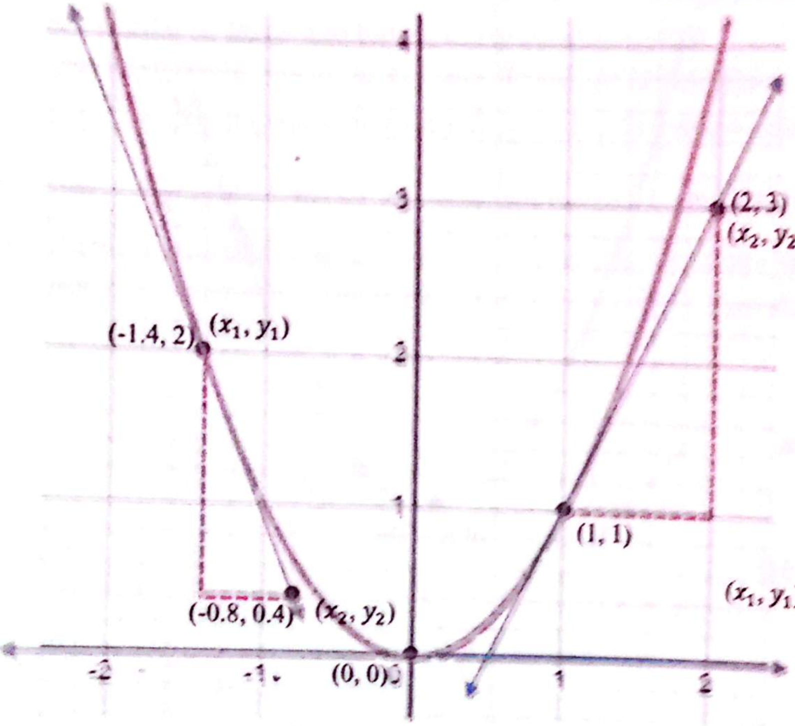
Caution!
Make sure that the angle of tangent line with the curve below and above the given point must be same. The gradient of the curve can also be found by taking any two points on the tangent line.

Example:

Find the gradient to the curve $y = x^2$ drawn below at:

(i) $x = 1$ **(ii)** 0 **(iii)** -1.4.

Solution: The graph of the given curve is drawn here.



(i) For finding gradient, we first draw a tangent to the curve with equal space on either side at $x = 1$.Then draw right triangle by taking the point at which $x = 1$ and any other point on the tangent line as shown in the graph.

The gradient of the tangent = $\frac{y_2-y_1}{x_2-x_1} = \frac{\text{change in } y}{\text{change in } x} = \frac{3-1}{2-1} = 2$

ce, the gradient of the curve at $x = 1$ is 2.

The tangent at $x = 0$ is a horizontal line, so its slope is zero. Hence, the gradient of the curve at $x = 0$ is 0.

Similarly by drawing a tangent to the curve at $x = -1.4$, we have a right triangle as shown in the graph. By taking the point $(-1.4, 2)$ and any other point $(-0.8, 0.4)$ on the tangent line, we have,

The gradient of the tangent $= \frac{y_2 - y_1}{x_2 - x_1}$

$$= \frac{0.4 - 2}{-0.8 + 1.4} = \frac{-1.6}{0.6} \approx -2.7$$

The slope of the vertical line is undefined.

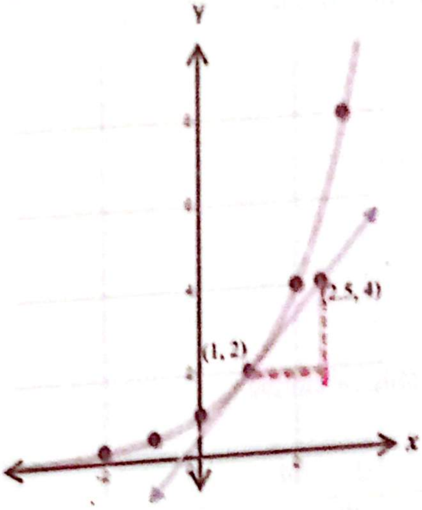
To Be Noted!

- If the tangent line comes down from left to right, the slope of tangent line will be negative.
- If the tangent line goes up from left to right, the slope of tangent line will be positive.

Example:

Estimate gradient to the given curve at (1, 2).

Solution: We first draw a tangent to the curve with equal space on either side at (1, 2).Then draw right triangle at the point at (1, 2).and any other point on the tangent line as shown in the graph.



The gradient of the tangent $= \frac{y_2 - y_1}{x_2 - x_1}$

$$= \frac{\text{change in } y}{\text{change in } x}$$

$$= \frac{4 - 2}{2.5 - 1} = 1.33$$

Hence, the gradient of the curve at (1, 2) is estimated as 1.33.

Real Life Problems Related to Functions and Graphs

Graphs and functions are essential tools in various aspects of daily life. They have wide applications often unconsciously in solving many real life problems including distance and time, supply and demand, temperature and time, population growth, health and fitness, weather forecasting, financial planning, time management, medical research, environmental monitoring, fitness tracking etc. By applying graphs and functions to daily life problems, we can develop a more analytical approach to decision making and problem solving.

Here are some problems involving functions and their graphs.

Example: A company's cost function is $C(x) = 2x + 300$, where x is the number of units produced and cost is given in dollars. Graph the cost function and find the cost of producing 150 units.

Solution:

Here, C(x) = 2x+300

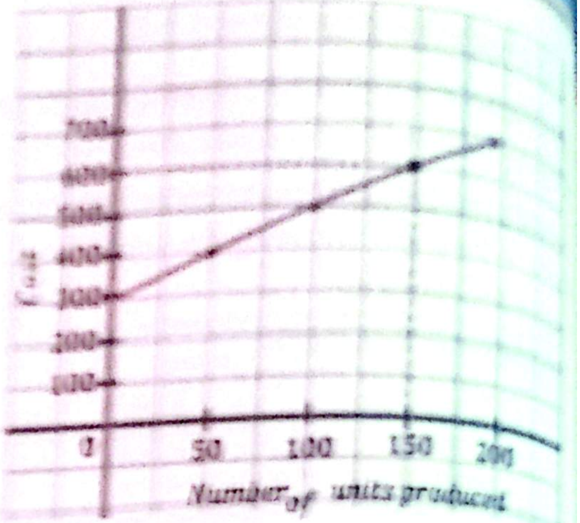


Table of values

x	1	50	100	150
C(x)	300	400	500	600

From the graph, cost of producing 150 units is 600 dollars.

Example: A researcher studies the spread of a disease and finds that the number of infected people grows according to the function f(t) = 10(1.2)^t, where t is time in days. Determine the number of people to become infected in next 6 days.

Solution:

Here, f(t) = 20(1.2)^t

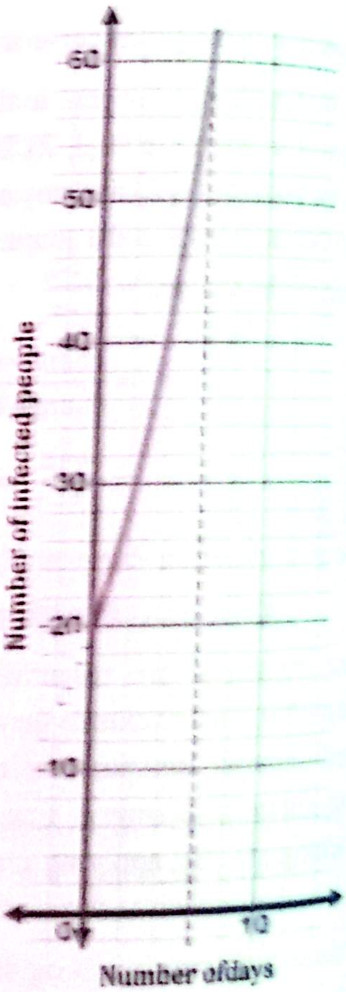


Table of values

t	1	2	4	6
f(t)	20	29	41	60

From the graph, number of people to be infected in next 6 days = 60

Exercise 6.3

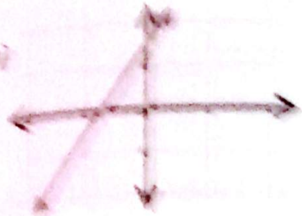
1. Identify the graphs of linear and non-linear functions from the following? Also indicate the graphs which are graphs of functions by applying vertical line test.

(i)

(ii)

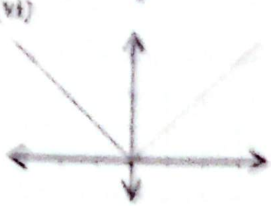
(iii)

(iv)



(v)

(vi)



2. Sketch the graph of the following functions and identify the graphs which are linear.

(i) $y = 3x + 2$ (ii) $y = x$ (iii) $y = x^2$

3. Sketch the graph of the following functions.

(i) $y = 2^x$ (ii) $y = |x|$ (iii) $y = x^3$

(iv) $y = \sqrt{x+2}$ (v) $y = \frac{1}{x-1}$

4. Check whether the curve(parabola) drawn for the following functions will open upwards or downwards. Also find x-intercept, y-intercept and sketch the graph of the given functions.

(i) $f(x) = (x-1)^2$ (ii) $f(x) = -x^2 + 2$

(iii) $f(x) = 3 - (x+2)^2$ (iv) $f(x) = -x^2 + 2x + 3$

5. Find the values of h and k if $y = hx^2 + 2x + k$ cuts x-axis and y-axis at $(-3, 0)$ and $(0, 2)$ respectively.

6. Solve the following equations graphically.

(i) $f(x) = x + 2$, $g(x) = x^2 - 3x + 2$ (ii) $f(x) = 2x + 5$, $g(x) = 2x^2 + 1$

7. If demand function of a product is $D(x) = 100 - 5x$ and supply function is $S(x) = x - 200$. Draw graphs of both functions on same graph by using same scale and find the value of x for which the supply will become equal to demand?

8. The growth of the population of a town is given by the function, $P(t) = 50{,}000(1.05)^t$, where t is the time in years. Draw the graph of this function and find the population growth over 5years.

9. Draw gradient of the curve $y = \frac{1}{2}x^2$ by drawing tangent line at $x = 2$.

10. A logistics company charges a base fee of 50 dollars and additional charges of 0.5 dollars per mile for transporting goods. If a shipment travels 250 miles, what is the total cost?

- Cartesian product of any two non-empty sets A and B is a set containing all ordered pairs (x, y) such that $x \in A$ and $y \in B$.
- A function f from set A to set B is a relation, rule or mapping which maps each element of set A to a unique element of set B.
    - A function f : A→B will be into if range(f) ⊂ B.
    - A function f : A→B will be onto if range(f) = B.
    - A function f : A→B will be one-one if unique element of A have unique image in B.
    - A function f : A→B will be injective if it is both into and one-one.
    - A function f : A→B will be bijective if it is both onto and one-one.
- Every bijective function has one-one correspondence between its elements.
- The process of finding the value of dependent variable by substituting any specific value of the independent variable is called evaluation of a function.
- For any two functions f and g:

  $(f+g)(x) = f(x) + g(x)$, $(f-g)(x) = f(x) - g(x)$

  $(f \times g)(x) = f(x) \times g(x)$, $(f \div g)(x) = f(x) \div g(x)$
- The composition of any two functions, $f(x)$ and $g(x)$ is denoted by $(fog)(x) = f(g(x))$.
- If $y = f(x)\ \forall\ x \in X$, then $x = f^{-1}(y)\ \forall\ y \in Y$.
- The domain of $f(x)$ = range of $f^{-1}(x)$ and range of $f(x)$ = domain of $f^{-1}(x)$.
- A polynomial function of no degree is called a zero polynomial function.
- A polynomial function of degree zero is called a constant function.
- A polynomial function of degree one is called a linear function.
- A polynomial function of degree two is called a quadratic function.
- The graph of the curve, $y = ax^2 + bx + c$ is a parabola with vertex at (h , k), where $h = \frac{-b}{2a}$ and $k = c - \frac{b^2}{4a}$. It will open upward when $a > 0$ and open downward when $a < 0$.
- A polynomial function of degree three is called a cubic function.
- The function of the form, $f(x) = \sqrt{x}$, where $x \geq 0$ is called a square root function.
- The function of the form, $f(x) = |x|$ is called the absolute valued function.
- The function defined by $f(x) = \frac{1}{x}$, where $x \neq 0$ is called a reciprocal function.
- The exponential function is of the form $f(x) = a^x$, where $a > 0$ and $a \neq 1$.
- The point of intersection of two graphs is called their solution point.

MISCELLANEOUS EXERCISE-6

1. Encircle the correct option in each of the following.

i. If $A = \{-2, 0, 2\}$ and $B = \{0, 2\}$ and $f : A \to B$ is defined as $f = \{(-2, 2), (0, 0), (2, 0)\}$ what type of function is f ?

(a) into (b) onto (c) injective (d) bijective

ii. If number of elements in set X is 3 and number of elements in set Y is 2, how many binary relations are possible from Y to X?

(a) 4 (b) 6 (c) 2^6 (d) 2^9

iii. What is the domain of the relation g = {(1, 0), (2, 2), (3, 4)}?

(a) {0, 1, 2, 3} (b) {0, 2, 4} (c) {1, 2, 3} (d) {0, 1, 2, 3, 4}

iv. What is the x-intercept of every point on y-axis?

(a) 0 (b) 1 (c) −1 (d) undefined

v. At what point will the graph of $y = 2x^2 - 1$ cuts y-axis?

(a) $(\pm\frac{1}{\sqrt{2}}, 0)$ (b) −1 (c) (0, −1) (d) (−1, 0)

vi. If $y = 2x - 1$, what is $f^{-1}(x)$?

(a) $\frac{1+y}{2}$ (b) 2y−1 (c) $\frac{1+x}{2}$ (d) y+1

vii. If $f(x) = \frac{1}{2}x$, what is $f^2(x)$?

(a) $\frac{1}{4x}$ (b) $\frac{1}{4}x$ (c) $2x$ (d) $\frac{1}{4}x^2$

viii. If $y = \frac{x}{x-2}$, for what value of x will the function become undefined?

(a) 0 (b) −2 (c) 2 (d) ±2

ix. Which of the following is an exponential function?

(a) $\left(\frac{1}{3}\right)^x$ (b) e^x (c) 2^x (d) All of these

x. If $f(x) = \frac{2}{3}x^2 - 5$, what is the value of f(−3)?

(a) 0 (b) 1 (c) −3 (d) −1

2. Find inverse of $f(x) = \frac{3x}{2x-1}$. At what value of the variable, $f^{-1}(x)$ will be undefined? Also find $f^{-1}(-1)$.

3. Sketch the graph of the quadratic function $y = -x^2 - 4x$. Will it open upward or downward?

4. Find the equation in the form of $y = ax^2 + bx + c$ which cuts x-axis at (−1, 0) and (1, 0) and y-axis at (0, 10).

5. For the function f defined by $f(x) = x^2 - x - 6$, find the value of x if $f(x) = f(3)$.

6. Find the inverse function to convert Celsius to Fahrenheit, if $C = \frac{5}{9}(F - 32)$ and use it to convert 25°C to Celsius graphically.

7. Find the point of intersection of the graphs of $f(x) = x - 2$ and $g(x) = x^2 - 4x + 2$.

8. When medication is taken by a patient, it is slowly used by the body. After n hours, the amount of medicine remaining in the body is given by $f(n) = 120 \times (0.9)^n$mg. Draw the graph of f(n) against n and use the graph to find when there is 20mg of the medicine left in the body.

9. A sound wave's amplitude (A) is given by the function, $A(t) = |2\cos(t)|$, where t is time in seconds. If t = 5 seconds, what is the amplitude?

07

Vectors in Plane

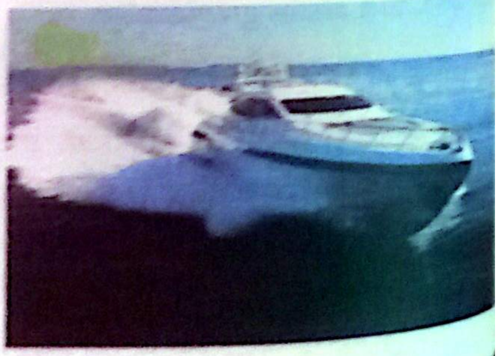
After studying this unit students will be able to:

- Introduce rectangular coordinate system in a plane.
- Represent vector as a directed line segment.
- Express a vector in terms of two non-zero and non-parallel coplanar vector.
- Express a vector in terms of position vector.
- Express translation by a vector.
- Find the magnitude of a vector.
- Add and subtract vectors.
- Multiply a vector by a scalar.
- Solve geometrical problems involving the use of a vector.
- Apply concepts of vectors in geometrical problems such as parallel and perpendicular lines in geometrical shapes, vector projectile motion, crosswinds aviation, military usage, designing roller coasters.

Vectors are fundamental mathematical concept that have a wide range of applications in the vario fields from the physics and engineering to the computer graphics and navigation systems. Vecto are utilized to assist in the localization of people, places and things, and to describe things that a acting in response to an external force being applied to them.

Do you know?

- Navigating by air and by boat is generally done using vectors.
- Planes are given a vector to travel, and they use their speed to determine how far they need to g before turning or landing. Flight plans are made using a series of vectors.
- Sports instructions are based on using vectors.
- Vectors are used in electrical engineering for analyzing and designing circuits, signals, a electromagnetic systems.



Introduction of Vectors

There are two main types of physical quantities used in Physics, Mathematics and Engineering. They are known as, 'scalers' and 'vectors'

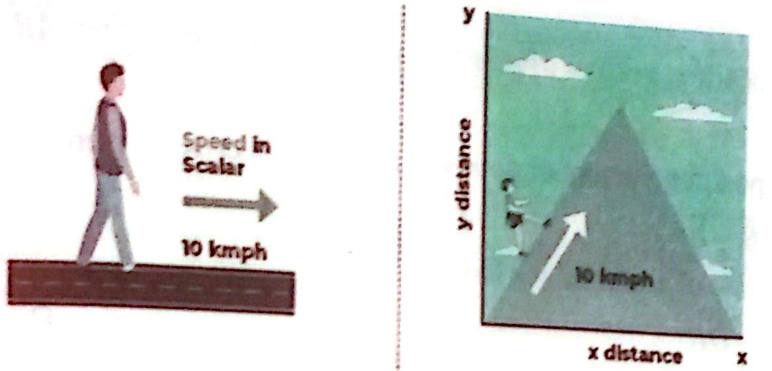
Scalar

A physical quantity that possesses magnitude only, is called a scalar quantity. It can be completely specified by a number along with unit. For example, mass, time, speed, density, work, distance, length, perimeter, area, volume, etc.

Vector

A physical quantity that possesses both magnitude and direction, is called a vector quantity. For example, weight, displacement, momentum, force, velocity, acceleration, electric field etc.

Scalar and Vector



Velocity is vector which means direction is also included

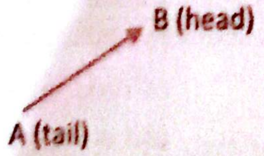
Class Activity:

Look at the table below and categorize the quantities as scalar or vector.

S. No	Quantity	Category
1	7 m	
2	40 m/sec, West	
3	144 square metre	
4	32 °C	
5	98 Newtons	
6	6540 Calories	
7	5 km, East	

Geometrical Representation of a Vector

Geometrically a vector is represented by a line segment with an arrow head at its one end (the ray). The length of the line segment describes the magnitude and the arrow head indicates the direction of the vector.



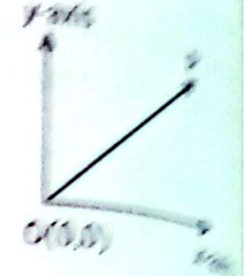
In the figure, the end A of the ray AB is called the tail or the initial point of the vector and the end B is called the terminal point with the arrow head denoted by $\overrightarrow{AB}$.

Usually, the vectors are denoted by bold face letters

a, b, c or $\vec{a}, \vec{b}, \vec{c}$ or $\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}$ etc.

Position Vector

The vector used to specify the position of a point P with respect to origin O is called position vector of P. The tail of this vector is at origin and head at the point P. Thus $\overrightarrow{OP}$ is the position vector of point P with respect to O.



Result:

If $\vec{a}$ and $\vec{b}$ are position vectors of points A and B respectively, then $\overrightarrow{AB} = \vec{b} - \vec{a}$

Proof:

If $\vec{a}$ and $\vec{b}$ are position vectors of points A and B respectively, then:

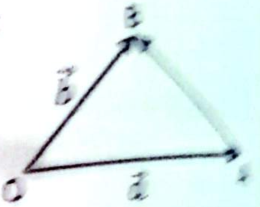
$\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$

Using triangle law of addition, we have:

$$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$$

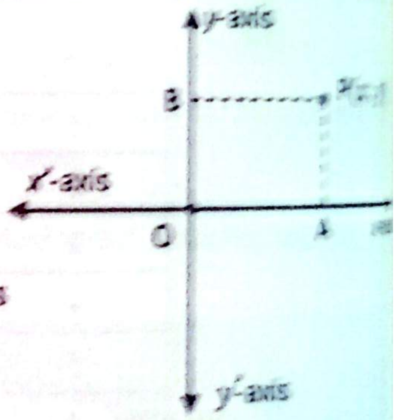
$$\Rightarrow \quad \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$\Rightarrow \quad \overrightarrow{AB} = \vec{b} - \vec{a}$$



Rectangular or Cartesian Coordinate System

A rectangular or Cartesian coordinate system consists of two number lines which are mutually perpendicular. Their point of intersection is called the origin, usually denoted by the letter 'O'. Horizontal line is called x-axis and vertical line is called y-axis. Ordered pairs of numbers can be represented on a Cartesian plane, which is sometimes called the Coordinate plane or *xy*-plane. Each point in the plane corresponds to an ordered pair (x, y) of real numbers x and y, called the coordinates of the point.



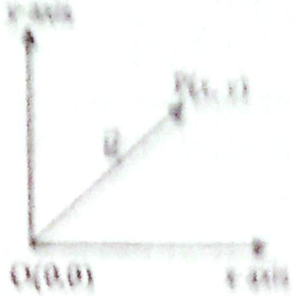
Key Fact

Cartesian plane has got this name after the French mathematician Rene Des (1596 – 1650). He developed the idea of the coordinate plane and the correspond of ordered pairs of numbers to points in the coordinate plane. Each point in the pl can corresponds to only one ordered pair and vice versa.

Vector in a Plane

Let R be the set of real numbers, then the Cartesian plane is defined as:

$$R^2 = \{(x, y): x, y \in R\}$$



Any point P(x, y) is uniquely determined by its coordinates x and y. For a given vector $\vec{u}$ in the plane, there exists a unique point P(x, y) in the plane such that:

$$\vec{u} = \overrightarrow{OP}$$

For the point P(x, y), a unique ordered pair [x, y] is used for the vector $\vec{u}$ called position vector of P(x, y). Thus:

$$\vec{u} = \overrightarrow{OP} = [x, y]$$

Negative of a Vector

A vector having the same magnitude but opposite in direction of a given vector $\vec{b}$ is called the negative of $\vec{b}$ and is denoted by $-\vec{b}$.

If $\vec{u} = [x, y]$, then negative vector of $\vec{u}$ is defined as:

$$-\vec{u} = [-x, -y]$$

Example:

P(3, – 4) is a point in the plane. Find the position vector $\overrightarrow{OP}$ and then its negative vector.

Solution:

Given is P(3, – 4) in the plane.

Position vector of P = $\overrightarrow{OP} = [3, -4]$

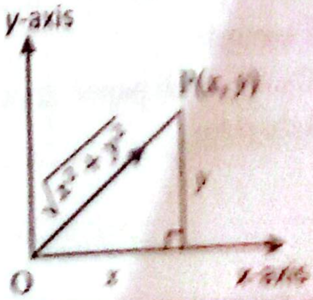
Negative vector of $\overrightarrow{OP} = -\overrightarrow{OP} = -[3, -4] = [-3, 4]$

Zero or Null Vector

If the initial and terminal points of a vector coincide then the vector has zero length. This vector is called zero vector and is denoted by $\vec{0}$. The zero vector has no direction. So, it can be assigned as convenient direction according to the situation. The zero vector is denoted by:

$$\vec{0} = [0, 0]$$

Magnitude (Norm) of a Vector



Given the vector $\vec{u} = \overrightarrow{AB}$. The magnitude or the length or the norm of the vector $\vec{u}$ is denoted by $|\vec{u}| = |\overrightarrow{AB}|$.

If $\vec{u} = [x, y]$, then:

$$|\vec{u}| = \sqrt{x^2 + y^2}$$

Key Facts:

(i) If $\vec{u}$ is any vector, then $|\vec{u}| \geq 0$. (ii) $|\vec{u}| = 0$ if and only if $\vec{u} = \vec{0}$.

(iii) $|c\vec{u}| = |c||\vec{u}|$ where c is a scalar. (iv) $|-\vec{u}| = |\vec{u}|$

Unit Vector

A vector which is in the direction of a non-zero vector $\vec{u}$ and has magnitude 1 is called a vector of $\vec{u}$ and is denoted by $\hat{u}$. If $\vec{u}$ is non-zero vector of arbitrary length $|\vec{u}|$ then:

$$\hat{u} = \frac{\vec{u}}{|\vec{u}|} \quad \Rightarrow \quad \vec{u} = |\vec{u}|\hat{u}$$

This mean any vector $\vec{u}$ can be constructed by multiplying the magnitude of the vector to a unit vector. The process of finding the unit vector of a vector $\vec{u}$ is called normalizing the vector $\vec{u}$.

If $\vec{u} = [x, y]$, then its unit vector is written as:

$$\hat{u} = \frac{[x,\ y]}{\sqrt{x^2+y^2}} = \left[\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right]$$

Example:

Find magnitude of the vector $\vec{u} = [-5, 12]$. Also find unit vector of $\vec{u}$.

Solution: $\vec{u} = [-5, 12]$

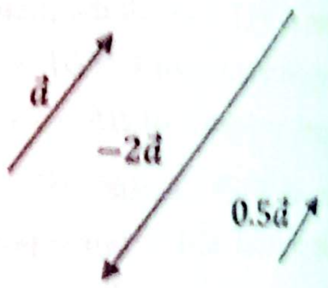
Here, $x = -5, y = 12$

Magnitude of $\vec{u} = |\vec{u}| = \sqrt{x^2 + y^2} = \sqrt{(-5)^2 + (12)^2} = \sqrt{25 + 144} = \sqrt{169} = 13$ units

Unit vector of $\vec{u} = \hat{u} = \frac{\vec{u}}{|\vec{u}|} = \frac{[-5,\ 12]}{13} = \left[\frac{-5}{13}, \frac{12}{13}\right]$

Scalar Multiplication

If λ is a non-zero scalar and $\vec{a}$ is a non-zero vector then the scalar multiple $\lambda\vec{a}$ is a vector whose magnitude is λ times the magnitude $\vec{a}$.



- $\vec{a}$ and $\lambda\vec{a}$ have the same direction if $\lambda > 0$.
- $\vec{a}$ and $\lambda\vec{a}$ have the opposite direction if $\lambda < 0$.
- If $\lambda\vec{a} = 0$ then either $\lambda = 0$ or $\vec{a} = 0$.
- If $\lambda = 0$, then $\lambda\vec{a}$ is a zero vector.

Note: If $\vec{u} = [x, y]$ then product of λ with $\vec{u}$ is $\lambda\vec{u} = \lambda[x, y] = [\lambda x, \lambda y]$.

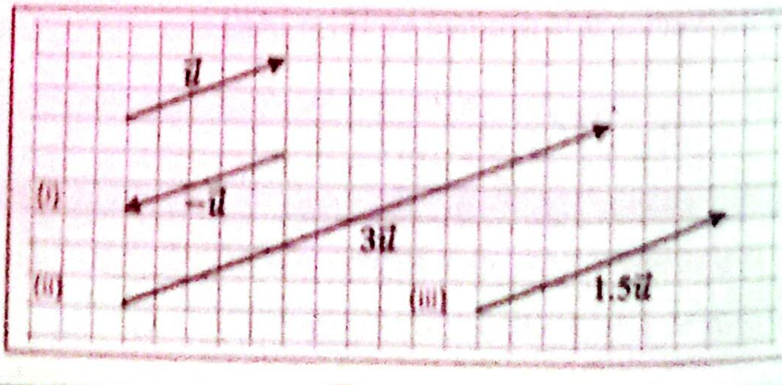
For example, if $\vec{u} = [-6, 4]$ and $\lambda = 3$, then:

$3\vec{u} = 3[-6, 4] = [-18, 12]$

Example:

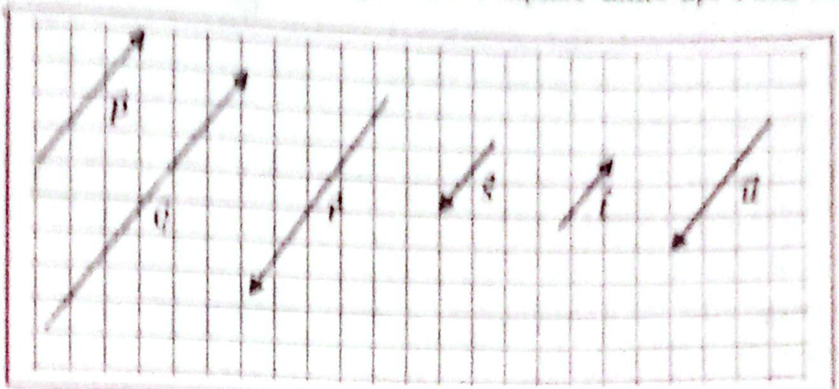
Using graph paper draw the vectors: (i) $-\vec{u}$ (ii) $3\vec{u}$ (iii) $1.5\vec{u}$, when $\vec{u}$ is given.

Solution:



Example:
In the figure, direction of $\vec{p}$ is 3 square units to the right and 4 square units up. Find the relation of other vectors with $\vec{p}$.

Solution:



(i) $\vec{q} = 2\vec{p}$

(ii) $\vec{r} = -1.5\vec{p}$

(iii) $\vec{s} = -0.5\vec{p}$

(iv) $\vec{t} = 0.5\vec{p}$

(v) $\vec{u} = -\vec{p}$

Translation by Vector

Translation by a vector is a process that moves a point or figure in space by a given vector. It is a type of transformation that slides a figure without rotating it, so the shape, size, and orientation of the figure do not change.

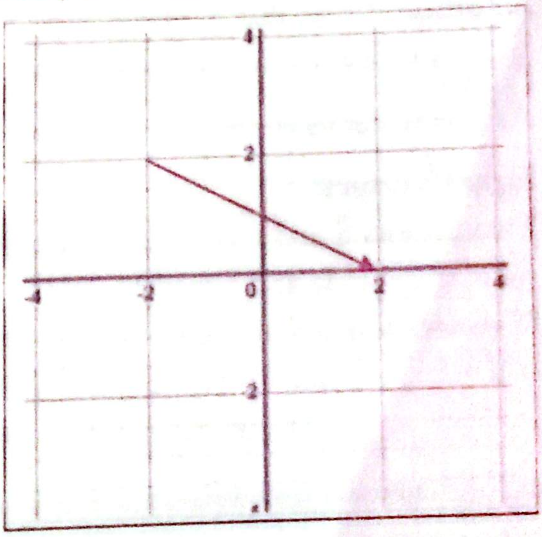
Geometrically

Place the tail of the arrow at the point, and the translated point is at the head of the arrow.

Algebraically

Represent the points and the vector by their coordinates. The translation vector's coordinates indicate how far to move the points along each coordinate axis.

For example, the vector $[a, b]$ moves points a units horizontally and b units vertically.



For example, a translation vector that moves a figure 4 units right and 2 units down can be represented mathematically as $[4, -2]$, or graphically as shown in the adjoining figure. It doesn't matter where the vector is positioned in the plane. In this figure, the vector starts at $(-2, 2)$ and ends at $(2, 0)$. But the initial point and terminal point of the vector is irrelevant. What matters is the length of the vector and the direction in which it points, so all you have to look at is how many units the vector moves in the y-direction and how many units the vector moves in the x-direction.

Example:

Use the translation vector $[2, -1]$ to find the new coordinates of triangle ABC when:
$A = (-3, 2)$, $B = (-1, 0)$ and $C = (-2, -1)$

Solution:

The vector indicates a translation of 2 units to the right and 1 unit down. We can therefore add 2 to all of the x-values and subtract 1 from all of the y-values to find the vertices of the image.

First let's write down the coordinates of the pre-image, triangle ABC.

A = (−3, 2), B = (−1, 0), C = (−2, −1)

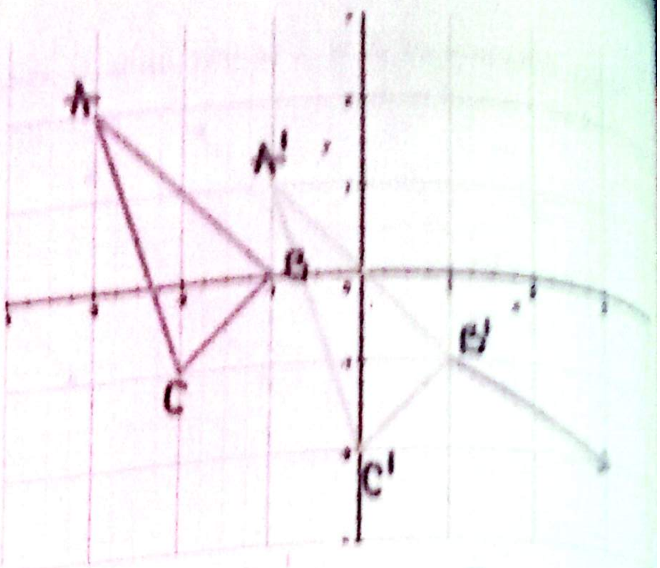
Now we can make the calculations to translate each vertex.

A' = (−3 + 2, 2 − 1) = (−1, 1)

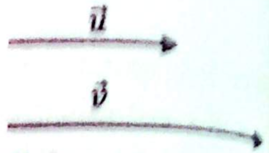
B' = (−1 + 2, 0 − 1) = (1, −1)

C' = (−2 + 2, −1 −1) = (0, −2)

The image after the translation is shown in the adjoining figure.



Parallel Vectors



Two non-zero vectors $\vec{u}$ and $\vec{v}$ are said to be parallel if $\vec{u} = \lambda\vec{v}$, where λ is a scalar. If $\lambda > 0$ then both vectors have the same direction and if $\lambda < 0$ then both are in the opposite direction.
The vectors which are in the opposite direction are known as anti-parallel vectors.
The vectors which are in the opposite direction are known as anti-parallel vectors, then:

In component form, when $\vec{u} = [x_1, y_1]$ and $\vec{v} = [x_2, y_2]$ are parallel vectors, then:

$$\vec{u} = \lambda\vec{v} \Rightarrow [x_1, y_1] = \lambda[x_2, y_2]$$

$$\Rightarrow [x_1, y_1] = [\lambda x_2, \lambda y_2] \quad \Rightarrow \quad x_1 = \lambda x_2 \text{ and } y_1 = \lambda y_2$$

Example:

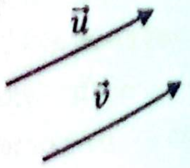
Prove that $\vec{a} = [-12, -15]$ and $\vec{b} = [4, 5]$ are parallel vectors.

Solution: Given that $\vec{a} = [-12, -15]$ and $\vec{b} = [4, 5]$

$\vec{a} = [-12, -15] = -3[4, 5] = -3\vec{b}$

As $\vec{a}$ is scaler multiple of $\vec{b}$, therefore $\vec{a} \parallel \vec{b}$.

Equal Vectors



Two vectors $\vec{u}$ and $\vec{v}$ are said to be equal if both have the same magnitude and direction.

If vectors $\vec{u}$ and $\vec{v}$ are equal then we write $\vec{u} = \vec{v}$.

If $\vec{u} = [x_1, y_1]$ and $\vec{v} = [x_2, y_2]$, then $\vec{u} = \vec{v}$ implies that:

$$[x_1, y_1] = [x_2, y_2] \quad \Rightarrow \quad x_1 = x_2 \text{ and } y_1 = y_2$$

Key Facts

- It is not necessary for the equal vectors to have the same position.
- Geometrically two vectors are equal if they are translation of one another.
- If in the relation $\vec{a} = \lambda\vec{b}$ and $\lambda = 1$ then $\vec{a} = \vec{b}$.
- Equal vectors are also parallel.

Example:

If $\vec{p} = [-c+4, -2]$ and $\vec{q} = [4, 2d]$ are equal vectors, find the values of c and d.

Solution:

Given that, $\vec{p} = \vec{q} \Rightarrow [-c+4, -2] = [4, 2d]$

$\Rightarrow -c+4 = 4,\ -2 = 2d$

$\Rightarrow -c = 4-4,\ -2 \div 2 = d$

$\Rightarrow c = 0,\ d = -1$

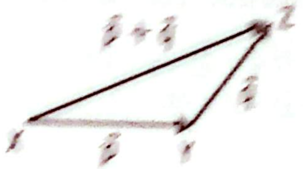
[illegible]

Are the vectors equal? If not why?

Addition of Vectors

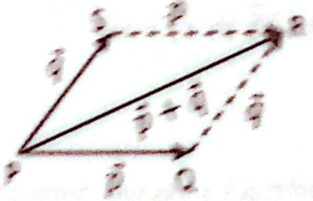
Head to Tail Rule

To add non-zero vectors $\vec{p}$ and $\vec{q}$ join the tail of the second vector $\vec{q}$ with the head of the first vector $\vec{p}$. Now the vector obtained by joining the tail of the first vector to the head of the second vector is the vector $\vec{p} + \vec{q}$ called the resultant vector of $\vec{p}$ and $\vec{q}$.



This method for the addition of two vectors is called head to tail rule of addition. Since $\vec{p}, \vec{q}$ and $\vec{p} + \vec{q}$ are along the sides of a triangle XYZ, so this method of addition of vectors is also known as triangle law of addition.

Parallelogram Law of Addition



Consider any parallelogram PQRS. Let $\overrightarrow{PQ} = \vec{p}$ and $\overrightarrow{QR} = \vec{q}$. Since the vector $\overrightarrow{QR}$ has the same magnitude and direction as that of $\overrightarrow{PS}$. Similarly, $\overrightarrow{SR}$ has the same magnitude and direction as that $\overrightarrow{PQ}$. Therefore:

$$\overrightarrow{PQ} = \overrightarrow{SR} \text{ and } \overrightarrow{QR} = \overrightarrow{PS}$$

Using triangle law of addition, we have:

$$\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$$

$$\Rightarrow \vec{p} + \vec{q} = \overrightarrow{PR}$$

Showing that the diagonal vector $\overrightarrow{PR}$ of the parallelogram is the sum of the vectors of $\vec{p}$ and $\vec{q}$. This is known as parallelogram law of addition. Furthermore:

From ΔPQR, $\overrightarrow{PR} = \vec{p} + \vec{q}$ (i)

From ΔPSR, $\overrightarrow{PR} = \vec{q} + \vec{p}$ (ii)

$\therefore\ \vec{p} + \vec{q} = \vec{q} + \vec{p}$ [from (i) and (ii)]

This shows that vector addition is commutative.

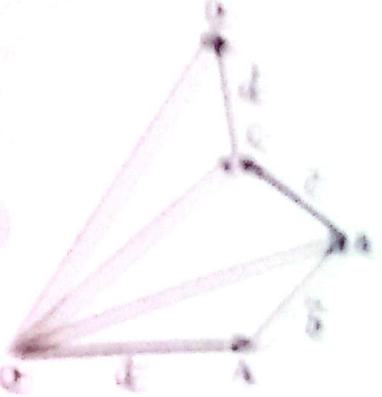
Key Facts

For any vector $\vec{a}$:

- $\vec{a} + \vec{0} = \vec{0} + \vec{a} = \vec{a}$
- $\vec{a} + (-\vec{a}) = (-\vec{a}) + \vec{a} = \vec{0}$

Polygon Law of Vector Addition

The triangle law of the addition of 2 vectors can be extended to any number of vectors. For example, let we have four vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ and we need to find their sum.



From the figure,

$\overrightarrow{OA} = \vec{a}$, $\overrightarrow{AB} = \vec{b}$, $\overrightarrow{BC} = \vec{c}$, $\overrightarrow{CD} = \vec{d}$

Now, $\vec{a} + \vec{b} + \vec{c} + \vec{d} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$

$= (\overrightarrow{OA} + \overrightarrow{AB}) + \overrightarrow{BC} + \overrightarrow{CD}$

$= \overrightarrow{OB} + \overrightarrow{BC} + \overrightarrow{CD}$ $(\because \overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB})$

$= (\overrightarrow{OB} + \overrightarrow{BC}) + \overrightarrow{CD}$

$= \overrightarrow{OC} + \overrightarrow{CD}$ $(\because \overrightarrow{OB} + \overrightarrow{BC} = \overrightarrow{OC})$

$= \overrightarrow{OD}$

Here, $\overrightarrow{OD}$ is the sum of all these four vectors. The same method is adopted to find the sum of any number of vectors. This method is called polygon law of addition of vectors.

In component form: If $\vec{a} = [x_1, y_1]$ and $\vec{b} = [x_2, y_2]$, $\vec{c} = [x_3, y_3]$, then;

$\vec{a} + \vec{b} + \vec{c} = [x_1, y_1] + [x_2, y_2] + [x_3, y_3] = [x_1 + x_2 + x_3, y_1 + y_2 + y_3]$

Example:

If $\vec{a} = [4, 0]$ and $\vec{b} = [-3, 1]$, $\vec{c} = [2, -2]$ then find $\vec{a} + 2\vec{b} + \vec{c}$. Also find $|\vec{a} + 2\vec{b} + \vec{c}|$.

Solution:

Given that, $\vec{a} = [4, 0]$, $\vec{b} = [-3, 1]$ and $\vec{c} = [2, -2]$

Now, $\vec{a} + 2\vec{b} + \vec{c} = [4, 0] + 2[-3, 1] + [2, -2] = [4, 0] + [-6, 2] + [2, -2]$

$= [4 - 6 + 2, 0 + 2 - 2] = [0, 0]$

$|\vec{a} + 2\vec{b} + \vec{c}| = \sqrt{0^2 + 0^2} = \sqrt{0} = 0$ units

Subtraction of Two Vectors

To subtract a vector from the other we find the negative vector of the vector to be subtracted and then add it to the other vector.

Consider two non-zero vectors $\vec{a}$ and $\vec{b}$ then:

$\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$

From the figure, $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{BC} = -\vec{b}$ then

$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ (head to tail rule of addition)

$\vec{a} + (-\vec{b}) = \vec{a} - \vec{b} = \overrightarrow{AC}$

Thus, $\overrightarrow{AC}$ is the vector which represents $\vec{a} - \vec{b}$.

In component form, if $\vec{a} = [x_1, y_1]$ and $\vec{b} = [x_2, y_2]$, then:

$\vec{a} - \vec{b} = [x_1, y_1] - [x_2, y_2]$

$= [x_1 - x_2, y_1 - y_2]$

Example:

If $\vec{a} = [7, 10]$ and $\vec{b} = [-3, 10]$, then find $\vec{a} - \vec{b}$ and a unit vector along $\vec{a} - \vec{b}$.

Solution:

Given that, $\vec{a} = [7, 10]$ and $\vec{b} = [-3, 10]$

Now, $\vec{a} - \vec{b} = [7, 10] - [-3, 10] = [7 + 3, 10 - 10] = [10, 0]$

$|\vec{a} - \vec{b}| = \sqrt{10^2 + 0^2} = \sqrt{100} = 10$ units

Unit vector along $\vec{a} - \vec{b} = \frac{\vec{a} - \vec{b}}{|\vec{a} - \vec{b}|} = \frac{[10,\ 0]}{10} = \left[\frac{10}{10}, \frac{0}{10}\right] = [1, 0]$

If $\vec{a} = [3, 4]$ and $\vec{b} = [-3, 4]$, then find:

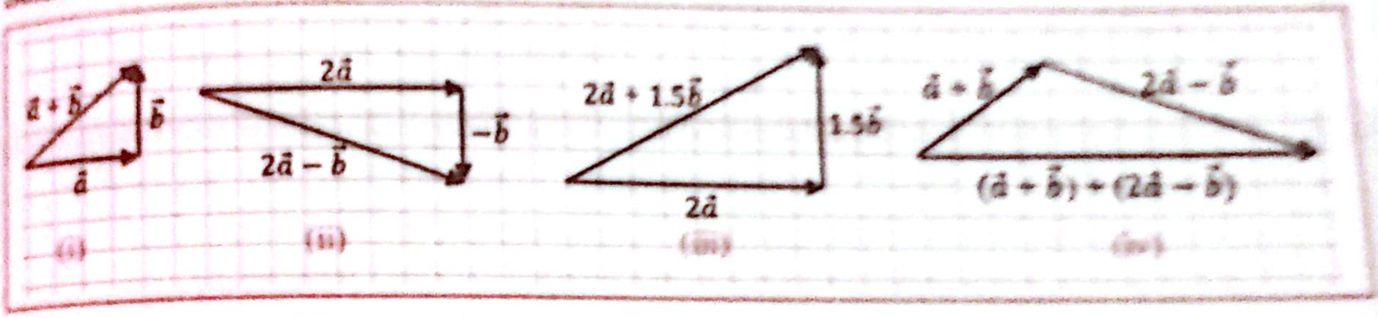
$5\vec{a} + \vec{b}$ and $3\vec{a} - 2\vec{b}$

Example:

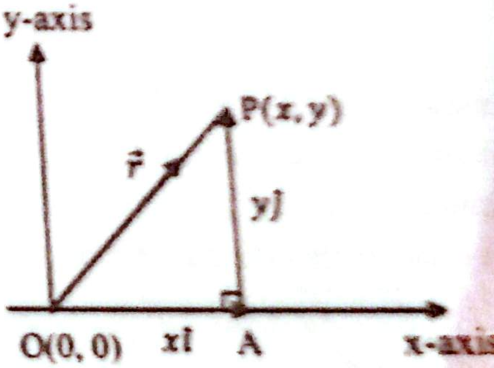
Draw vectors $\vec{a} = 4$ units east and $\vec{b} = 3$ units north on graph paper. Then, draw the following vectors on the graph paper.

(i) $\vec{a} + \vec{b}$ (ii) $2\vec{a} - \vec{b}$ (iii) $2\vec{a} + 1.5\vec{b}$ (iv) $(\vec{a} + \vec{b}) + (2\vec{a} - \vec{b})$

Solution:



Representation of a Vector in Cartesian Plane



We denote the unit vector along x-axis by $\hat{i}$ and along y-axis by $\hat{j}$.

Let us consider a point P(x, y) in the Cartesian plane.

We want to find the position vector $\overrightarrow{OP}$ in terms of unit vectors $\hat{i}$ and $\hat{j}$.

Draw perpendicular from P on x-axis meeting the x-axis at point A. As we know that for any vector $\vec{u}$:

$$\hat{u} = \frac{\vec{u}}{|\vec{u}|} \text{ or } \vec{u} = |\vec{u}|\hat{u}$$

i.e., Every vector is equal to the product of magnitude with its unit vector. So,

$$\overrightarrow{OA} = |\overrightarrow{OA}|\hat{i} = x\hat{i}$$

$$\overrightarrow{AP} = |\overrightarrow{AP}|\hat{j} = y\hat{j}$$

Now, $$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$$

$$\Rightarrow \overrightarrow{OP} = x\hat{i} + y\hat{j}$$

$$\Rightarrow \vec{r} = x\hat{i} + y\hat{j}$$

This is the position vector of the point $P(x, y)$ lying in the Cartesian plane.

Components of Vector When its Tail is not at Origin

Consider a vector $\overrightarrow{PQ}$ withend points $P(x_1, y_1)$ and $Q(x_2, y_2)$.

Draw perpendiculars from P and Q on x-axis meeting at S and T respectively.

Now from figure:

$$|\overrightarrow{PR}| = |\overrightarrow{ST}| = |\overrightarrow{OT}| - |\overrightarrow{OS}| = x_2 - x_1$$

$$|\overrightarrow{RQ}| = |\overrightarrow{TQ}| - |\overrightarrow{TR}|$$

$$= |\overrightarrow{TQ}| - |\overrightarrow{SP}| = y_2 - y_1$$

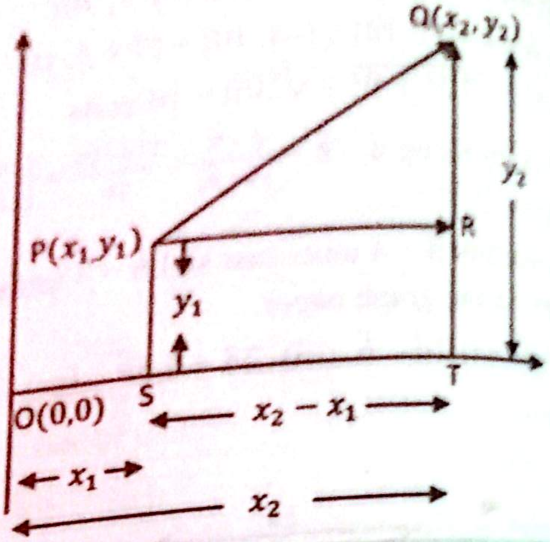
$$\therefore \quad \overrightarrow{PR} = |\overrightarrow{PR}|\hat{\imath} = (x_2 - x_1)\hat{\imath}$$

and $\overrightarrow{RQ} = |\overrightarrow{RQ}|\hat{\jmath} = (y_2 - y_1)\hat{\jmath}$

As, $\overrightarrow{PQ} = \overrightarrow{PR} + \overrightarrow{RQ}$

Therefore:

$$\overrightarrow{PQ} = (x_2 - x_1)\hat{\imath} + (y_2 - y_1)\hat{\jmath}$$



Key Fact:

- In the relation, $\vec{r} = x\hat{\imath} + y\hat{\jmath}$:
  x is called x-component of $\vec{r}$ and y is called y-component of $\vec{r}$.
- $x\hat{\imath}$ and $y\hat{\jmath}$ are known as parts of vector $\vec{r}$ in the direction of $\hat{\imath}$ and $\hat{\jmath}$ respectively.
- When the tail of a vector is not at origin then it is not a position vector.

Rectangular Components of a Vector

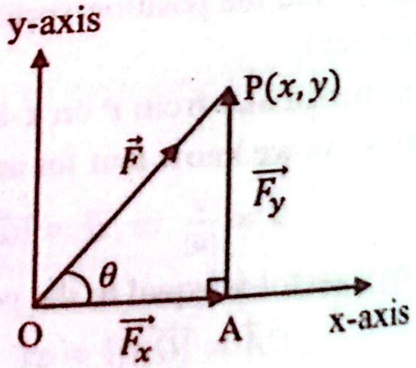
Consider a vector $\overrightarrow{OP} = \vec{F}$ making angle θ with x-axis as shown in the adjoining figure. We want to find its components that are perpendicular to each other.

If we draw perpendicular AP on x-axis, then from figure:

$$\vec{F} = \overrightarrow{F_x} + \overrightarrow{F_y} = F_x\hat{\imath} + F_y\hat{\jmath} \quad \ldots\ldots \quad \text{(i)}$$

Here, $\overrightarrow{F_x}$ and $\overrightarrow{F_y}$ are rectangular components of $\vec{F}$ while F_x and F_y are magnitudes of $\overrightarrow{F_x}$ and $\overrightarrow{F_y}$ respectively.

Now if the magnitude of $\vec{F}$ is given, then we can find F_x and F_y using trigonometric ratios.



From right triangle OAP:

$$\frac{F_x}{|\vec{F}|} = \cos\theta \quad \text{and} \quad \frac{F_y}{|\vec{F}|} = \sin\theta$$

$$\Rightarrow \quad F_x = |\vec{F}|\cos\theta \quad \ldots\ldots\text{(ii)}$$

$$F_y = |\vec{F}|\sin\theta \quad \ldots\ldots\text{(iii)}$$

Key Fact:

$\overrightarrow{F_x}$ and $\overrightarrow{F_y}$ are called horizontal and vertical components of $\vec{F}$ respectively.

Equations (ii) and (iii) are used to find the rectangular components of a given vector making angle θ with x-axis.

Conversely, if the rectangular components of $\vec{F_x}$ and $\vec{F_y}$ of a vector are given then we can find the magnitude and angle of resultant vector by using following relations.

$$|\vec{F}| = \sqrt{(F_x)^2 + (F_y)^2} \quad \ldots\ldots\text{(iv)} \quad \text{and} \quad \theta = tan^{-1}\left(\frac{F_y}{F_x}\right) \quad \ldots\ldots\text{(v)}$$

Example:

A ball is thrown with an initial velocity of 25 m/s at an angle of 45° with the horizontal. Find the horizontal and vertical components of the velocity.

Solution:

Let V represents the velocity.

Horizontal component of velocity is given as:

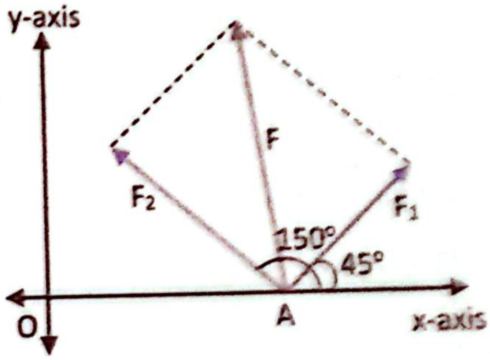
$V_x = Vcos45° = 25 \times 0.707 \ = 17.68$ m/s

Vertical component of velocity is given as:

$V_y = Vsin45° = 25 \times 0.707 \ = 17.68$ m/s

Example:

Two forces F_1 and F_2 with magnitudes 30 N and 40 N, respectively, act on an object at a point A as shown in the figure. Find the resultant force **F** acting at A and its magnitude.



Solution:

First, we write F_1 and F_2 in component form:

$F_1 = F_1\,cos45°\hat{\imath} + F_1\,sin45°\hat{\jmath}$

$= 30 \times 0.707\,\hat{\imath} + 30 \times 0.707\,\hat{\jmath} = 21.21\,\hat{\imath} + 21.21\,\hat{\jmath}$

$F_2 = F_2\,cos150°\hat{\imath} + F_2\,sin150°\hat{\jmath}$

$= 40 \times (-0.866)\,\hat{\imath} + 40 \times 0.5\,\hat{\jmath} = -34.64\,\hat{\imath} + 20\,\hat{\jmath}$

So, the resultant force **F** is:

$F = F_1 + F_2 = (21.21\,\hat{\imath} + 21.21\,\hat{\jmath}) + (-34.64\,\hat{\imath} + 20\,\hat{\jmath}) = (21.21 - 34.64)\,\hat{\imath} + (21.21 + 20)\,\hat{\jmath}$

$= -13.43\hat{\imath} + 41.21\hat{\jmath}$

Magnitude of resultant force $= |\mathbf{F}| = \sqrt{(-13.43)^2 + (41.21)^2} = \sqrt{180.36 + 1{,}698.26}$

$= \sqrt{1{,}878.62} = 43.34$ units

Concept Related to Vectors Using Analytical Representation

Magnitude of a Vector

In Cartesian plane a vector $\vec{u}$ is expressed as:

$$\vec{u} = x\hat{\imath} + y\hat{\jmath}$$

The magnitude of vector $\vec{u}$ is defined as:

$$|\vec{u}| = \sqrt{x^2 + y^2}$$

$$|\vec{u}| = \sqrt{(x\text{-component})^2 + (y\text{-component})^2}$$

Equal Vectors

Two vectors are said to be equal if and only if they have same x and y components.

If $\vec{u} = x_1\hat{i} + y_1\hat{j}$ and $\vec{v} = x_2\hat{i} + y_2\hat{j}$ then $\vec{u} = \vec{v}$ if $x_1 = x_2$ and $y_1 = y_2$

Negative of a Vector

If $\vec{p} = x\hat{i} + y\hat{j}$ then negative of $\vec{p}$ is:

$$-\vec{p} = (-1)\vec{p} = (-1)(x\hat{i} + y\hat{j}) = (-x)\hat{i} + (-y)\hat{j}$$

Unit Vector

If $\vec{w} = x\hat{i} + y\hat{j}$ then $\vec{w}$ is called a unit vector if:

$$|\vec{w}| = 1 \Rightarrow \sqrt{x^2 + y^2} = 1 \text{ or } x^2 + y^2 = 1$$

Parallel Vectors

The vectors $\vec{u} = x_1\hat{i} + y_1\hat{j}$ and $\vec{v} = x_2\hat{i} + y_2\hat{j}$ are said to be parallel if for some scala

have:

$$\vec{u} = \lambda\vec{v}$$

$$\Rightarrow (x_1\hat{i} + y_1\hat{j}) = \lambda(x_2\hat{i} + y_2\hat{j})$$

$$\Rightarrow x_1\hat{i} + y_1\hat{j} = \lambda x_2\hat{i} + \lambda y_2\hat{j}$$

Comparing the components, we have:

$$x_1 = \lambda x_2 \text{ and } y_1 = \lambda y_2$$

$$\Rightarrow \frac{x_1}{x_2} = \lambda \text{ and } \frac{y_1}{y_2} = \lambda$$

$$\text{or } \frac{x_1}{x_2} = \frac{y_1}{y_2} = \lambda$$

Which is the condition for the two vectors to be parallel. If λ is positive then $\vec{u}$ and $\vec{v}$ have same direction and if λ is negative then $\vec{u}$ and $\vec{v}$ are in the opposite direction.

Key Facts

A vector having both x-componen and y-component zero is called a null vector denoted by $\vec{O}$.

$$\vec{O} = 0\hat{i} + 0\hat{j}$$

Addition and Subtraction of Vectors

If $\vec{u} = x_1\hat{i} + y_1\hat{j}$ and $\vec{v} = x_2\hat{i} + y_2\hat{j}$ are any two vectors then:

(i) $\vec{u} + \vec{v} = (x_1\hat{i} + y_1\hat{j}) + (x_2\hat{i} + y_2\hat{j}) = (x_1 + x_2)\hat{i} + (y_1 + y_2)\hat{j}$

(ii) $\vec{u} - \vec{v} = (x_1\hat{i} + y_1\hat{j}) - (x_2\hat{i} + y_2\hat{j}) = (x_1 - x_2)\hat{i} + (y_1 - y_2)\hat{j}$

Unit Vector in the Direction of another Vector

Let we have a non-zero vector $\vec{u} = x\hat{i} + y\hat{j}$

$$\therefore \quad |\vec{u}| = \sqrt{x^2 + y^2}$$

Now the unit vector in the direction of $\vec{u}$ is

$$\hat{u} = \frac{\vec{u}}{|\vec{u}|} = \frac{x\hat{i} + y\hat{j}}{\sqrt{x^2+y^2}}$$

$$\hat{u} = \left(\frac{x}{\sqrt{x^2+y^2}}\right)\hat{i} + \left(\frac{y}{\sqrt{x^2+y^2}}\right)\hat{j}$$

Key Facts

If λ is a scalar and $\vec{a} = x\hat{i} + y\hat{j}$ then product of λ with $\vec{a}$ is

$$\lambda\vec{a} = \lambda(x\hat{i} + y\hat{j}) = \lambda x\hat{i} + \lambda y\hat{j}$$

Example:

If $\vec{a} = 2\hat{i} - \hat{j}$ then find unit vector along the vector $\vec{a}$.

Solution:

$$|\vec{a}| = \sqrt{2^2 + (-1)^2} = \sqrt{4+1} = \sqrt{5}$$

The unit vector in the direction of $\vec{a}$ is:

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{2\hat{i} - \hat{j}}{\sqrt{5}} = \frac{2}{\sqrt{5}}\hat{i} + \frac{1}{\sqrt{5}}\hat{j}$$

Example: If $\vec{u} = 5\hat{i} + \hat{j}$ and $\vec{v} = 2\hat{i} - 9\hat{j}$, then find (i) $2\vec{u} + 3\vec{v}$ (ii) $4\vec{u} - 2\vec{v}$

(i) $2\vec{u} + 3\vec{v} = 2(5\hat{i} + \hat{j}) + 3(2\hat{i} - 9\hat{j}) = 10\hat{i} + 2\hat{j} + 6\hat{i} - 27\hat{j} = 16\hat{i} - 25\hat{j}$

(ii) $4\vec{u} - 2\vec{v} = 4(5\hat{i} + \hat{j}) - 2(2\hat{i} - 9\hat{j}) = 20\hat{i} + 4\hat{j} - 4\hat{i} + 18\hat{j} = 16\hat{i} + 22\hat{j}$

Example: Check whether the vectors $\vec{u} = 3\hat{i} + 2\hat{j}$ and $\vec{v} = 6\hat{i} - 4\hat{j}$, are parallel or not.

Solution:

Given that $\vec{u} = 3\hat{i} + 2\hat{j}$ and $\vec{v} = 6\hat{i} - 4\hat{j}$

$\vec{v} = 6\hat{i} - 4\hat{j} = 2(3\hat{i} - 2\hat{j}) \neq 2\vec{u}$.

Therefore $\vec{u}$ and $\vec{v}$ are not parallel.

[illegible]

If $\vec{a} = x\hat{i} + 2\hat{j}$ and $\vec{b} = 6\hat{i} - (x + y)\hat{j}$ are equal vectors, find the values of x and y.

Exercise 7.1

1. Draw the following vectors.

(i) 10 N force along x-axis (ii) 50 m/s velocity at 150° with x-axis

(iii) 220 m displacement towards north (iv) 24 m/s^2 acceleration at 45° with x-axis

2. Draw the vectors on the graph paper when $\vec{p}$ is 4cm long making an angle of 45° with x-axis.

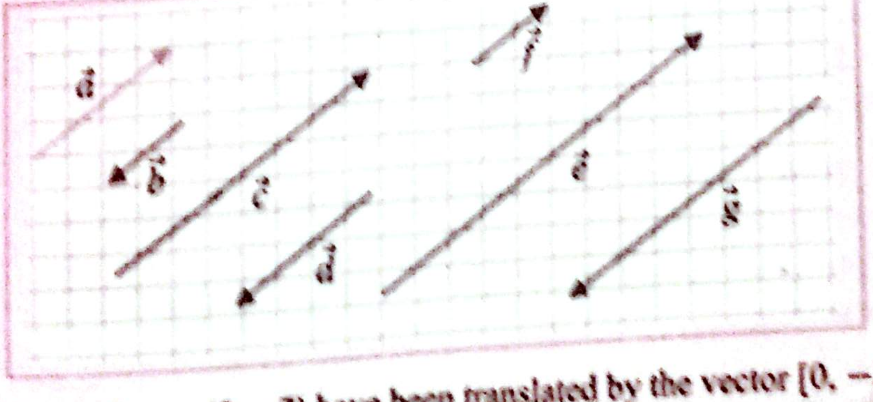
(i) $2\vec{p}$ (ii) $-\vec{p}$ (iii) $0.5\,\vec{p}$ (iv) $-1.5\vec{p}$ (v) $-0.5\vec{p}$

3. Draw vectors $\vec{a}$ = 3 units west and $\vec{b}$ = 3 units north on graph paper. Draw the following vectors on the graph paper.

(i) $2\vec{a} + \vec{b}$ (ii) $\vec{a} - 2\vec{b}$ (iii) $3\vec{a} + 1.5\vec{b}$ (iv) $(2\vec{a} + \vec{b}) + (\vec{a} - 2\vec{b})$

(v) $0.5(\vec{a} + \vec{b})$ (vi) $3\vec{a} - 2\vec{b}$ (vii) $2\vec{a} - 2.5\vec{b}$ (viii) $(2\vec{a} + \vec{b}) - (\vec{a} - 2\vec{b})$

4. In the figure, direction of $\vec{a}$ is 4 square units to the right and 3 square units up. Find the relation of other vectors with $\vec{a}$.



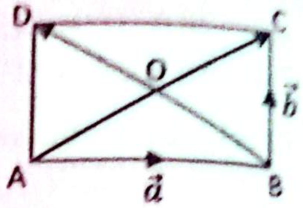
5. A point with coordinates (5, −7) have been translated by the vector [0, −3]. Find new position of the point.

6. A vector [−5, 4] have been translated by another vector [4, −3]. Find new location of the given vector.

7. Triangle ABC is a pre-image with its vertices at A = (−4, 6), B = (−1, 4) and C = (−6, 1).
Find the coordinates of the image if:
(i) the triangle is translated by [5, 0] (ii) the triangle is translated by [−3, −4]

8. What translation by vector would be needed to bring the point E(−6, 5) to the origin?

9. ABCD is a rectangle as shown in the adjoining figure.
Represent the following vectors in terms of $\vec{a}$ and $\vec{b}$.



(i) $\overrightarrow{DC}$ (ii) $\overrightarrow{DA}$ (iii) $\overrightarrow{AC}$ (iv) $\overrightarrow{BD}$
(v) $\overrightarrow{AO}$ (vi) $\overrightarrow{BO}$ (vii) $\overrightarrow{AB}+\overrightarrow{AC}$

10. Given that $\overrightarrow{OP}=\vec{p}+\vec{q}$, $\overrightarrow{OQ}=\vec{p}-\vec{q}$ and M is mid-point of $\overrightarrow{PQ}$. Find in terms of $\vec{p}$ and $\vec{q}$
(i) $\overrightarrow{PQ}$ (ii) $\overrightarrow{PM}$ (iii) $\overrightarrow{QM}$ (iv) $\overrightarrow{OM}$

11. Express $\overrightarrow{AB}$ in the form of $[x, y]$ and $x\hat{\imath}+y\hat{\jmath}$, when:
(i) A (1, 3), B(4, 6) (ii) A (−2, 5), B(2, −5) (iii) A (4, −7), B(−1, −9)

12. Find the unit vectors of the following vectors.
(i) $\overrightarrow{PQ}$ when P(1, 4), Q(4, 8) (ii) $\vec{u}=5\hat{\imath}-12\hat{\jmath}$ (iii) $\vec{b}=-6\hat{\imath}-8\hat{\jmath}$

13. If $\vec{a}=3\hat{\imath}-2\hat{\jmath}$ and $\vec{b}=\hat{\imath}+4\hat{\jmath}$, then find x and y such that $x\vec{a}+y\vec{b}=4\hat{\imath}-12\hat{\jmath}$.

14. Given that $D=(3,-2), E=(-3,-5), F=(0,5)$ and $G=(3,6)$. Find the following vectors.
(i) $\overrightarrow{DE}$ (ii) $\overrightarrow{DE}+\overrightarrow{FG}$ (iii) $3\overrightarrow{EF}-2\overrightarrow{EG}$
(iv) $\frac{3}{2}\overrightarrow{DF}+\frac{1}{2}\overrightarrow{EG}$ (v) $\overrightarrow{DF}-\overrightarrow{EF}+2\overrightarrow{FG}$

15. If $\vec{a}=3\hat{\imath}+5\hat{\jmath}$ and $\vec{b}=2\hat{\imath}-4\hat{\jmath}$, then find:
(i) $\vec{a}+2\vec{b}$ (ii) $2\vec{a}-3\vec{b}$ (iii) $5(2\vec{a}+\vec{b})$ (iv) $0.5(\vec{a}-\vec{b})$
(v) $|\vec{a}+\vec{b}|$ (vi) $|\vec{a}-\vec{b}|$ (vii) $|\vec{a}|+2|\vec{b}|$ (viii) $|\vec{a}|-|\vec{b}|$

16. Find the value of p for which the vector $\vec{u}=3\hat{\imath}+6\hat{\jmath}$ is parallel to $\vec{v}=\hat{\imath}+p\hat{\jmath}$.

17. If $\vec{a}=2\hat{\imath}-4\hat{\jmath}$, $\vec{b}=\hat{\imath}-2\hat{\jmath}$ and $\vec{c}=[1,3]$, then find a unit vector in the direction of:
(i) $\vec{a}+\vec{b}$ (ii) $2\vec{a}-\vec{c}$ (iii) $2\vec{a}-\vec{b}$ (iv) $3(\vec{a}+\vec{c})$
(v) $\vec{a}+\vec{b}+2\vec{c}$ (vi) $4\vec{a}-2\vec{b}+\vec{c}$

18. If $\vec{a}=\hat{\imath}-3\hat{\jmath}$ and $\vec{b}=2\hat{\imath}+4\hat{\jmath}$ then find:
(i) A vector of magnitude 4 in the direction $2\vec{a}-\vec{b}$.
(ii) A vector of magnitude 6 in the opposite direction of $\vec{a}+2\vec{b}$.

19. An airplane leaves the airport on the bearing of 45° traveling at 380 mph. The wind is blowing at a bearing of 135° at a speed of 50 mph. What is the actual velocity of the airplane?

20. A truck leaves dry port on a bearing of 30° and travels 15 km. It then turns due west and travels 8 km. How far is the truck from the dry port and what is its bearing?

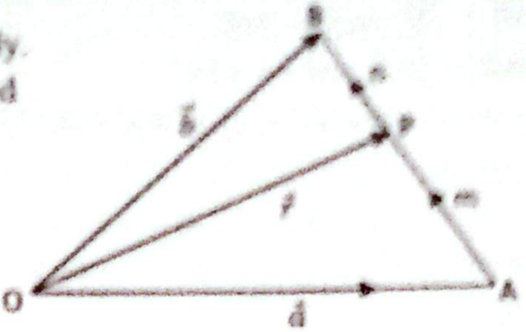
atio Formula for Finding Position Vector of a Point on a Vector

$\overline{AB}$ be any line segment and P be the point which ide this line segment in the given ratio m : n internally. e position vectors of the given points A and B are $\vec{a}$ and spectively. Let $\vec{r}$ be the position vector of point P.

that:

$|\overrightarrow{AP}| : |\overrightarrow{PB}| = m : n$

$\frac{|\overrightarrow{AP}|}{|\overrightarrow{PB}|} = \frac{m}{n} \Rightarrow n|\overrightarrow{AP}| = m|\overrightarrow{PB}|$



ause $\overrightarrow{AP}$ and $\overrightarrow{PB}$ have the same direction. so

$$n\overrightarrow{AP} = m\overrightarrow{PB} \qquad \text{(i)}$$

m figure, $\overrightarrow{OA} + \overrightarrow{AP} = \overrightarrow{OP}$

$\Rightarrow \vec{a} + \overrightarrow{AP} = \vec{r} \quad \Rightarrow \quad \overrightarrow{AP} = \vec{r} - \vec{a}$

ilarly, $\overrightarrow{OP} + \overrightarrow{PB} = \overrightarrow{OB}$

$\Rightarrow \vec{r} + \overrightarrow{PB} = \vec{b} \quad \Rightarrow \quad \overrightarrow{PB} = \vec{b} - \vec{r}$

Key Facts

When the point P divides the line segment $\overline{AB}$ in the ratio $m : n$ externally then:

$$\vec{r} = \frac{n\vec{a} - m\vec{b}}{n - m}$$

stituting the values in equation (i), we get:

$$n(\vec{r} - \vec{a}) = m(\vec{b} - \vec{r}) \Rightarrow n\vec{r} - n\vec{a} = m\vec{b} - m\vec{r}$$

$$\Rightarrow n\vec{r} + m\vec{r} = m\vec{b} + n\vec{a} \Rightarrow (n+m)\vec{r} = m\vec{b} + n\vec{a} \Rightarrow \vec{r} = \frac{m\vec{b} + n\vec{a}}{m + n}$$

ticular Case:

If m = n, then P will be the midpoint of $\overline{AB}$ and position vector of P in this case is:

$$\vec{r} = \frac{n\vec{b} + n\vec{a}}{n + n} = \frac{n(\vec{b} + \vec{a})}{2n} = \frac{\vec{a} + \vec{b}}{2}$$

Example:

The position vectors of points A and B are $2\hat{\imath} - \hat{\jmath}$ and $3\hat{\imath} + 2\hat{\jmath}$ respectively. Find the position vector of point P dividing the line segment joining A and B in the ratio 3 : 4 internally.

Solution:

Let $\vec{a} = 2\hat{\imath} - \hat{\jmath}, \vec{b} = 3\hat{\imath} + 2\hat{\jmath}$ and m : n = 3 : 4

If $\vec{r}$ be the position vector of P, then by using ratio formula, we have:

$$\vec{r} = \frac{m\vec{b} + n\vec{a}}{m + n} = \frac{3(3\hat{\imath} + 2\hat{\jmath}) + 4(2\hat{\imath} - \hat{\jmath})}{3 + 4} = \frac{9\hat{\imath} + 6\hat{\jmath} + 8\hat{\imath} - 4\hat{\jmath}}{7} = \frac{17\hat{\imath} + 2\hat{\jmath}}{7} = \frac{17}{7}\hat{\imath} + \frac{2}{7}\hat{\jmath}$$

Application to Geometry

There are many theorems and geometrical problems that can be solved by using vector methods.

We solve some example here..

Example:

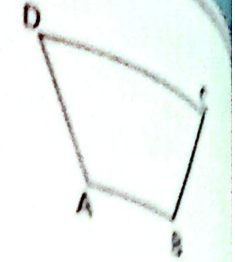
What type of a quadrilateral ABCD is if $2\overrightarrow{AB} = \overrightarrow{DC}$?

Solution:

$2\overrightarrow{AB} = \overrightarrow{DC}$ implies that:

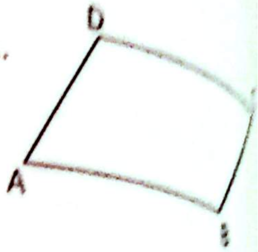
$2|\overrightarrow{AB}| = |\overrightarrow{DC}|$ and $\overrightarrow{AB} \parallel \overrightarrow{DC}$

These two conditions show that ABCD is a trapezium.



Example:

A(x, y), B(−2, 3), C(−3, −4) and D(4, −5) are vertices of a parallelogram taken in order. Use vector method to find vertex A.



Solution:

Position vectors of vertices of parallelogram are:

$\overrightarrow{OA} = x\hat{\imath} + y\hat{\jmath}$, $\overrightarrow{OB} = -2\hat{\imath} + 3\hat{\jmath}$

$\overrightarrow{OC} = -3\hat{\imath} - 4\hat{\jmath}$, $\overrightarrow{OD} = 4\hat{\imath} - 5\hat{\jmath}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (-2\hat{\imath} + 3\hat{\jmath}) - (x\hat{\imath} + y\hat{\jmath}) = (-2 - x)\hat{\imath} + (3 - y)\hat{\jmath}$

$\overrightarrow{DC} = \overrightarrow{OC} - \overrightarrow{OD} = (-3\hat{\imath} - 4\hat{\jmath}) - (4\hat{\imath} - 5\hat{\jmath}) = (-3 - 4)\hat{\imath} + (-4 + 5)\hat{\jmath} = -7\hat{\imath} + \hat{\jmath}$

As, ABCD is a parallelogram, therefore:

$\overrightarrow{AB} = \overrightarrow{DC} \Rightarrow (-2 - x)\hat{\imath} + (3 - y)\hat{\jmath} = -7\hat{\imath} + \hat{\jmath}$

$\Rightarrow -2\hat{\imath} - x\hat{\imath} + 3\hat{\jmath} - y\hat{\jmath} = -7\hat{\imath} + \hat{\jmath} \quad \Rightarrow -2\hat{\imath} + 7\hat{\imath} + 3\hat{\jmath} - \hat{\jmath} = x\hat{\imath} + y\hat{\jmath}$

$\Rightarrow 5\hat{\imath} + 2\hat{\jmath} = x\hat{\imath} + y\hat{\jmath} \quad \Rightarrow x = 5, y = 2$

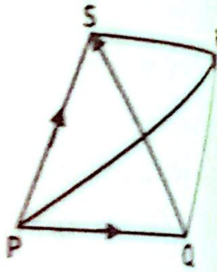
∴ Fourth vertex of parallelogram is A(5, 2).

Example:

Use vectors to prove that the diagonals of a rhombus bisect each other.

Solution:

Consider any rhombus PQRS. Let $\vec{p}, \vec{q}, \vec{r}$ and $\vec{s}$ be the position vectors of the vertices P, Q, R and S respectively.



Let M and N be the mid points of diagonal vectors $\overrightarrow{PR}$ and $\overrightarrow{QS}$ respectively.

p.v. of M $= \frac{\vec{p} + \vec{r}}{2}$ (i) p.v. of N $= \frac{\vec{q} + \vec{s}}{2}$ (ii)

Since PQRS is a rhombus, therefore:

$$\overrightarrow{PQ} = \overrightarrow{SR}$$

$$\Rightarrow \vec{q} - \vec{p} = \vec{r} - \vec{s}$$

$$\Rightarrow \vec{q} + \vec{s} = \vec{p} + \vec{r} \qquad \text{(iii)}$$

Using equation (iii) in (i) and (ii), we get:

p.v. of M = p.v. of N

the position vectors of the mid-points of both the diagonals are same.

efore, the diagonals of a rhombus bisect each other.

Check Poi

Prove that the diagon
of parallelogram, squa
and rectangle bisect ea
other.

Exercise 7.2

What type of a quadrilateral EFGH is if $\overline{EF} = \overline{HG}$ and $\overline{EH} = \overline{FG}$?

Use vectors to find the length of diagonals of a parallelogram having adjacent sides $2i + j$ and $i - 3j$.

(i) Show that $A(0, 2), B(\sqrt{3}, -1), C(0, -2)$ are the vertices of a right-angled triangle.

(ii) Show that $A(3, 1), B(-2, -3), C(2, 2)$ are the vertices of an isosceles triangle.

The position vectors of points C and D are $3i + 2j$ and $i - 4j$ respectively. Find the position vector of point O dividing the line segment joining C and D in the ratio 2 : 3 internally.

If $M(2, 4)$ is mid-point of $\overline{XY}$ such that $X(-1, 3)$ and $Y(x, y)$. Find the point Y.

If $\overline{OA} = -i + 3j$, $\overline{OB} = 6i + 2j$, find the point trisecting the join of $\overline{AB}$.

Find x such that the points $P(-1, x)$, $Q(3, 2)$ and $R(7, 3)$ are collinear.

Find the coordinates of vertex D of a parallelogram ABCD if $A(-3, 0)$, $B(1, -2)$ and $C(5, 0)$ are its three vertices.

If $\overline{AB} = \overline{CD}$ and $A(0, 2)$, $C(-2, 4)$, $D(-1, 5)$, then find vertex B.

Vertices of a quadrilateral are $U(9, 4)$, $V(-7, 7)$, $W(-4, -7)$ and $X(5, -5)$. Find mid points of its sides and prove that the figure obtained by joining mid points consecutively is a parallelogram.

Prove that the line segments joining the midpoints of two sides of a triangle are half in length of third side.

Prove that the joining the midpoints of the two non-parallel sides of a trapezium is parallel to its parallel sides.

I have Learnt

- Introduction of rectangular coordinate system in a plane.
- Representing vector as a directed line segment.
- Expressing a vector in terms of two non-zero and non-parallel coplanar vector.
- Expressing a vector in terms of position vector.
- Expressing translation by a vector and finding the magnitude of a vector.
- Adding and subtracting vectors, and multiplying a vector by a scalar.
- Solving geometrical problems involving the use of a vector.
- Applying concepts of vectors in geometrical problems such as parallel and perpendicular lines in geometrical shapes, vector projectile motion, crosswinds aviation, military usage, designing roller coasters.

MISCELLANEOUS EXERCISE-7

1. Chose the correct option.

i. Which of the following is a scalar quantity?
a. velocity b. speed c. torque d. force

ii. Which of the following is a vector quantity?
a. velocity b. speed c. distance d. work

iii. If $\vec{a}$ and $\vec{b}$ are position vectors of points A and B respectively, then $\overrightarrow{AB}$ is:
a. $\vec{b}+\vec{a}$ b. $\vec{a}-\vec{b}$ c. $\vec{b}-\vec{a}$ d. $-\vec{b}-\vec{a}$

iv. Which of the following is not a symbol of vector a?
a. $\vec{a}$ b. $\underline{a}$ c. $\boldsymbol{a}$ d. $|\vec{a}|$

v. If $\overrightarrow{OP}=[-6, 7]$, then $-\overrightarrow{OP}$ is equal to:
a. $[-6, 7]$ b. $[6, 7]$ c. $[6, -7]$ d. $[-6, -7]$

vi. If $\vec{u}=-5\hat{i}+12\hat{j}$, then $|\vec{u}|$ is equal to:
a. 17 b. 7 c. 169 d. 13

vii. Given that $\vec{u}$ is any vector. Which of the following is true?
a. $|-\vec{u}|=|\vec{u}|$ b. $-|\vec{u}|=|\vec{u}|$ c. $|\vec{u}|+|-\vec{u}|=0$ d. $|\vec{u}| =$

viii. The unit vector of the vector $\vec{u}=6\hat{i}+10\hat{j}-2\hat{j}$ is:
a. $\frac{3\hat{i}}{5}-\frac{4\hat{j}}{5}$ b. $\frac{3\hat{i}}{5}+\frac{4\hat{j}}{5}$ c. $-\frac{3\hat{i}}{5}-\frac{4\hat{j}}{5}$ d. $-\frac{3\hat{i}}{5}+\frac{4\hat{j}}{5}$

ix. If $\vec{a}=\lambda\vec{b}$, $\vec{a}=12\hat{i}-18\hat{j}$ and $\vec{b}=-2\hat{i}+3\hat{j}$, then λ is equal to:
a. 3 b. -3 c. 6 d. -6

x. If $\vec{u}=[-5x, 8]$ and $\vec{v}=[10, 4y]$ are equal vectors, then:
a. $x=2, y=2$ b. $x=-2, y=-2$ c. $x=-2, y=2$ d. $x=2, y=$

xi. If $\vec{p}=[5, -6]$ and $\vec{q}=[2, 6]$, then $\vec{p}-2\vec{q}$ is:
a. $[1, -18]$ b. $[9, 18]$ c. $[1, 18]$ d. $[-9, -18]$

xii. If $\vec{u}=5\hat{i}+10\hat{j}$ and $\vec{v}=4\hat{j}$, then $|\vec{u}-\vec{v}|$ is:
a. $\sqrt{123}$ b. $\sqrt{61}$ c. $\sqrt{11}$ d. $-\sqrt{61}$

xiii. Which of the following vectors represents a position vector?
a. $\overrightarrow{OP}$ b. $-\overrightarrow{OP}$ c. $\overrightarrow{PO}$ d. $\overrightarrow{PQ}$

xiv. What type of a quadrilateral ABCD is, if $\overrightarrow{AB}=\frac{2}{3}\overrightarrow{DC}$?
a. kite b. rectangle c. trapezium d. rhombus

2. Given that $\vec{p}=3\hat{i}-4\hat{j}$ and $\vec{q}=-3\hat{i}-4\hat{j}$. Prove that $|\vec{p}|=|\vec{q}|$. Is $\vec{p}=\vec{q}$?

3. If $\vec{a}=2\hat{i}-4\hat{j}$ and $\vec{b}=-2\hat{i}+x\hat{j}$, then find the value of x if $|\vec{a}+2\vec{b}|=6$.

4. In ΔOAB, $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$ and M is mid point of $\overrightarrow{OA}$.
Write the following vectors in terms of $\vec{a}$ and $\vec{b}$.
(i) $\overrightarrow{OM}$ (ii) $\overrightarrow{AM}$ (iii) $\overrightarrow{BM}$ (iv) $\overrightarrow{OB} + \overrightarrow{BA}$

5. Two tractors are pulling a truck stuck in the mud. First tractor is pulling with a force of 250 N at an angle of 50° with the horizontal while the second tractor is pulling with a force of 300 N at an angle of 40° with the horizontal. What is the magnitude and direction of the resultant force?

6. Ammar and Javed are playing with a ball. Ammar throws two balls for Javed to catch. The path of the two balls is described by the vectors $\mathbf{V_1} = 120\hat{i} + 12\hat{j}$ and $\mathbf{V_2} = 90\hat{i} - 30\hat{j}$ where the distances are expressed in meters.

(i) How much farther did the first ball travel as compared with the second ball?
(ii) What is the distance between the two balls thrown?

7. Ahmad can swim in still water with a speed of 6 m/s. He goes swimming in a river Chenab which has a current flowing towards west with a speed of 1.5 m/s. Assuming his speed is the same, what is his resultant velocity if:

(i) He tries to swim due west along the current?
(ii) He tries to swim due east against the current?
(iii) He tries to swim north across the river?

8. A plane is travelling north with a speed of 150 km/h. A steady wind is blowing due east with a speed of 50 km/h.

(i) What is the resultant speed of the plane?
(ii) How far is the plane from its starting point after 10 hours?

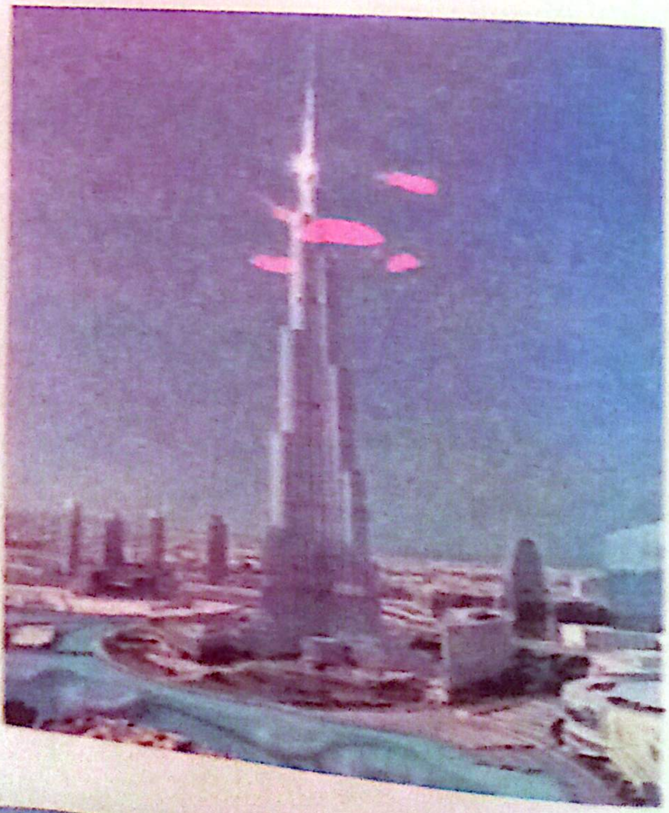
08

Application of Trigonometry

After studying this unit students will be able to:

- Extend sine and cosine functions to angles between 90° and 180°.
- Solve problems using laws of sines, cosines and the area formula for any triangle.
- Solve simple trigonometric problems in three dimensions.
- Apply concepts of trigonometry to real life world problems such as video games, flight engineering, navigation and sound waves.

Burj Khalifa at Dubai is the tallest building in the world with tallest free-standing structure highest number of stories. It is 828 metres (2,716.5 feet) high and consists of more tha stories. How do scientists know its true height? A common way to measure the height invo determining the angle of elevation, which is formed by the top of building and the groun point some distance away from the base of the building. This method is much more prac than climbing the building and dropping a very long tape measure or rope.



Trigonometric Ratios

Quadrantal Angles

In a coordinate plane the two axes divide the plane in four equivalent parts called quadrants.

If terminal ray of an angle in standard position coincides with any axis, then it is called quadrantal angle. The measure of a quadrantal angle is multiple of 90°.

For example; 0°, 90°, 180°, 270°, 360°, 450°, ... are quadrantal angles.

In radian measure quadrantal angle is a multiple of $\frac{\pi}{2}$.

For example; $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi, \frac{5\pi}{2}, 3\pi, \ldots$ are quadrantal angles.

[illegible]

Two angles having same terminal ray in standard position, are called coterminal angles.

Trigonometric Ratios

The ratio between any two sides of a right-angled triangle is called trigonometric ratio.

In triangle ABC, ∠C = 90° and ∠A = θ.

With respect to acute angle θ, BC = a is length of perpendicular, AC = b is length of base and AB = c is length of hypotenuse.

Various trigonometric ratios are defined as:

- The ratio of perpendicular and hypotenuse is sine of θ denoted by sin θ.

  $\therefore \sin\theta = \frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{a}{c}$

- The ratio of base and hypotenuse is cosine of θ denoted by cos θ.

  $\therefore \cos\theta = \frac{\text{Base}}{\text{Hypotenuse}} = \frac{b}{c}$

- The ratio of perpendicular and base is tangent of θ denoted by tan θ.

  $\therefore \tan\theta = \frac{\text{Perpendicular}}{\text{Base}} = \frac{a}{b}$

- Furthermore, $\operatorname{cosec}\theta = \frac{\text{Hypotenuse}}{\text{Perpendicular}} = \frac{c}{a} = \frac{1}{\sin\theta}$, $\sec\theta = \frac{\text{Hypotenuse}}{\text{Base}} = \frac{c}{b} = \frac{1}{\cos\theta}$

  and $\cot\theta = \frac{\text{Base}}{\text{Perpendicular}} = \frac{b}{a} = \frac{1}{\tan\theta}$

Trigonometric Ratios with the help of Unit Circle

A circle whose radius is 1 unit is called a unit circle.

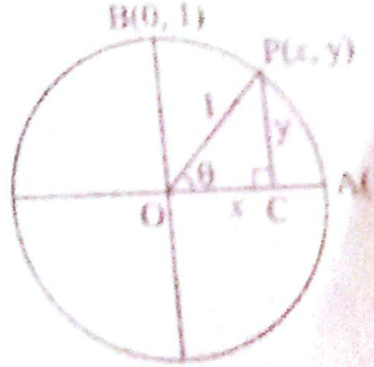
Consider a unit circle with centre O.

Take a point P(x, y) on the circle. Join P with O.

Draw PC perpendicular to x-axis.

Let OC = x, PC = y and OP = 1 (unit circle).

If ∠COP = θ where 0 < θ < 180°, then:



$\cos\theta = \frac{x}{1}$ and $\sin\theta = \frac{y}{1}$ ...... (i)

Other trigonometric ratios can be defined as:

$\tan\theta = \frac{y}{x}$, $\cot\theta = \frac{x}{y}$, $\sec\theta = \frac{1}{x}$ and $\operatorname{cosec}\theta = \frac{1}{y}$

Now from (i), we have:

$x = \cos\theta$ and $y = \sin\theta$

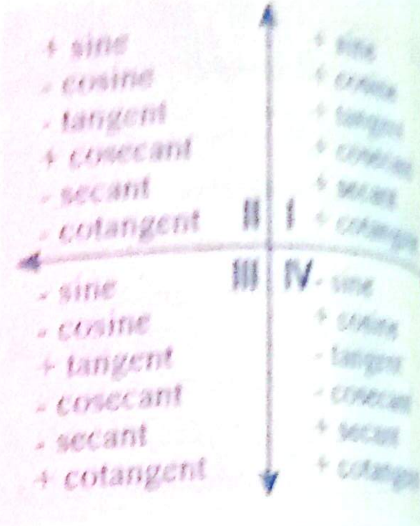
So, $P(x, y) = P(\cos\theta, \sin\theta)$

∴ If P is a point on a unit circle with centre O and $\angle COP = \theta$, then coordinates of P are

$(\cos\theta, \sin\theta)$

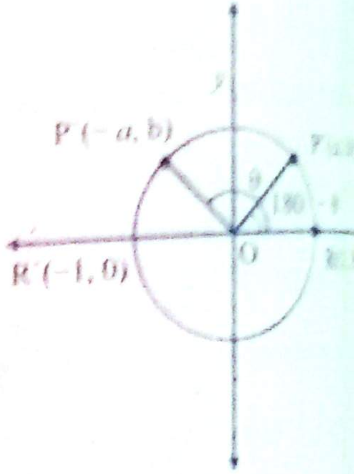
Signs of Trigonometric Ratios in Different Quadrants

As $\cos\theta = x$ and $\sin\theta = y$ in the unit circle. Therefore, depending upon the signs of x and y in different quadrants, the signs of trigonometric ratios change. The adjoining figure shows signs of trigonometric ratios in different quadrants.



Trigonometric Ratios between 90° and 180°

We have discussed finding the sine and cosine for angles in the first quadrant, but what angle is in second quadrant $(90° < \theta < 180°)$? In the diagram, $R(1, 0)$ and $P(a, b)$ are points on the unit circle. Under a reflection in the y-axis, the image of $P(a, b)$ is $P'(-a, b)$ and the image of $R(1, 0)$ is $R'(-1, 0)$. Since angle measure is preserved under a line reflection,



∴ $\angle ROP = \angle R'OP'$

Also $\angle ROP'$ and $\angle R'OP'$ are supplementary angles.

∴ $\angle ROP' + \angle R'OP' = 180°$ (i)

But, $\angle R'OP' = \angle ROP$

Therefore, (i) becomes:

$\angle ROP' + \angle ROP = 180°$

$\angle ROP = 180° - \angle ROP' = 180° - \theta$

Case 1: If $\angle ROP' = \theta$, then:

$\cos\theta = -a$ and $\sin\theta = b$ .... (ii) (using the properties of unit circle)

Case 2: If $\angle ROP = 180° - \theta$, then:

$\cos(180° - \theta) = a$ and $\sin(180° - \theta) = b$ .... (iii)

Comparing (ii) and (iii), we get:

$\cos(180° - \theta) = -\cos\theta$ and $\sin(180° - \theta) = \sin\theta$

As, $\frac{\sin(180° - \theta)}{\cos(180° - \theta)} = \tan(180° - \theta)$ and $\frac{\sin\theta}{-\cos\theta} = -\tan\theta$

∴ $\tan(180° - \theta) = -\tan\theta$

To find sine and cosine of angle lying between 90° and 180°, proceed as follows:

a. Measure the angle between the terminal side of the given angle and the horizontal axis. This is the reference angle.

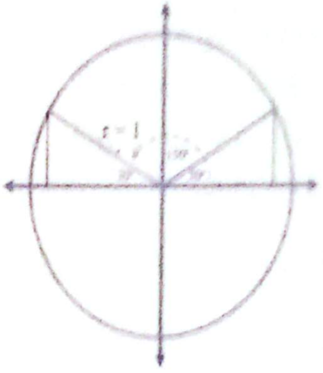
b. Determine the values of the cosine and sine of the reference angle.

c. Give the cosine the same sign as the x-values in the quadrant of the original angle.

d. Give the sine the same sign as the y-values in the quadrant of the original angle.

Example:

Find the exact values of sin 150°, cos 150°, and tan 150°.



Solution:

Using above relations, we have:

$$\sin 150^\circ = \sin(180^\circ - 150^\circ) = \sin 30^\circ = \frac{1}{2}$$

$$\cos 150^\circ = -\cos(180^\circ - 150^\circ) = -\cos 30^\circ = -\frac{\sqrt{3}}{2}$$

$$\tan 150^\circ = -\tan(180^\circ - 150^\circ) = -\tan 30^\circ = -\frac{1}{\sqrt{3}}$$

Key Facts:

- Angles have cosines and sines with the same absolute value as their reference angles.
- The sign (positive or negative) can be determined from the quadrant of the angle.

Exercise 8.1

1. Find the reference angles of the following angles.

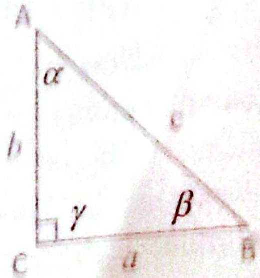
   (i) 125° (ii) 138° (iii) 111° (iv) 142°

2. If $\theta = 36^\circ$. Find equivalent angle α in the second quadrant for which $\sin\alpha = \sin\theta$.
3. If $\theta = 87^\circ$. Find equivalent angle β in the second quadrant for which $\cos\beta = -\cos\theta$.
4. Using a reference angle, find the exact values of sin 120°, cos 120° and tan 120°.
5. Using the reference angle, find $\cos\frac{3\pi}{4}$, $\sin\frac{3\pi}{4}$ and $\tan\frac{3\pi}{4}$.
6. If $\cos\theta = 0.559$, find the values of:

   (i) $\cos(180^\circ - \theta)$ (ii) $\sin(180^\circ - \theta)$ (iii) $\tan(180^\circ - \theta)$

   (iv) $\cot(180^\circ - \theta)$ (v) $\sec(180^\circ - \theta)$ (vi) $\mathrm{cosec}(180^\circ - \theta)$

Solution of Right Angled Triangles



We have studied about the solution of right triangles in the previous grade. We know that any triangle (in particular a right triangle) has six quantities, three sides and three angles. In the adjoining figure, a right triangle ABC contains three sides a, b, c and three angles α, β, γ. We can solve a right-angled triangle when its:

(i) Two sides are given,

(ii) One acute angle and one side are given.

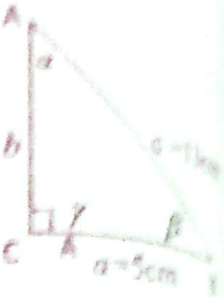
Let's solve some examples to discuss the two cases.

Case 1: When Measures of two Sides are Given

Example: Solve the right triangle ABC in which $\gamma = 90°, a = 5cm, c = 13cm$.

Solution: Given: A right triangle ABC in which $\gamma = 90°, a = 5cm, c = 13cm$

To Find: b, α and β.



Using Pythagoras theorem, we have:

$$a^2 + b^2 = c^2$$

Substituting values of a and c, we get:

$$5^2 + b^2 = 13^2 \Rightarrow 5^2 + b^2 = 13^2$$

$$\Rightarrow b^2 = 13^2 - 5^2 = 169 - 25 = 144$$

$$\Rightarrow b = 12cm$$

$$\text{Now } \cos\beta = \frac{5}{13} = 0.385$$

$$\Rightarrow \beta = \cos^{-1}(0.385) = 67.4°$$

$$\text{As } \alpha + \beta = 90°$$

$$\Rightarrow \alpha = 90° - \beta = 90° - 67.4° = 22.6°$$

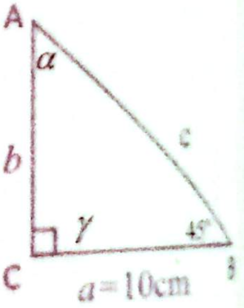
Case 2: When Measures of one Side and one Angle are Given

Example: Solve the right triangle ABC in which $\gamma = 90°$, $\beta = 45°$, $a = 10cm$.

Solution: Given: A right triangle ABC in which $\gamma = 90°$, $\beta = 45°$, $a = 10cm$.

To Find: b, c and α.

As $\alpha + \beta = 90°$



$$\text{As } \alpha + \beta = 90°$$

$$\Rightarrow \alpha = 90° - \beta = 90° - 45° = 45°$$

$$\text{Now } \frac{b}{a} = \tan 45°$$

$$\Rightarrow b = a \times \tan 45° = 10 \times 1 = 10cm$$

To find c, we use Pythagoras theorem as follows:

$$c^2 = a^2 + b^2$$

Substituting values of a and b, we get:

$$c^2 = 10^2 + 10^2 = 100 + 100 = 200$$

$$\Rightarrow c = 10\sqrt{2}cm$$

Example: From a small plane Z the angle of depression of a boat is 25°. The angle of depres of a boat on the other side of the plane is 50°. The plane is flying at an altitude of 1500 m. far apart are the boats?

Solution: In the figure,

distance between the boats = XY = XU + UY.

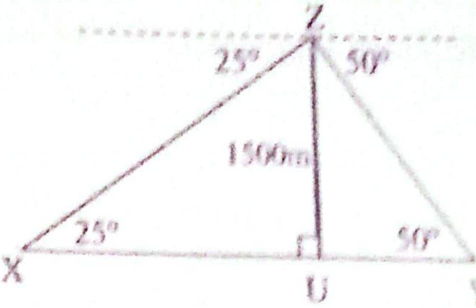
Using the properties of angles,

$\angle X = 25^\circ$ and $\angle Y = 50^\circ$

In right ΔXUZ,

$$\tan 25^\circ = \frac{ZU}{XU}$$

$$XU = \frac{ZU}{\tan 25^\circ} = \frac{1500}{0.466} = 3216.76m$$



In right ΔYUZ,

$$\tan 50^\circ = \frac{ZU}{YU}$$

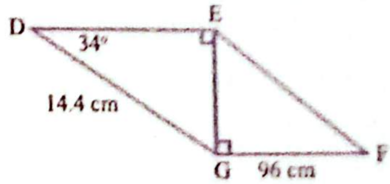
$$YU = \frac{ZU}{\tan 50^\circ} = \frac{1500}{1.192} = 1258.65m$$

$\therefore$ Distance between the boats = XY = XU + UY

= 3216.76 + 1258.65 = 4475.41 m

Example:

Find the measure of ∠F in the adjoining figure.



Solution:

To find ∠F, we need measure of $\overline{GE}$.

In right ΔDEG,

$$\sin 34^\circ = \frac{GE}{GD}$$

$\overline{GE} = GD \times \sin 34^\circ = 14.4 \times 0.559 = 8.05cm$

Now in right ΔEGF,

$$\sin(\angle F) = \frac{GE}{GD} = \frac{8.05}{9.6} = 0.839$$

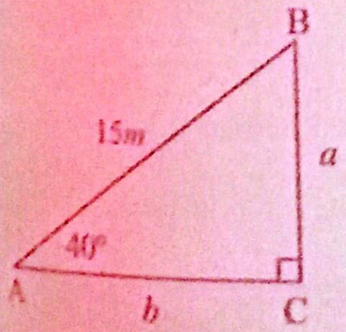
$$\angle F = \sin^{-1}(0.839) = 57^\circ 2'$$

Check Point:

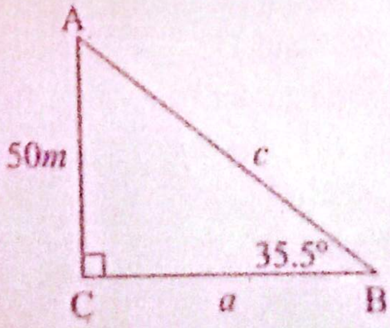
Abbas is flying a kite to which the angle of elevation is 70°. The string of the kite is 65 meters long. How far is the kite above the ground?

Exercise 8.2

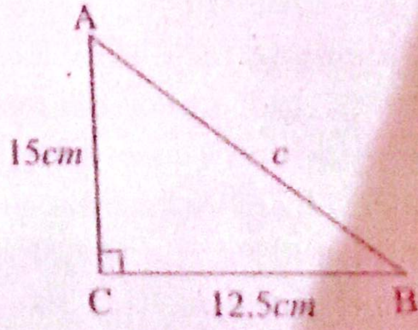
1. Solve the following right-angled triangles.



(i)



(ii)



(iii)

2. Solve right angled triangles ABC in which $\gamma = 90°$ and
(i) $a = 12cm, \beta = 35°$ (ii) $b = 30cm, \alpha = 25° 15'$ (iii) $a = 50cm, b = 25cm$
(iv) $a = 30cm, c = 40cm$ (v) $a = 24cm, \alpha = 36° 15'$ (vi) $b = 10cm, \alpha = 70$

3. A tower casts a shadow that is 20 *m* long when the angle of elevation of the sun is 6 How tall is the tower?

4. Arif is standing on top of a cliff 105 *m* above a lake. The measurement of the angle of depression to a boat on the lake is 42°. How far is the boat from Arif?

5. A ladder that is 20 ft. long is leaning against the side of a building. If the angle formed between the ladder and the ground is 75°, how far is the bottom of the ladder from the base of the building?

6. Uzma is standing 50 meters from a hot air balloon that is preparing to take off. The angle of elevation to the top of the balloon is 28°. Find the height of the balloon.

7. A man is in a boat that is floating 175 feet from the base of a 200-foot cliff. What is the angle of depression between the cliff and the boat?

8. A flagpole casts a shadow 40 feet long when the measurement of the angle of elevation of the sun is 31°. How tall is the flagpole?

9. A straight waterslide is 175 feet above ground and is 200 feet long. What is the angle of depression to the bottom of the slide?

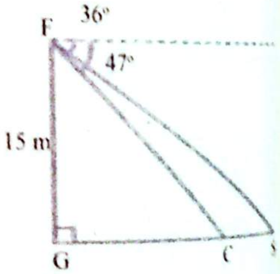
10. Zain wants to measure the height of a tree. He walks exactly 50 *m* from the base of the tree and looks up that the angle from the ground to the top of the tree is 33°. How tall is the tree?

11. From a 100*m* observation tower on the beach, a man sights a whale in difficulty. The angle of depression of the whale is 7°. How far is the whale from the shoreline?

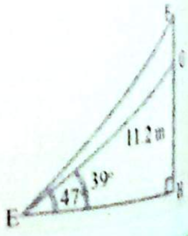
12. Urooba is sitting on the ground midway between two trees, 100 *m* apart. The angles of elevation of the tops of the trees are 13° and 18°. How much taller is one tree than the other?

13. The angle of elevation of the top of a tree, T is 27°. From the same point on the ground, the angle of elevation of a hawk H, flying directly above the tree is 43°. The tree is 12.7m tall. How high is the hawk above the ground?

14. This diagram shows a falcon F, on a tree, with a squirrel S, and a chipmunk C, on the ground. From the falcon, the angles of depression of the animals are 36° and 47°. How far apart are the animals on the ground?



15. Two guy wires support a flagpole FH. The first wire is 11.2 m long and has an angle of elevation of 39°. The second wire has an angle of elevation of 47°. How tall is the flagpole?



Oblique Triangles

An oblique triangle is any triangle that is not a right triangle. It could be an acute triangle or it could be an obtuse triangle.

Acute Triangle

Obtuse Triangle

The Pythagoras Theorem along with trigonometric ratios allow us to easily handle any given right triangle problem, but what if the triangle is not a right triangle i.e. the triangle is oblique?

To solve oblique triangles, we use certain laws known as "law of cosines", "law of sines" and "law of tangents".

Law of Cosines

When we know the measures of two sides and the included angle of a triangle (SAS), the size and shape of the triangle are determined. Therefore, we should be able to find the measure of the third side of the triangle. Sometimes three sides of oblique triangles are given (SSS) and we have to find the three angles of triangle. Such cases are easily handled by using law of cosines.

Statement:

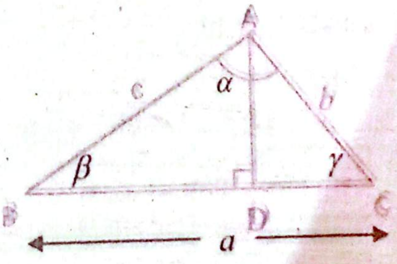
If a, b, c are three sides and α, β, γ are three angles of an oblique triangle with usual notation, then law of cosines is stated as follows:

$$a^2 = b^2 + c^2 - 2bc\cos\alpha \text{ .... } (1)$$

$$b^2 = a^2 + c^2 - 2ac\cos\beta \text{ .... } (2)$$

$$c^2 = a^2 + b^2 - 2ab\cos\gamma \text{ .... } (3)$$

Proof: Consider an acute angled triangle ABC with usual notations as shown in the figure.



Draw $\overline{AD} \perp \overline{BC}$.

In right ΔABD,

$$\frac{BD}{AB} = \cos\beta$$

$$\Rightarrow BD = AB\cos\beta = c\cos\beta \text{.....}(i)$$

Again, $$\frac{AD}{AB} = \sin\beta$$

$$\Rightarrow AD = AB\sin\beta = c\sin\beta \text{.....}(ii)$$

Now, $CD = BC - BD = a - c\cos\beta$

In right ΔACD,

$(AC)^2 = (AD)^2 + (CD)^2$ .....(Pythagoras Theorem)

$$b^2 = (c\sin\beta)^2 + (a - c\cos\beta)^2$$
$$b^2 = c^2\sin^2\beta + a^2 + c^2\cos^2\beta - 2ac\cos\beta$$
$$b^2 = a^2 + c^2(\sin^2\beta + \cos^2\beta) - 2ac\cos\beta$$
$$b^2 = a^2 + c^2 - 2ac\cos\beta$$

Key Fact:

Law of cosines is applicable when:

- Two sides and included angle are given.
- Three sides are given.

Similarly, we can prove that:

$a^2 = b^2 + c^2 - 2bc\cos\alpha$ and $c^2 = a^2 + b^2 - 2ab\cos\gamma$

From laws of cosines,

$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}, \ \cos\beta = \frac{a^2 + c^2 - b^2}{2ac}, \ \cos\gamma = \frac{a^2 + b^2 - c^2}{2ab}$$

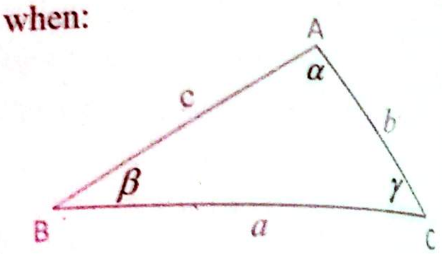
These laws are useful when three sides of oblique triangles are given and we have to find their angles.

Check Point:

Prove law of cosines by taking an obtuse triangle.

Example: Use the law of cosines to solve the triangle ABC when:

$a = 12cm, \ b = 7cm, \ \gamma = 59°30'$



Solution: Using law of cosines, we have:

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$
$$c^2 = 12^2 + 7^2 - 2\times12\times7\times\cos59°30'$$
$$c^2 = 144 + 49 - 168\times0.506 = 108$$
$$c = \sqrt{108} = 10.4cm$$

Now to find angle α, we use following formula.

$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc} = \frac{7^2 + 10.4^2 - 12^2}{2\times7\times10.4}$$
$$= \frac{13.16}{145.6} = 0.0904$$

$\alpha = \cos^{-1}(0.0904) = 84°49'$

Finally, $\alpha + \beta + \gamma = 180°$

$\beta = 180° - \alpha - \gamma \ = 180° - 84°49' - 59°30'$

$\beta = 35°41'$

Key Fact:

If in the law of cosines,

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

angle α is 90°, then

$$a^2 = b^2 + c^2 - 2bc\cos90°$$
$$a^2 = b^2 + c^2 - 2bc\times0 \Rightarrow a^2 = b^2 + c^2$$

which becomes Pythagoras Theorem. Thus Pythagoras Theorem is a special case of "law of cosines".

Check Point:

Two forces of magnitude 20N and 30N are inclined at angle of 105° with each other. Find the magnitude of resultant force. Can you find the angle between force of magnitude 20N and resultant force?

Law of Sines

...of cosines is not applicable when two angles and one side (AAS or ASA) or one angle ...sides (angle not included (SSA)) of an oblique triangle are given. In order to handle ...situations, we use another law known as "law of sines".

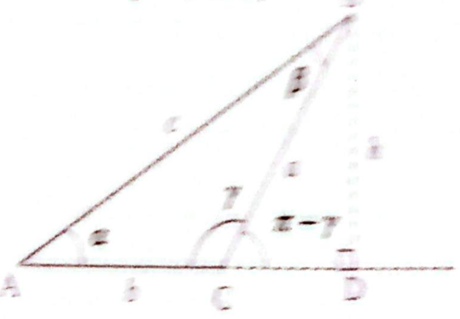
...ement:

...are three sides and α, β, γ are three angles of an oblique triangle with usual notation, ...law of sines is stated as follows:

$$\frac{a}{\sin\alpha}=\frac{b}{\sin\beta}=\frac{c}{\sin\gamma}$$

...of: Consider an obtuse triangle ABC as shown in the figure below.

...h be the height of triangle ABC with respect to base $\overline{AC}$. In right ΔABD,



$\frac{BD}{AB}=\sin\alpha$

$\Rightarrow BD=AB\sin\alpha=c\sin\alpha$

$\Rightarrow h=c\sin\alpha$.....(i)

In right ΔCBD,

$\frac{BD}{CB}=\sin(\pi-\gamma)$

$\Rightarrow BD=CB\sin(\pi-\gamma)$ $\quad$ [as $\sin(\pi-\gamma)=\sin\gamma$]

$\Rightarrow h=a\sin\gamma$.....(ii)

Check Point:

Prove law of sines by taking an acute triangle.

From (i) and (ii), we get:

$a\sin\gamma=c\sin\alpha$

Or $\frac{a}{\sin\alpha}=\frac{c}{\sin\gamma}$

Similarly, by drawing perpendiculars on other two sides of ΔABC, we get:

$\frac{a}{\sin\alpha}=\frac{b}{\sin\beta}$ and $\frac{c}{\sin\gamma}=\frac{b}{\sin\beta}$

Combining above results, we obtain the following law known as "law of sines".

$\frac{a}{\sin\alpha}=\frac{b}{\sin\beta}=\frac{c}{\sin\gamma}$ or $\frac{\sin\alpha}{a}=\frac{\sin\beta}{b}=\frac{\sin\gamma}{c}$

To Determine the Number of Solutions in ΔABC

Task: Given measure of sides a, b and angle α. Use the law of sines to solve the ΔABC for $\sin\beta$.

Case-1:

(i) When $\sin\beta>1$, there is no triangle, so no solution.

(iii) When $\sin\beta = 1$, there is one right triangle (one solution) if α is acute and no triangle (no solution) if α is obtuse.

Case-2:

When α is acute and $\sin\beta < 1$, there are two possible values of β, acute or obtuse

$0 < \beta < 90^\circ$ and $\beta' = 180^\circ - \beta$

(i) If $\alpha + \beta' < 180^\circ$, there are two possible triangles ΔABC and $\Delta AB'C$.

(ii) If $\alpha + \beta' \ge 180^\circ$, there is only one ΔABC.

Case-3:

When α is obtuse and $\sin\beta < 1$, then β must be acute.

(i) If $\alpha + \beta < 180^\circ$, there is only one ΔABC.

(ii) If $\alpha + \beta' \ge 180^\circ$, there is no triangle, so no solution.

Alternatively, if we let $h = b\sin\alpha$, the height of triangle, we can summarize the number of possible triangles given a, b and α in ΔABC.

Angle is α	Acute $a < h$	Acute $a = h$	Acute $h < a < b$	Acute $a > b$	Obtuse $a < h$	Obtuse $a > b$
Possible Triangles	none	one right triangle	two triangles	one triangle	none	one triangle

Example:
Use the law of sines to solve the triangle ABC when:
$a = 25cm,\ \alpha = 66^\circ 51',\ \gamma = 44^\circ 12'$

Solution:
Given: $a = 25cm,\ \alpha = 66^\circ 51',\ \gamma = 44^\circ 12'$

To Find: b, c and β

$$\alpha + \beta + \gamma = 180^\circ$$

$$66^\circ 51' + \beta + 44^\circ 12' = 180^\circ$$

$$\beta = 180^\circ - 66^\circ 51' - 44^\circ 12' = 68^\circ 57'$$

Using law of sines, we have:

$$\frac{a}{\sin\alpha} = \frac{c}{\sin\gamma} \Rightarrow \frac{25}{\sin 66^\circ 51'} = \frac{c}{\sin 44^\circ 12'}$$

$$c = \frac{25 \times \sin 44^\circ 12'}{\sin 66^\circ 51'} = 18.96cm$$

Again, by law of sines,

Solve the triangle ABC by using both law of sines and law of cosines when:
$a = 25cm, \alpha = 66^\circ 51',\ \gamma = 44^\circ 12'$

$$\frac{b}{\sin\beta}=\frac{a}{\sin\alpha} \Rightarrow \frac{b}{\sin 68°57'}=\frac{25}{\sin 66°51'}$$

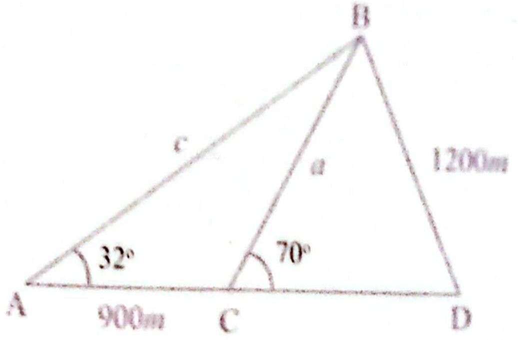
$$\Rightarrow b=\frac{25}{\sin 66°51'}\times\sin 68°57' = 25.37\,cm$$

Example:
Find the values of unknowns in the adjoining figure. Figure is not drawn according to scale.

Solution:
In ΔABC,

$$\angle ACB = 180° - 70° = 110°$$

$$\text{and } \angle ABC = 180° - 32° - 110° = 38°$$



Using law of sines for triangle ABC,

$$\frac{a}{\sin 32°}=\frac{900}{\sin 38°}$$

$$\Rightarrow a=\frac{900}{\sin 38°}\times\sin 32° = 774.66m$$

Again for the ΔABC,

$$\frac{c}{\sin 110°}=\frac{774.66}{\sin 32°}$$

$$c=\frac{774.66}{\sin 32°}\times\sin 110° = 1373.69m$$

Now in ΔBCD,

$$\frac{a}{\sin\angle D}=\frac{1200}{\sin 70°} \Rightarrow \frac{774.66}{\sin\angle D}=\frac{1200}{\sin 70°}$$

$$\sin\angle D=\frac{774.66\times\sin 70°}{1200}=0.607$$

$$\angle D=\sin^{-1}(0.607)=37°21'$$

Law of Tangents

Law of tangents is also used to solve oblique triangles. This law handles the same situations for which the law of cosines is used.

Statement:

If a, b, c are three sides and α,β,γ are three angles of an oblique triangle with usual notation, then law of tangents is stated as follows:

$$\frac{a+b}{a-b}=\frac{\tan\left(\frac{\alpha+\beta}{2}\right)}{\tan\left(\frac{\alpha-\beta}{2}\right)},\quad \frac{b+c}{b-c}=\frac{\tan\left(\frac{\beta+\gamma}{2}\right)}{\tan\left(\frac{\beta-\gamma}{2}\right)},\quad \frac{c+a}{c-a}=\frac{\tan\left(\frac{\gamma+\alpha}{2}\right)}{\tan\left(\frac{\gamma-\alpha}{2}\right)}$$

Proof:

For any oblique triangle ABC, the law of sines is given as follows:

$$\frac{a\ b}{\sin\alpha}=\frac{}{\sin\beta}$$

$$\frac{a}{b}=\frac{\sin\alpha}{\sin\beta}$$

Using componendo and dividendo property, we get:

$$\frac{a+b}{a-b}=\frac{\sin\alpha+\sin\beta}{\sin\alpha-\sin\beta}$$

$$\frac{a+b}{a-b}=\frac{2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}}{2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}}$$

$$\frac{a+b}{a-b}=\frac{\tan\frac{\alpha+\beta}{2}}{\tan\frac{\alpha-\beta}{2}}$$

Similarly, we can prove that:

$$\frac{b+c}{b-c}=\frac{\tan\left(\frac{\beta+\gamma}{2}\right)}{\tan\left(\frac{\beta-\gamma}{2}\right)},\quad \frac{c+a}{c-a}=\frac{\tan\left(\frac{\gamma+\alpha}{2}\right)}{\tan\left(\frac{\gamma-\alpha}{2}\right)}$$

[illegible]

Prove that

$$\frac{a+c}{a-c}=\frac{\tan\left(\frac{\alpha+\gamma}{2}\right)}{\tan\left(\frac{\alpha-\gamma}{2}\right)}$$

Example:

Use the law of tangents to solve the triangle ABC when $a = 500mm$, $b = 600mm$, $\gamma = 50°20'$

Solution:

Given: $a = 500cm, b = 600cm, \gamma = 50°20'$

To Find: c, α and β

As in any triangle,

$$\alpha+\beta+\gamma=180° \Rightarrow \beta+\alpha=180°-\gamma$$

$$\Rightarrow \beta+\alpha=180°-50°20'=129°40' \quad .....(i)$$

Now using law of tangents for $b > a$,

$$\frac{b+a}{b-a}=\frac{\tan\frac{\beta+\alpha}{2}}{\tan\frac{\beta-\alpha}{2}}$$

Substituting the values of b, c and $\beta+\alpha$, we get:

[illegible]

It is better to use formula

$$\frac{a+c}{a-c}=\frac{\tan\left(\frac{\alpha+\gamma}{2}\right)}{\tan\left(\frac{\alpha-\gamma}{2}\right)}$$

rather than the formula

$$\frac{c+a}{c-a}=\frac{\tan\left(\frac{\gamma+\alpha}{2}\right)}{\tan\left(\frac{\gamma-\alpha}{2}\right)}$$

when $a > c$ in the given situ

$$\frac{600+500}{600-500}=\frac{\tan\frac{129^\circ 40'}{2}}{\tan\frac{\beta-\alpha}{2}} \Rightarrow \frac{1100}{100}=\frac{\tan 64^\circ 50'}{\tan\frac{\beta-\alpha}{2}}$$

$$\tan\frac{\beta-\alpha}{2}=\frac{\tan 64^\circ 50'}{11} \Rightarrow \tan\frac{\beta-\alpha}{2}=0.193$$

$$\frac{\beta-\alpha}{2}=\tan^{-1}(0.193)=10^\circ 57'$$

$$\beta-\alpha=21^\circ 54' \text{ ....} (ii)$$

Check Point

Solve the triangle ABC by using both law of tangents and law of sines when:

$b=50cm, c=40, \alpha=80^\circ 30'$

Solving (i) and (ii), we get:

$$\beta=75^\circ 47', \alpha=53^\circ 53'$$

To find c, we use law of sines (we can also use law of cosines).

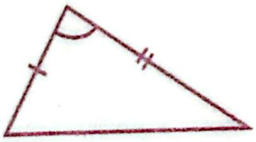
$$\frac{c}{\sin\gamma}=\frac{b}{\sin\beta} \Rightarrow c=\frac{b\times\sin\gamma}{\sin\beta}$$

$$c=\frac{600\times\sin 50^\circ 20'}{\sin 75^\circ 47'}=476.45mm$$

When Law of Sines and Law of Cosines are Used

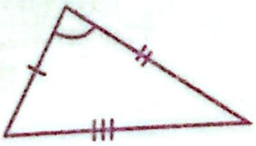
The Law of Cosines and the Law of Sines can be used to find the remaining three measures of any triangle when we know the measure of a side and the measures of any two other components (two sides, two angles, one side one angle) of the triangle.

1. **Given:** Two sides and the included angle.



- Use the Law of Cosines to find the measure of the third side.
- Use the Law of Sines or the Law of Cosines to find the measure of another angle.
- Use the sum of the angles of a triangle to find the measure of the third angle.

2. **Given:** Three sides.



- Use the Law of Cosines to find the measure of an angle.
- Use the Law of Sines or the Law of Cosines to find the measure of another angle.
- Use the sum of the angles of a triangle to find the measure of the third angle.

3. **Given:** Two angles and a side

- Use the sum of the angles of a triangle to find the measure of the third angle.
- Use the Law of Sines to find the remaining sides.

4. **Given:** Two sides and an angle opposite to one of them

- Use the Law of Sines to find the possible measure(s) of another angle.
- Determine if there are two, one, or no possible triangles.
- If there is a triangle, use the sum of the angles of a triangle to find the measure(s) of the third angle.
- Use the Law of Sines or the Law of Cosines to find the measure(s) of the third side.

Half Angle Formulae

Sometimes it is convenient to use half angle formulae for the solution of oblique triangles when measures of three sides of a triangle are given. In this section, we will study half angle formulae for sines, cosines and tangents. Let us derive them one by one.

(a) Cosines of Half Angle

Statement: In any triangle ABC with usual notations,

$$\cos\frac{\alpha}{2}=\sqrt{\frac{S(S-a)}{bc}},\quad \cos\frac{\beta}{2}=\sqrt{\frac{S(S-b)}{ac}},\quad \cos\frac{\gamma}{2}=\sqrt{\frac{S(S-c)}{ab}}$$

where $S=\frac{a+b+c}{2}$.

Proof: We know that:

$$\cos\alpha=\frac{b^2+c^2-a^2}{2bc}$$

$$2\cos^2\frac{\alpha}{2}-1=\frac{b^2+c^2-a^2}{2bc}\Rightarrow 2\cos^2\frac{\alpha}{2}=\frac{b^2+c^2-a^2}{2bc}+1$$

$$\Rightarrow 2\cos^2\frac{\alpha}{2}=\frac{b^2+c^2+2bc-a^2}{2bc}=\frac{(b+c)^2-(a)^2}{2bc}$$

$$\Rightarrow 2\cos^2\frac{\alpha}{2}=\frac{(b+c+a)(b+c-a)}{2bc}$$

As $a+b+c=2S \Rightarrow b+c-a=2S-2a=2(S-a)$

$$\therefore 2\cos^2\frac{\alpha}{2}=\frac{2S\times 2(S-a)}{2bc}$$

$$\cos^2\frac{\alpha}{2}=\frac{S(S-a)}{bc}$$

$$\cos\frac{\alpha}{2}=\sqrt{\frac{S(S-a)}{bc}}$$

Check Point:

Find the diameter of the circle shown in the figure below.

Similarly, it can be proved that:

$$\cos\frac{\beta}{2}=\sqrt{\frac{S(S-b)}{ac}}\quad\text{and}\quad\cos\frac{\gamma}{2}=\sqrt{\frac{S(S-c)}{ab}}$$

(b) Sines of Half Angle

Statement: In any triangle ABC with usual notations,

$$\sin\frac{\alpha}{2}=\sqrt{\frac{(S-b)(S-c)}{bc}},\quad \sin\frac{\beta}{2}=\sqrt{\frac{(S-a)(S-c)}{ac}},\quad \sin\frac{\gamma}{2}=\sqrt{\frac{(S-a)(S-b)}{ab}}$$

where $S=\frac{a+b+c}{2}$.

Proof: We know that

$$\cos\alpha=\frac{b^2+c^2-a^2}{2bc}$$

$$1-2\sin^2\frac{\alpha}{2}=\frac{b^2+c^2-a^2}{2bc}\Rightarrow 2\sin^2\frac{\alpha}{2}=1-\frac{b^2+c^2-a^2}{2bc}$$

$$\Rightarrow 2\sin^2\frac{\alpha}{2}=\frac{(a)^2-(b-c)^2}{2bc}=\frac{(a-b+c)(a+b-c)}{2bc}$$

As $a+b+c=2S\Rightarrow a+c-b=2S-2b=2(S-b)$ and $a+b-c=2S-2c=2(S-c)$

$$\therefore 2\sin^2\frac{\alpha}{2}=\frac{2(S-b)\times 2(S-c)}{2bc}=\frac{2(S-b)(S-c)}{bc}$$

$$\sin^2\frac{\alpha}{2}=\frac{(S-b)(S-c)}{bc}$$

$$\sin\frac{\alpha}{2}=\sqrt{\frac{(S-b)(S-c)}{bc}}$$

Check Point:

Prove that

$$\cos\frac{\gamma}{2}=\sqrt{\frac{S(S-c)}{ab}}\ \text{and}\ \sin\frac{\gamma}{2}=\sqrt{\frac{(S-a)(S-b)}{ab}}$$

Similarly, it can be proved that:

$$\sin\frac{\beta}{2}=\sqrt{\frac{(S-a)(S-c)}{ac}}\quad\text{and}\quad\sin\frac{\gamma}{2}=\sqrt{\frac{(S-a)(S-b)}{ab}}$$

(c) Tangents of Half Angle

Statement: In any triangle ABC with usual notations,

$$\tan\frac{\alpha}{2}=\sqrt{\frac{(S-b)(S-c)}{S(S-a)}},\quad \tan\frac{\beta}{2}=\sqrt{\frac{(S-a)(S-c)}{S(S-b)}},\quad \tan\frac{\gamma}{2}=\sqrt{\frac{(S-a)(S-b)}{S(S-c)}}$$

where $S=\frac{a+b+c}{2}$.

Proof: $\tan\frac{\alpha}{2}=\frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}=\frac{\sqrt{\frac{(S-b)(S-c)}{bc}}}{\sqrt{\frac{S(S-a)}{bc}}}=\sqrt{\frac{(S-b)(S-c)}{S(S-a)}}$

$\therefore \tan\frac{\alpha}{2}=\sqrt{\frac{(S-b)(S-c)}{S(S-a)}}$

Similarly, it can be proved that:

$$\tan\frac{\beta}{2}=\sqrt{\frac{(S-a)(S-c)}{S(S-b)}},\quad \tan\frac{\gamma}{2}=\sqrt{\frac{(S-a)(S-b)}{S(S-c)}}$$

Solve the triangle AB when $a = 5cm, b =$ $c = 7cm$ using half an formulae for sines an tangents. Which one method do you think

Example:

Use half angle formulae to solve the triangle ABC when $a = 5cm, b = 6cm, c = 7cm$.

Solution:

Given: $a = 5cm, b = 6cm, c = 7cm$.

To Find: α, β and γ

We use half angle formula for cosines here.

Now $S=\frac{a+b+c}{2}=\frac{5+6+7}{2}=9$

$\therefore \cos\frac{\alpha}{2}=\sqrt{\frac{S(S-a)}{bc}}=\sqrt{\frac{9(9-5)}{6\times7}}=\sqrt{\frac{36}{42}}=0.926$

$\frac{\alpha}{2}=\cos^{-1}(0.926)=22.18^\circ \Rightarrow \alpha=44.36^\circ$

Now $\cos\frac{\beta}{2}=\sqrt{\frac{S(S-b)}{ac}}=\sqrt{\frac{9(9-6)}{5\times7}}$

$=\sqrt{\frac{27}{35}}=0.878$

$\frac{\beta}{2}=\cos^{-1}(0.878)=28.58^\circ \Rightarrow \beta=57.20^\circ$

To find third angle, we have:

$\alpha+\beta+\gamma=180^\circ \Rightarrow \gamma=180^\circ-\alpha-\beta$

$\Rightarrow \gamma=180^\circ-44.36^\circ-57.20^\circ=78.44^\circ$

Check Point:

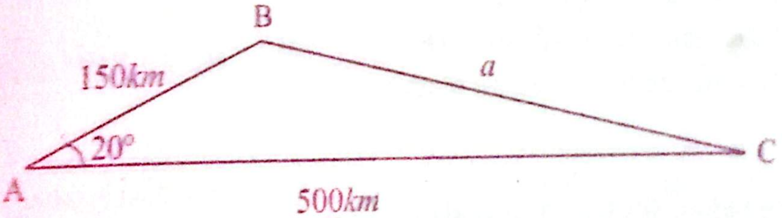
The base of an isosceles triangle measures 14.5 *cm* and the vertex a measures 110°.

a. Find the measure of one of the congruent sides of the triangle t nearest hundredth.

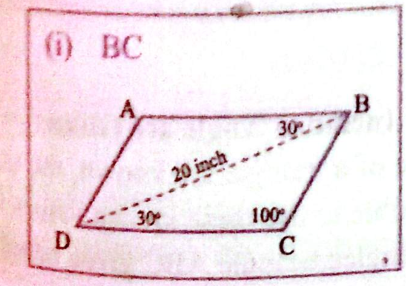
b. Find the perimeter of the triang

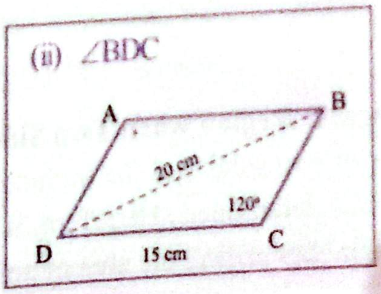
Exercise 8.

1. Use the Law of Cosines to find the remaining side(s) and angle(s) in triangle ABC.
   - (i) $a=7, b=12, \gamma=58.5^\circ$ (ii) $b=20, c=15, \alpha=36^\circ$ (iii) $a=150, c=150, \beta=15^\circ$
   - (iv) $\beta=45^\circ, a=30, c=42$ (v) $a=7, b=10, c=12$ (vi) $a=3, b=3, c=5$
   - (vii) $a=30, b=45, c=50$ (viii) $a=6, b=6, c=6$ (ix) $a=5, b=12, c=13$
2. Use the half angles formulae to solve parts (v) to (ix) in Q.1 where possible.
3. Use the Law of Sines to solve triangle ABC in the following (where possible).
   - (i) $a=10, \alpha=40^\circ, \beta=60^\circ$ (ii) $b=20, \alpha=50^\circ, \beta=70^\circ$
   - (iii) $c=12, \alpha=45^\circ, \beta=75^\circ$ (iv) $b=18, \beta=40^\circ35', \gamma=120^\circ$
   - (v) $a=14.6, \alpha=25^\circ10', \beta=85.5^\circ$ (vi) $c=52, \alpha=42.3^\circ, \gamma=85^\circ14'$
4. Solve parts (i) to (iv) of Q.1 by using the Law of Tangents.
5. Solve by using appropriate law.
   - (i) $a=10, b=8, \beta=80^\circ$ (ii) $b=20, c=14, \beta=70^\circ$
   - (iii) $c=12, b=10, \gamma=64^\circ$ (iv) $b=18, \alpha=55^\circ5', a=37$
   - (v) $a=14.6, b=10.6, c=17.2$ (vi) $c=88, \beta=23.2^\circ, \gamma=73^\circ14'$
6. A pilot is flying from city A to city C, 500 *km* apart. He starts his flight 20° off course and flies on this course for 150 *km* and is above city B. How far is he from city C?

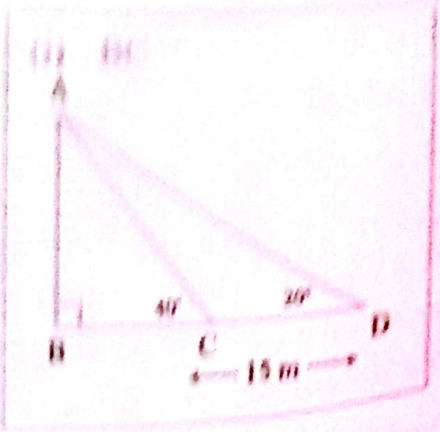


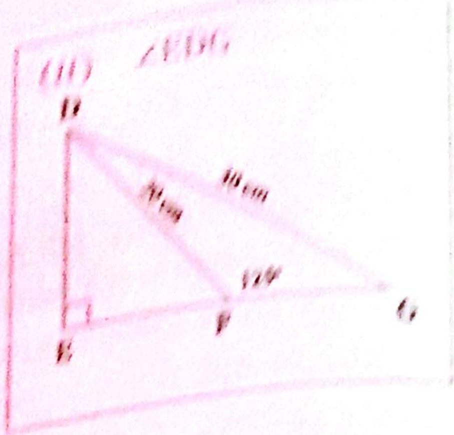
7. Two sides of a triangular plot have lengths 400m and 600m. The measurement of angle between the sides is 45°. Find the perimeter and area of the plot.
8. The sides of a triangle are 6.5cm, 8.2cm and 5.8cm. Find the measurement of smallest and largest angles.
9. The sides of a parallelogram are 50cm and 70cm. Find the length of each diagonal if the larger angle measures 110 °.
10. For parallelogram ABCD, find :


(i) BC

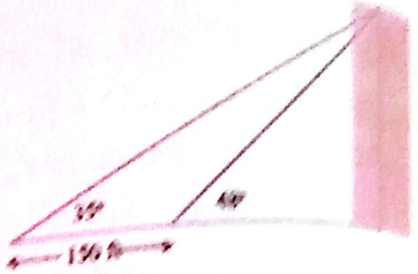

(ii) ∠BDC

11. For the figure below find:

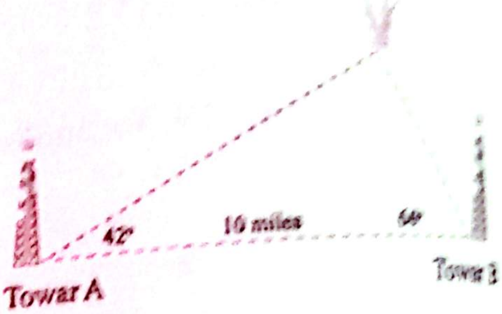




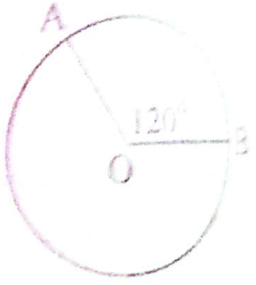
12. Find the height of the building in the figure below.



13. Fire towers A and B are located 10 miles apart at the same level of ground. Rangers at fire tower A spots a fire at 42°, and rangers at fire tower B spot the same fire at 64°. How far from tower A is the fire to the nearest tenth of a mile?



14. Circle O has a radius of 15cm. The angle between radii $\overline{OA}$ and $\overline{OB}$ is 120°. Find the length of chord $\overline{AB}$.



15. Two lighthouses are 12 miles apart along a straight shore. A ship is 15 miles from one lighthouse and 20 miles from the other. Find, to the nearest degree, the mea of the angle between the lines of sight from the ship to each lighthouse.

Area of Triangular Region

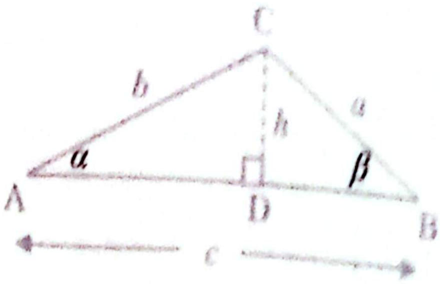
(a) Area of Triangular Region when Two Sides and Included Angle are Given

When the measures of two sides and the included angle of a triangle are known, the size an shape of the triangle is determined. Therefore, it is possible to use these known values to fin the area of the triangle. Let us find the area of an acute angled triangle ABC given below.

We know that:

Area (Δ) of triangle ABC $= \frac{1}{2} \times$ base$\times$ height

Area (Δ) of triangle ABC $= \frac{1}{2} \times AB \times CD = \frac{1}{2} \times c \times h$ .... (i)



Now in right angled triangle ACD, $AB = c$

and $\frac{h}{b} = \sin\alpha \Rightarrow h = b\sin\alpha$

Therefore from (i),

$\Delta = \frac{1}{2} \times c \times b\sin\alpha = \frac{1}{2}bc\sin\alpha$

Again, in right angled triangle BCD

$\frac{h}{a} = \sin\beta \Rightarrow h = a\sin\beta$

Therefore from (i),

$\Delta = \frac{1}{2} \times c \times a\sin\beta = \frac{1}{2}ac\sin\beta$

Similarly, we can prove that:

$\Delta = \frac{1}{2}ab\sin\gamma$

Hence $\Delta = \frac{1}{2}bc\sin\alpha = \frac{1}{2}ac\sin\beta = \frac{1}{2}ab\sin\gamma$

Check Point:

1. Rashid found the area of parallelogram *ABCD* by using $(AB)(BC)(\sin\angle B)$.
Shehzad found the area of parallelogram *ABCD* by using $(AB)(BC)(\sin\angle A)$. Explain why Rashid and Shehzad both got the correct answer.

Example:

Find the area of triangle DEF if $DE = 10$, $EF = 8$ and $\angle E = 30°$.

Solution:

$$\text{Area of triangle DEF} = \frac{1}{2} \times DE \times EF \times \sin(\angle E)$$
$$= \frac{1}{2} \times 10 \times 8 \times \sin 30°$$
$$= 40 \times 0.5 = 20\ sq.\ units$$

Challenge:

Take an obtuse angled triangle and prove that

$\Delta = \frac{1}{2}ab\sin\gamma$

(b) Area of Triangular Region when Two Angles and One Side are Given

If *a* is measure of side of a triangle ABC and any two angles let's say β and γ are given then:

$$\text{Area of triangle ABC} = \Delta = \frac{1}{2}a^2\frac{\sin\beta\sin\gamma}{\sin\alpha}$$

Let us prove this formula.

We have proved in the previous section that:

Check Point:

Urwa said that the area of rhombus *PQRS* is $(PQ)^2(\sin P)$. Do you agree with Urwa? Explain why or why not.

Area of triangle ABC $= \Delta = \frac{1}{2}ab\sin\gamma$ .....(i)

But in the current situation, b is not given. We can replace b by using law of sines as follows:

$$\frac{b}{\sin\beta} = \frac{a}{\sin\alpha} \Rightarrow b = \frac{a\sin\beta}{\sin\alpha}$$

Substituting the value of b in (i), we get

$$\Delta = \frac{1}{2}a\left(\frac{a\sin\beta}{\sin\alpha}\right)\sin\gamma = \frac{1}{2}a^2\frac{\sin\beta\sin\gamma}{\sin\alpha}$$

In the same way, we can prove that

$$\Delta = \frac{1}{2}b^2\frac{\sin\alpha\sin\gamma}{\sin\beta} \quad \text{and} \quad \Delta = \frac{1}{2}c^2\frac{\sin\alpha\sin\beta}{\sin\gamma}$$

Hence, $\Delta = \frac{1}{2}a^2\frac{\sin\beta\sin\gamma}{\sin\alpha}$, $\Delta = \frac{1}{2}b^2\frac{\sin\alpha\sin\gamma}{\sin\beta}$, $\Delta = \frac{1}{2}c^2\frac{\sin\alpha\sin\beta}{\sin\gamma}$

[illegible]

Prove that

(i) $\Delta = \frac{1}{2}b^2\frac{\sin\alpha\sin\gamma}{\sin\beta}$

(ii) $\Delta = \frac{1}{2}c^2\frac{\sin\alpha\sin\beta}{\sin\gamma}$

Example:

Calculate the cost of grass cutting @ Rs.10 per sq. unit of a triangular plot ABC if $a = 16.5$, $\alpha = 40.5°$, $\beta = 65°$.

Solution: To find the cost of grass cutting, we need area of plot and to find area, we need the third angle γ.

$$\gamma = 180° - \alpha - \beta$$

$$\Rightarrow \gamma = 180° - 40.5° - 65° = 74.5°$$

Now, area of triangle ABC $= \frac{1}{2}a^2\frac{\sin\beta\sin\gamma}{\sin\alpha}$

$$= \frac{1}{2}\times(16.5)^2\times\frac{\sin 65°\times\sin 74.5°}{\sin 40.5°}$$

$$= 183.05\ sq.\ units$$

Therefore, the cost of grass cutting @ Rs.10 per sq. unit = Rs. 10 × 183.05 = Rs.1830.50

(c) Area of Triangular Region when Three Sides are Given

If a, b, c are measure of sides of a triangle ABC then:

Area of triangle ABC $= \Delta = \sqrt{S(S-a)(S-b)(S-c)}$ where $S = \frac{a+b+c}{2}$

Check Point

Two sides of a triangle are 20cm long each. Find the angle between both sides if area of triangle is 200 sq. unit.

Let us prove this formula.

We have proved in the previous section that:

Area of triangle ABC $= \Delta = \frac{1}{2}bc\sin\alpha$

Using half angle formula, we have:

$$\Delta = \frac{1}{2}bc\times 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} = bc\times\sqrt{\frac{(S-b)(S-c)}{bc}}\times\sqrt{\frac{S(S-a)}{bc}}$$

$$\Delta = bc\times\sqrt{\frac{(S-b)(S-c)}{bc}}\times\sqrt{\frac{S(S-a)}{bc}}=\sqrt{S(S-a)(S-b)(S-c)}$$

This formula is known as hero's formula or Heron's formula.

Example:

Find area of a triangle with dimensions 16, 18, 20.

Solution: Let $a=16, b=18, c=20$.

Area of triangle = $\sqrt{S(S-a)(S-b)(S-c)}$

Where $S=\frac{16+18+20}{2}=\frac{54}{2}=27$

Now, area of triangle = $\sqrt{27(27-16)(27-18)(27-20)}=\sqrt{18711}$

$=136.79$ *sq.units*

Exercise 8.4

1. Find the area of triangular region ABC in the following.

(i) $a=8, b=14, \gamma=68.7°$ (ii) $b=30, c=25, \alpha=46°$

(iii) $a=20, c=15, \beta=25°$ (iv) $\beta=46°50', a=43, c=52$

(v) $b=4.5, c=2.5, \alpha=65.2°$ (vi) $a=2, b=2, c=5$

(vii) $a=12, \alpha=44°, \beta=60°$ (viii) $b=30, \alpha=40°, \beta=80°$

(ix) $c=10, \alpha=75°, \beta=45°$ (x) $b=21, \beta=30°15', \gamma=110°$

(xi) $a=18.4, \alpha=65°10', \beta=95.5°$ (xii) $c=25, \alpha=52.7°, \gamma=79°24'$

(xiii) 18, 21, 32 (xiv) 20, 26, 37 (xv) $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$

(xvi) 12.4, 13.7, 20.2 (xvii) 1.6, 2.6, 4.1

2. The adjacent sides of parallelogram ABCD measure 12 and 15. The measure of one angle of the parallelogram is 135°. Find the area of the parallelogram.

3. Three streets intersect in pairs enclosing a small triangular park. The measures of the distances between the intersections are 30 *m*, 34 *m*, and 27 *m*. Find the area of the park.

4. A field is bordered by two pairs of parallel roads so that the shape of the field is a parallelogram. The lengths of two adjacent sides of the field are 2 *km* and 3 *km*, and the length of the shorter diagonal of the field is 3 *km*.
   a. Find the cosine of the acute angle of the parallelogram.
   b. Find the exact value of the sine of the acute angle of the parallelogram.
   c. Find the exact value of the area of the field.
   d. Find the area of the field to the nearest integer.

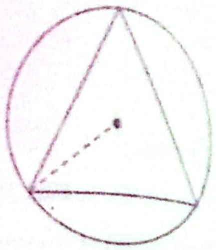
5. The roof of a shed consists of four congruent isosceles triangles. The length of each equal side of one triangular section is 22.0 feet and the measure of the vertex angle of each triangle is 75°. Find the area of one triangular section of the roof.

6. A garden is in the shape of an isosceles trapezoid. The lengths of the parallel sides of the garden are 30 feet and 20 feet, and the length of each of the other two sides is 10 feet. If a base angle of the trapezoid measures 60°, find the exact area of the garden.

7. In triangle ABC, $\angle B = 30^\circ$ and in triangle DEF, $\angle E = 150^\circ$. Show that if AB = DE and BC = EF, the areas of the two triangles are equal.

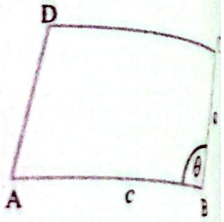
8. Area of a triangular garden is $150 m^2$. If two corner angles of a side measure 40° and 60°, find the length of that side. Also find the third angle of garden.

9. An equilateral triangle is inscribed in a circle of radius $6cm$. Find the area of triangle.



10. Laiba wants to draw triangle ABC with AB = 15 cm, BC = 8 cm, and an area of 40 cm^2.
   a. What must be the sine of $\angle B$?   b. Find, the measure of $\angle B$.

11. Let ABCD be a parallelogram with $AB = c$, $BC = a$, and $\angle B = \theta$.
   a. Write a formula for the area of parallelogram in terms of c, a, and θ.
   b. For what value of θ does parallelogram have the greatest area?



Circles Connected with Triangles

(a) Circumcircle

A circle passing through the vertices of any triangle (polygon) is called circumcircle. The centre of this circle is called the circumcentre and its radius is called the circumradius. The radius of circumcircle is denoted by R.

(b) Incircle

A circle drawn inside and touching the sides of a triangle (polygon) is called incircle or inscribed circle. It is the largest circle contained in the triangle. The centre of this circle is called the incentre usually denoted by I and its radius is called the inradius denoted by r.

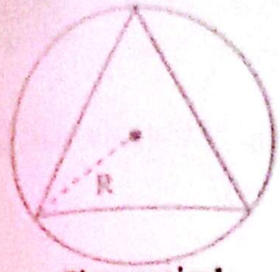
(c) Escribed Circle

A circle touching one of the sides of a triangle externally and the extensions of its two other sides internally, is called an escribed circle (e-circle or ex-circle). The centre of this circle is called the e-centre and its radius is called the e-radius.

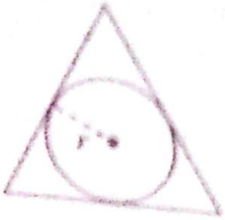
If a circle touches extended arms of angle A (circle is drawn opposite to vertex A) of a triangle ABC, then its centre and radius are respectively denoted by I_1 and r_1. Similarly,

(i) Centre and radius of e-circle opposite to vertex B are denoted by I_2 and r_2.

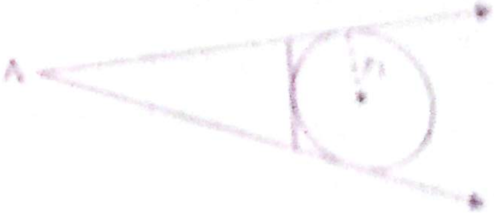
(iii) Centre and radius of e-circle opposite to vertex C are denoted by I_3 and r_3.



Circumcircle



Incircle



Escribed circle

Key Fact:

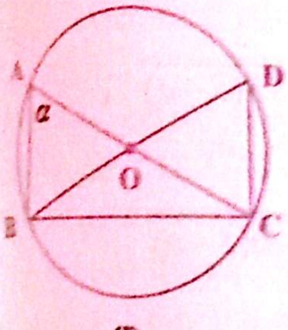
(i) Circumcentre is the point of intersection of right bisectors of sides of a triangle.

(ii) Incentre is the point of intersection of angle bisectors of a triangle.

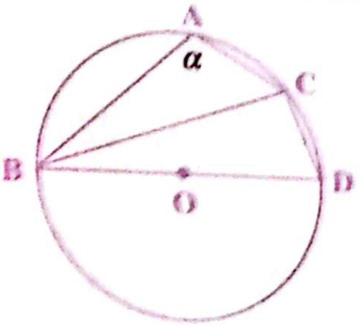
(iii) E-centre is the point of intersection of one internal bisector of angle and external bisectors of remaining angles of a triangle.

Circum-Radius of Circle In-terms of Side and Opposite Angle of Triangle

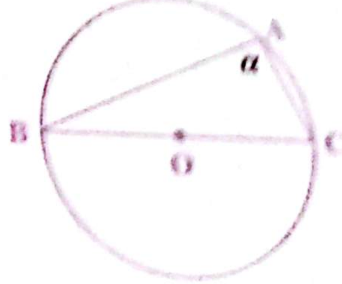
Let O be the circumcentre of triangle ABC. Draw a diameter BD. Also join C to D.



(i)



(ii)



(iii)

Case-1: When α is Acute

In figure (i), α and $\angle BDC$ are angles in the same segment associared with chord BC.

$\therefore \angle BDC = \alpha$

Now in right angled triangle BCD,

$$\frac{BC}{BD} = \sin(\angle BDC) = \sin\alpha$$

$$\frac{a}{2R} = \sin\alpha \Rightarrow R = \frac{a}{2\sin\alpha}$$ where BD = 2R and BC = a.

Case-2: When α is Obtuse

In figure (ii), $\alpha + \angle BDC = 180°$ because ABDC is a cyclic quadrilateral.

$\therefore \angle BDC = \pi - \alpha$

Now in right angled triangle BCA

[illegible] where $\sin(\pi-\alpha)=\sin\alpha$ $\quad \frac{a}{2R}=\sin\alpha \Rightarrow R=\frac{a}{2\sin\alpha}$

Case-3: **When α is Right**

As angle in a semicircle is right angle. Therefore, ABC is a right angled triangle and BC is

diameter. Hence $a = BC = 2R$

$$\therefore \frac{a}{2R}=1=\sin\frac{\pi}{2}=\sin\alpha$$

$$\frac{a}{2R}=\sin\alpha \Rightarrow R=\frac{a}{2\sin\alpha}$$

Similarly, we can prove that:

$$R=\frac{b}{2\sin\beta} \text{ and } R=\frac{c}{2\sin\gamma}$$

Combining all results, we get:

$$R=\frac{a}{2\sin\alpha}=\frac{b}{2\sin\beta}=\frac{c}{2\sin\gamma}$$

Circum-Radius of Circle In-terms of Sides of Triangle

We have proved in the previous section, that:

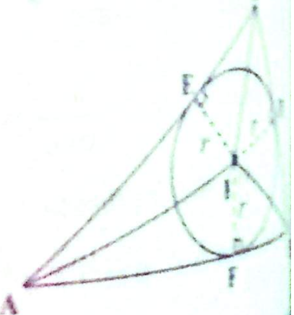
$$R=\frac{a}{2\sin\alpha}=\frac{a}{2\times 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}=\frac{a}{4\times\frac{\sqrt{(S-b)(S-c)}}{bc}\frac{\sqrt{S(S-a)}}{bc}}$$

$$R=\frac{a}{4\times\sqrt{\frac{(S-b)(S-c)}{bc}}\sqrt{\frac{S(S-a)}{bc}}}=\frac{abc}{4\sqrt{S(S-a)(S-b)(S-c)}}$$

$$R=\frac{abc}{4\Delta}$$

Where $\Delta=\sqrt{S(S-a)(S-b)(S-c)}$ and $\sin\alpha=2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$.

In-Radius of a Circle in a Triangle



Consider a triangle ABC. Draw the three angle bisectors of triangle which intersect at point I which is in-centre of circle.

By drawing angle bisectors, the given triangle ABC has been divided into three triangles IAB, IBC and IAC.

Draw altitude ID of triangle IBC, altitude IE of triangle IAC and altitude IF of triangle IAB. Note that

$ID = IE = IF =$ radius of in-circle $= r$

Now if $AB = c$, $BC = a$, $AC = b$ and Δ is the area of angle ABC, then from the figure:

of triangle ABC = area of triangle IBC + area of triangle IAC + area of triangle IAB

$$\Delta = \frac{1}{2}\times BC\times ID + \frac{1}{2}\times AC\times IE + \frac{1}{2}\times AB\times IF$$

$$\Delta = \frac{1}{2}\times a\times r + \frac{1}{2}\times b\times r + \frac{1}{2}\times c\times r = \frac{1}{2}\times r\times (a+b+c)$$

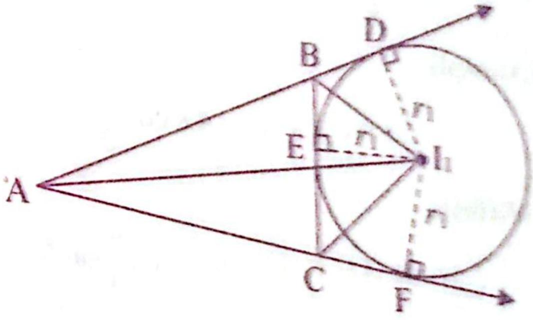
$$\Delta = r\times \frac{a+b+c}{2} = rS$$

$$r = \frac{\Delta}{S}$$

Key Fact

The two geometrical figures, circle and triangle are made up of distinct conception. The triangle is made with straight lines, whereas circle is made by curve lines (arcs). The beauty of circum- circle and in-circle of a triangle is a combination of a triangle and a circle. Engineers use this combination in making parts of machines.

E-Radius of a Circle of a Triangle



Consider a triangle ABC. Draw internal bisector of ∠A and external bisectors of ∠B and ∠C of triangle intersecting at I_1 which is the centre of ex-circle.

By drawing angle bisectors, the given triangle ABC has been divided into three triangles I_1AB, I_1BC and I_1AC.

Draw altitude I_1D of triangle I_1AB, altitude I_1E of triangle I_1BC and altitude I_1F of triangle I_1AC. Note that

$I_1D = I_1E = I_1F$ = radius of e-circle = r_1

Now if AB = c, BC = a, AC = b and Δ is the area of triangle ABC, then from the figure:

area of triangle ABC = area of triangle I_1AB + area of triangle I_1AC + area of triangle I_1BC

$$\Delta = \frac{1}{2}\times AB\times I_1D + \frac{1}{2}\times AC\times I_1F - \frac{1}{2}\times BC\times I_1E$$

$$\Delta = \frac{1}{2}\times c\times r_1 + \frac{1}{2}\times b\times r_1 - \frac{1}{2}\times a\times r_1 = \frac{1}{2}\times r_1\times (c+b-a)$$

$$\Delta = r_1\times \frac{2S-a-a}{2} = r_1\times \frac{2S-2a}{2} = r_1\times \frac{2(S-a)}{2}$$

$$r_1 = \frac{\Delta}{S-a}$$

Similarly, we can prove that:

$$r_2 = \frac{\Delta}{S-b} \quad \text{and} \quad r_3 = \frac{\Delta}{S-c}$$

Example:

Find r, R and r_2 when $a = 8$, $b = 9$, $c = 11$.

Solution: **Given:** $a = 8$, $b = 9$, $c = 11$

To find: r, R and r_2

First of all, we find S and Δ.

$$S = \frac{8+9+11}{2} = \frac{28}{2} = 14$$

and $\Delta = \sqrt{S(S-a)(S-b)(S-c)}$

$$= \sqrt{14(14-8)(14-9)(14-11)} = \sqrt{1260} = 35.5$$

Now, $r = \dfrac{\Delta}{S} = \dfrac{35.5}{14} = 2.54$

$$R = \frac{abc}{4\Delta} = \frac{792}{142} = 5.58$$

$$r_2 = \frac{\Delta}{S-b} = \frac{35.5}{14-9} = \frac{35.5}{5} = 7.1$$

Key Facts:

We know that

$$R = \frac{a}{2\sin\alpha} = \frac{b}{2\sin\beta} = \frac{c}{2\sin\gamma}$$

$$\Rightarrow \frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2R$$

$$\Rightarrow \frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

Which is law of sines.

Check Point:

Find r_1, r_2 and r_3 when $a = 5.4$, $b = 7.6$ and $c = 10.2$

Example:

Prove that $r_2 = 4R\cos\frac{\alpha}{2}\cos\frac{\gamma}{2}\sin\frac{\beta}{2}$

Solution:

$$\text{R.H.S} = 4R\cos\frac{\alpha}{2}\cos\frac{\gamma}{2}\sin\frac{\beta}{2}$$

$$= 4\times\frac{abc}{4\Delta}\times\sqrt{\frac{S(S-a)}{bc}}\times\sqrt{\frac{S(S-c)}{ab}}\times\sqrt{\frac{(S-a)(S-c)}{ac}}$$

$$= \frac{abc}{\Delta}\times\sqrt{\frac{S^2(S-a)^2(S-c)^2}{a^2b^2c^2}} = \frac{abc}{\Delta}\times\frac{S(S-a)(S-c)}{abc}$$

$$= \frac{1}{\Delta}\times\frac{S(S-a)(S-b)(S-c)}{S-b} = \frac{\Delta^2}{\Delta(S-b)}$$

$$= \frac{\Delta}{S-b} = r_2 = \text{L.H.S}$$

Check Point:

Show that

$$4R\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\sin\frac{\gamma}{2} = r_3$$

Exercise 8.5

1. Find r, R, r_1, r_2 and r_3 when
   - (i) $a = 10, b = 13, c = 17$
   - (ii) $a = 22, b = 24, c = 30$
   - (iii) $a = 3, b = 4, c = 5$
   - (iv) $a = 50, b = 60, c = 70$
2. If ABC is an equilateral triangle, prove that:
   - (i) $r : \text{R} : r_1 = 1 : 2 : 3$
   - (ii) $r_1 : r_2 : r_3 = 1 : 1 : 1$

3. Show that:

(i) $\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}=\frac{2S}{abc}=\frac{1}{2rR}$ (ii) $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{s}{\Delta}$

(iii) $r_1r_2+r_2r_3+r_1r_3=S^2$ (iv) $r_1+r_2+r_3-r=4R$

(v) $\sqrt{rr_1r_2r_3}=\Delta$ (vi) $r_1r_2r_3=rS^2$

4. Show that: $(\sin\alpha+\sin\beta+\sin\gamma)=\frac{S}{R}$

5. Prove that: $(r_1+r_2)\tan\frac{\gamma}{2}=(r_3-r)\cot\frac{\gamma}{2}=c$

6. Show that:

(i) $4rR\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}=r^2\cot\frac{\alpha}{2}\cot\frac{\beta}{2}\cot\frac{\gamma}{2}=\Delta$

(ii) $r=4R\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}=S\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$

(iii) $r=a\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\sec\frac{\alpha}{2}=b\sin\frac{\alpha}{2}\sin\frac{\gamma}{2}\sec\frac{\beta}{2}=c\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sec\frac{\gamma}{2}$

7. Prove that:

(i) $4R\sin\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}=r_1$ (ii) $4R\cos\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\gamma}{2}=r_2$

8. Prove that:

(i) $r_1=S\tan\frac{\alpha}{2}$ (ii) $r_2=S\tan\frac{\beta}{2}$ (iii) $r_3=S\tan\frac{\gamma}{2}$

9. Prove that:

(i) $r=(S-a)\tan\frac{\alpha}{2}=(S-b)\tan\frac{\beta}{2}=(S-c)\tan\frac{\gamma}{2}$

10. Find the radius of in-circle and circum-circle of a triangle having sides: 7*cm*, 12*cm* and 15*cm*.

11. The measures of sides of a triangle are 5*cm*, 9*cm*, 10*cm*. Find the circumference of ex-circle:

(i) opposite to smaller side (ii) opposite to larger side

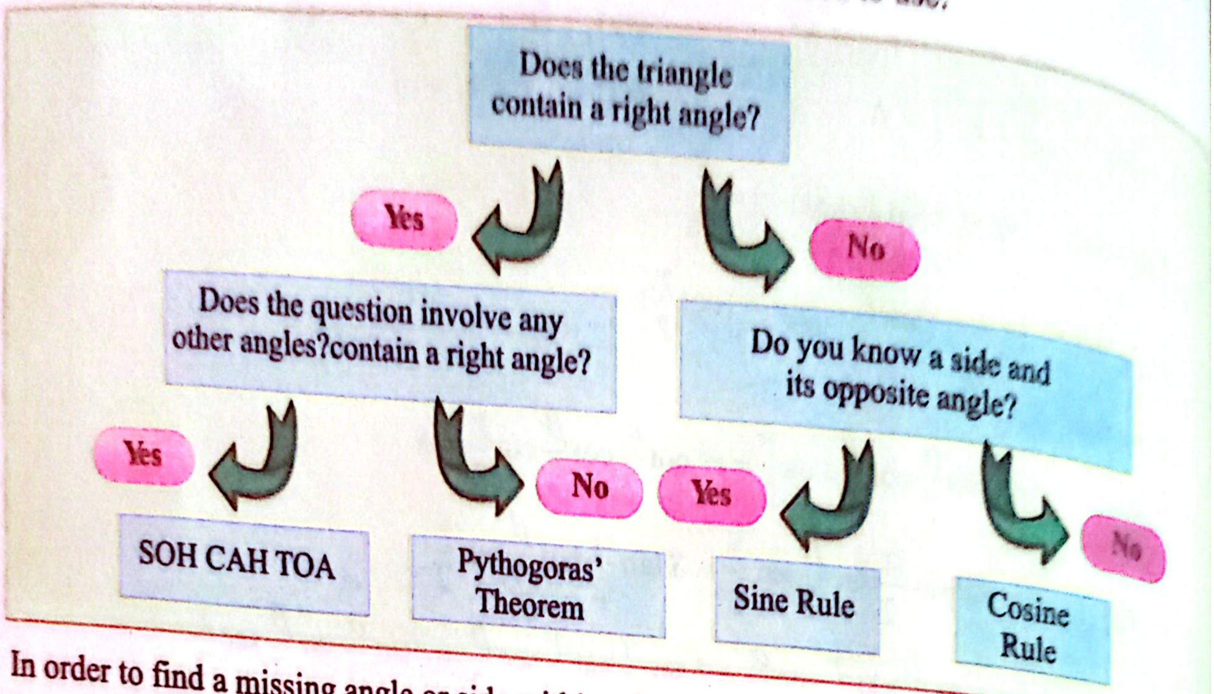
12. Find the area and circumference of in-circle of a triangle having sides: 2.5*cm*, 7.3 *cm* and 6.2cm.

Trigonometric Problems in Three Dimensions

3-D Trigonometry

3-D trigonometry is an application of the trigonometric skills developed for 3-dimensional shapes in order to find unknown sides and angles.

In this topic, we will learn about 3-D trigonometry. We will use combined knowledge of Pythagoras' theorem, trigonometric ratios, the law of sines (sine rule) and the law of cosines (cosine rule) to find missing angles and sides of triangles in 3-dimensional shapes. The flowchart below can help to determine which path we need to use:



In order to find a missing angle or side within a 3-dimensional shape:

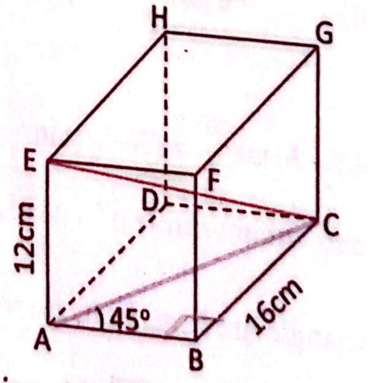
Step 1: Find the missing side or angle of a triangle.

Step 2: Sketch and label the second triangle using information from Step 1.

Step 3: Calculate the missing side or angle of the final triangle.

Let us solve some examples to understand the method.

Example: Find the length of $\overline{EC}$ in the given cuboid.



Step 1: Finding the length of missing side $\overline{AC}$ of right triangle ABC.

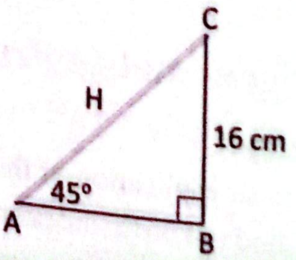
$\overline{AC}$ is hypotenuse of right triangle ABC. We can find its length by using simple trigonometric ratios.

In right triangle ABC:

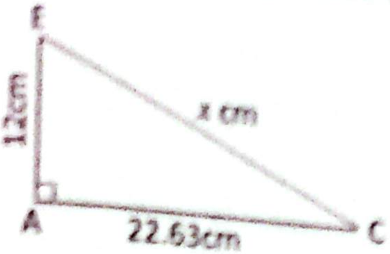
$\sin\theta = \frac{16}{H}$

$\Rightarrow H = \frac{16}{\sin\theta} = \frac{16}{\sin 45^\circ}$

$= \frac{16}{0.707} = 22.63$



Sketching and labeling the second right triangle EAC using information from step 1.



EAC is a right-angled triangle, so we can use Pythagoras' Theorem to find the length of EC

Let EC = x cm

$x^2 = (12)^2 + (22.63)^2$

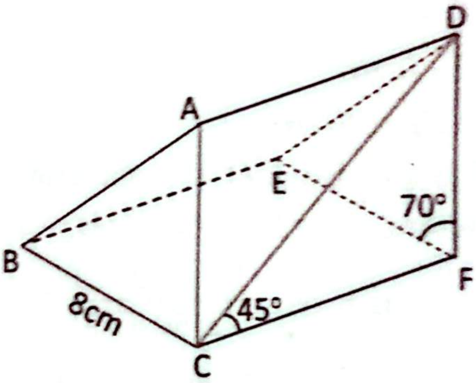
$x^2 = 144 + 512.12 = 656.12$

Taking square root on both sides, we get:

$x = \sqrt{656.12} = 25.61$

Therefore, EC = 25.61 cm

Example: ABCDEF is an isosceles triangular prism. BC is 8cm, ∠DFE = 70° and angle FCD = 45°. Find CD.



Solution:

To find CD, we are required DF.

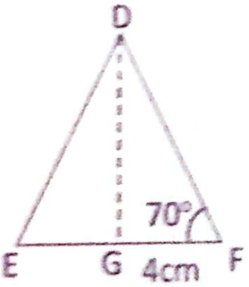
Therefore, first consider ΔDEF to find DF.

In right ΔDGF:

$$\cos 70^\circ = \frac{4}{DF}$$

$$\Rightarrow DF = \frac{4}{\cos 70^\circ}$$

$$= 11.695 \text{ cm}$$

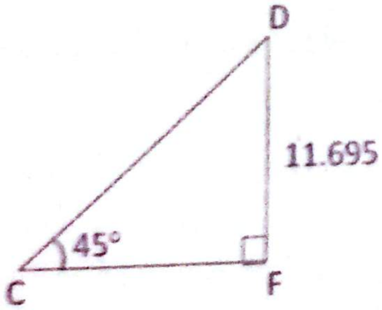


Now in right ΔCFD:

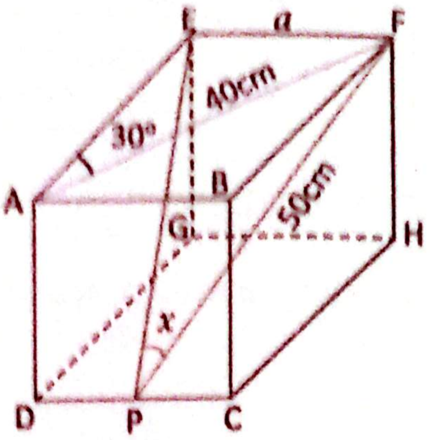
$$\sin 45^\circ = \frac{11.695}{CD}$$

$$\Rightarrow CD = \frac{11.695}{\sin 45^\circ}$$

$$= 16.54 \text{ cm}$$



Example: Find $\angle x$ in the following figure where P is mid-point of $\overline{CD}$.



Solution: To find the value of $\angle x$, we need EF = a which is length of a side of ΔAEF.

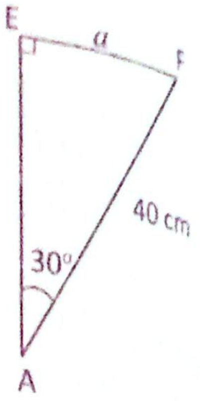
In right ΔAEF:

$$\frac{EF}{AF} = \sin 30°$$

$$\Rightarrow \quad \frac{a}{40} = \frac{1}{2}$$

$$\Rightarrow \quad a = 20$$

$$\therefore \quad EF = 20 \text{ cm}$$



As P is a midpoint of $\overline{DC}$, therefore ΔPEF is an isosceles triangle.

Therefore, using law of cosines, we have:

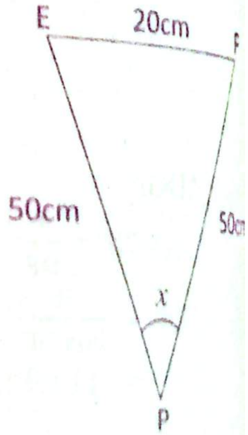
$$\text{Cos } x = \frac{50^2 + 50^2 - 20^2}{2(50)(50)}$$

$$\text{Cos } x = \frac{2500 + 2500 - 400}{5000}$$

$$\text{Cos } x = \frac{4600}{5000} = 0.92$$

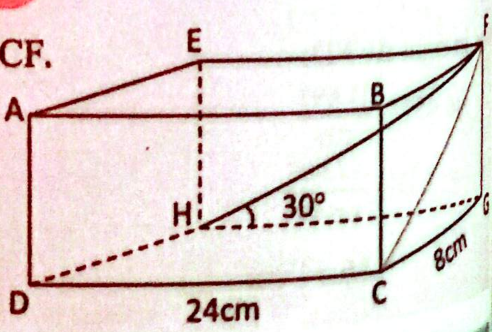
$$\Rightarrow x = \text{Cos}^{-1}(0.92) = 23.1°$$

$$\therefore \angle EPF = 23.1°$$

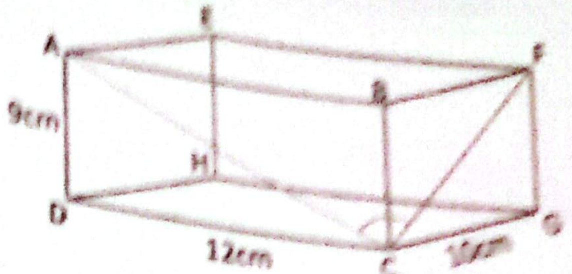


Exercise 8.6

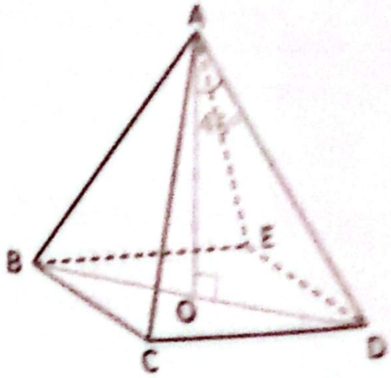
1. In the figure, ABCDEFGH is a cuboid. Calculate CF.



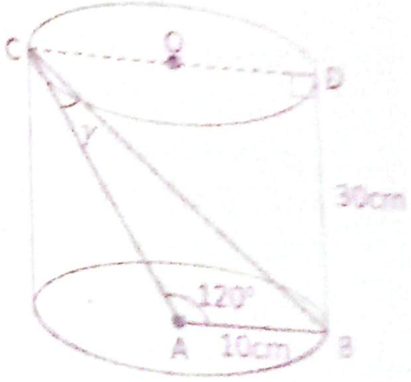
2. Find the angle ACF in the given cuboid in the following figure.



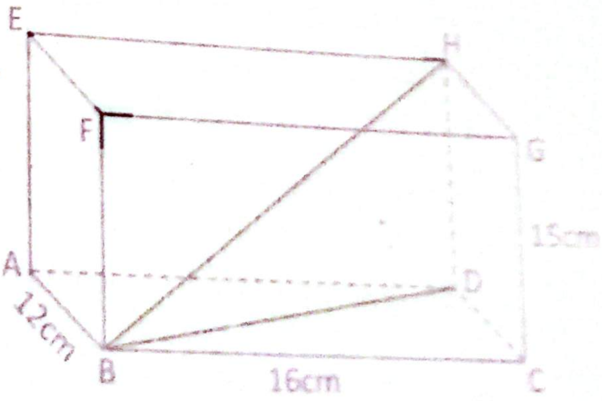
3. In the figure, perimeter of square based pyramid is 36 cm. Find the length of diagonal of the base and height of the pyramid.



4. The diagram shows a cylinder.
Find the value of γ in triangle ABC.

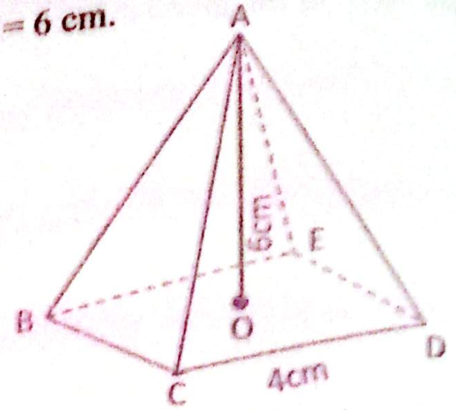


5. In the figure, three workers are standing at positions B, D and H of a container of cuboid shape.



(i) Find the distance between workers at B and D.
(ii) Find the distance between workers at B and H.
(iii) Find ∠DBH.

6. In the figure, O is the centre of the square based pyramid ABCDE. Calculate the angle between side AB and plane BCDE if OA = 6 cm.



7. Floor of a room is 9 m long and 6 m wide. The angle of elevation from the bottom left corner to the top right corner of the room is 49°.
   - (i) Find the distance from one corner of the floor to the opposite corner of the floor.
   - (ii) Find the height of the room.
   - (iii) Find the angle of elevation from the bottom corner of the 9 m long wall to the opposite top corner of the wall.
   - (iv) Find the angle of depression from the top corner of the 6 m long wall to the opposite bottom corner of the wall.

8. Three satellites A, B, and C, used for GPS navigation are orbiting the Earth in the same plane. The distance between satellites A and B is 15 km. If D is a house on Earth, such that $\angle BAD = 70°$ and ACBD is a rhombus then:
   - (i) Determine the distance between satellite A and satellite C.
   - (ii) Determine the distance between satellites B and C.
   - (iii) Find the distance from satellite C to the house D.

MISCELLANEOUS EXERCISE-8

1. Select the correct option in the following.

i. In right ΔABC, $a = 2cm$, $c = 4cm$, what is α ?

(a) 30^o (b) 45^o (c) 60^o (d) 120^o

ii. If in a triangle, $a = 10, b = 15, \alpha = 32^o$ then $\beta = ...$

(a) 42.5^o (b) 46.5^o (c) 52.7^o (d) 62.8^o

iii. Area of an equilateral triangle having side a is:

(a) $\frac{\sqrt{3}}{8}a$ (b) $\frac{\sqrt{3}}{4}a^2$ (c) $\frac{\sqrt{3}}{16}a^2$ (d) $\frac{\sqrt{3}}{16}a^2$

iv. If a, a and b are length sides of an isosceles triangle, then $S = ...$

(a) $\frac{a}{2} + b$ (b) $a + \frac{a+b}{2}$ (c) $a - \frac{a+b}{2}$ (d) $a + \frac{b}{2}$

v. Area of a triangle ABC with $a = 20, b = 30, \gamma = 90°$ is:

(a) 0 (b) 30 (c) 300 (d) 600

vi. For an equilateral triangle, $r : r_1 : R = \ldots$

(a) 1 : 2 : 3 (b) 3 : 2 : 1 (c) 1 : 1 : 2 (d) 1 : 3 : 2

vii. Radius of circum-circle for sides 6, 8, 10 is:

(a) 6 (b) 5 (c) 4 (d) 2

viii. If $a = 5, b = 10, c = 20$ are sides of a triangle ABC, then angle α is:

(a) not possible (b) acute (c) obtuse (d) 0°

ix. Radius of circum-circle $R = \ldots$

(a) $\frac{a}{2}\sec\frac{\alpha}{2}$ (b) $\frac{b}{2}\sec\frac{\beta}{2}$ (c) $\frac{c}{2}\csc\frac{\gamma}{2}$ (d) $\frac{c}{2}\cos\frac{\gamma}{2}$

x. If in a triangle ABC, $a = b = c$, then $\tan\frac{\alpha}{2} = \ldots$

(a) $\sqrt{\frac{S-a}{a}}$ (b) $\sqrt{\frac{S-b}{b}}$ (c) $\sqrt{\frac{S-c}{c}}$ (d) all (a), (b) & (c)

xi. Shadow of a man 5.6ft tall is making an angle of elevation of 45° with the Sun. What is the length of shadow?

(a) 2.8ft (b) 5.6ft (c) 8.4ft (d) 11.2ft

xii. $R(\sin\alpha + \sin\beta + \sin\gamma) = \ldots$

(a) S (b) S^2 (c) $\frac{1}{S}$ (d) $\frac{1}{S^2}$

Solve the following oblique triangles.

(i) $a = 8, \alpha = 15°, \beta = 20°$

(ii) $a = 34, b = 41, \alpha = 115°$

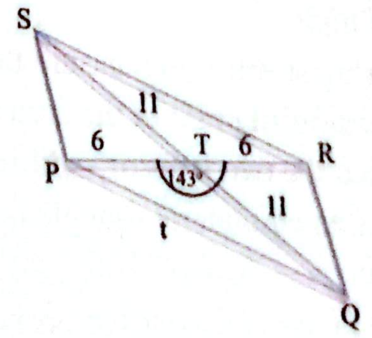
(iii) $b = 17.3, c = 14.5, \beta = 98.2°$

(iv) $c = 55.4, \alpha = 115°, \beta = 45°$

(v) $a = 34, c = 48, \beta = 108°$

(vi) $a = 44, b = 33, c = 55$

The diagonals of a parallelogram measure 12 centimeters and 22 centimeters and intersect at an angle of 143 degrees. Find the length of the longer sides of the parallelogram.



4. Usman and Abubakar follow a triangular path when they take a walk. They walk from home for 1.5*km* along a straight road, turn at an angle of 100°, walk for another 0.9km and then return home.
   a. Find the length of the last portion of their walk.
   b. Find the total distance that they walk.

5. A kite is in the shape of a quadrilateral with two pair of congruent adjacent sides. The lengths of two sides are 20.0 inches and the lengths of the other two sides are 35.0 inches. The two shorter sides meet at an angle of 115°.
   a. Find the length of the diagonal between the points at which the unequal sides meet.
   b. Using the answer to part **a**, find the measure of the angle at which the two longer sides meet.

6. Three streets intersect in pairs enclosing a small triangular park. The measures of the distances between the intersections are 55 *m*, 63 *m*, and 77 *m*. Find the area of the park.

7. The base of an isosceles triangle measures 14.5 centimeters and the vertex angle measures 110°.
   a. Find the measure of one of the congruent sides of the triangle.
   b. Find the perimeter of the triangle.

8. Three streets intersect in pairs enclosing a small park. Two of the angles at which the streets intersect measure 80° and 60°. The length of the longest side of the park is 90 m. Find the lengths of the other two sides of the park.

9. Aamir wants to draw a parallelogram with the measure of one side 12 *cm*, the measure of one diagonal 10 *cm* and the measure of one angle 120°. Is this possible? Explain why or why not. (Hint: Use law of sines.)

10. The angle of depression from an observer in an apartment complex to a pipe on the building next door is 55°. From a point five stories below the original observer, the angle of elevation to the pipe is 20°. Find the distance from each observer to the pipe and the distance from the pipe to the apartment complex. It is given that one story of a building is 9 feet high.

11. A geologist wants to measure the diameter of a crater. From her camp, it is 4 miles to the northern-most point of the crater and 2 miles to the southern-most point. If the angle between the two lines of sight is 117°, what is the diameter of the crater?

12. Side of an equilateral triangle is 6cm long. Find the circumference of circum-circle and ex-circle.

13. Use the Law of Cosines to prove that if the angle between two congruent sides of a triangle measures 60°, the triangle is equilateral.

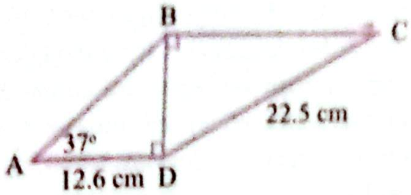
14. Prove that: $\frac{1}{r^2}+\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}=\frac{a^2+b^2+c^2}{\Delta^2}$

Prove that: (i) $r_1 = a\cos\frac{\beta}{2}\cos\frac{\gamma}{2}\sec\frac{\alpha}{2}$ (ii) $r_2 = b\cos\frac{\alpha}{2}\cos\frac{\gamma}{2}\sec\frac{\beta}{2}$

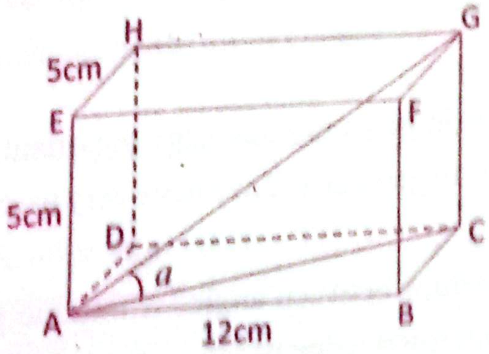
(iii) $r_3 = c\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\sec\frac{\gamma}{2}$

A mountain climber is on top of a mountain that is 680 *m* high. The angles of depression of two points on opposite sides of the mountain are 48° and 32°. How long would a tunnel be that runs between the two points?

Find the measure of $\angle C$ in the adjoining figure.



Sides of a square prism are 12cm, 5cm and 5cm long as shown below. Find the measure of angle *a*.



09

Chord and Arcs of a Circle

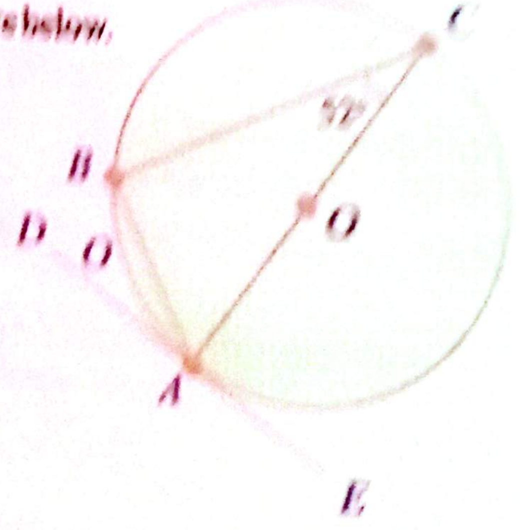
After studying this unit students will be able to:

- One and only one circle can pass through three non-collinear points.
- A straight line drawn from the centre of a circle to bisect the chord is perpendicular to the chord.
- Perpendicular drawn from the centre of a circle on a chord, bisects it.
- Two congruent chords of a circle are equidistant from the centre.
- Two chords of a circle equidistant from the centre are congruent.
- If two arcs of a circle (or of congruent circles) are congruent, then the corresponding chords are equal.
- If two chords of a circle (or of congruent circles) are equal, then their corresponding arcs (minor, major or semicircular) are congruent.
- Equal chords of a circle (or of congruent circles) subtend equal angles at the centre (at the corresponding centres).
- If the angles subtended by two chords of a circle (or congruent circles) at the centre (corresponding centres) are equal, the chords are equal.
- Apply concepts of chords and arcs to real life word problems such as decorative features, archery, bridges and roller coaster track.

Circle theorems are statements in geometry that state important results related to circles for a
used to solve various questions in geometry. They have very useful applications within fields of
engineering.

These theorems show relationships between angles within the geometry of a circle. We can u
these theorems along with prior knowledge of other angle properties to calculate missing angl
without the use of a protractor.

Can you find the value of angle in the figure below.



Chords of a Circle

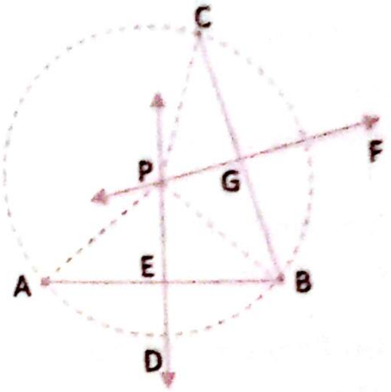
Definition

A line segment joining two points of a circle is called chord of that circle.

Theorem 9.1

Statement: One and only one circle can pass through three non collinear points.

Class Activity:



(i) Take three non-collinear points A, B and C on the paper and join them to obtain the triangle.

(ii) Draw right bisectors of the sides. They will meet at only one point P (say).

(iii) With centre P, draw a circle of radius PA.

What do you notice? Does the circle pass through B and C also?

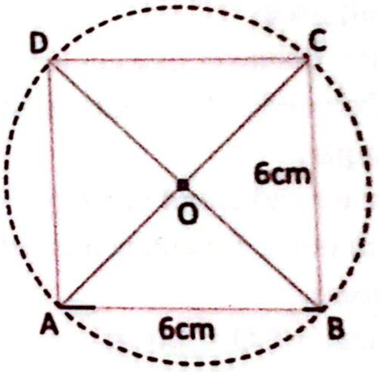
Corollaries: (i) One and only one circle can pass through two points lying on the same line.

(ii) Two circles cannot intersect each other at more than two points.

Example:

Construct a square with sides 6 cm long. Show that one and only one circle can be drawn through vertices of the square. Measure the radius of the circle.

Solution:



(i) Construct a square ABCD having side equal to 6 cm long.

(ii) Draw diagonals $\overline{AC}$ and $\overline{BD}$.

(iii) The two diagonals bisect each other at point O.

(iv) With centre O, construct a circle with radius OA.

(v) You will see that the circle will also pass through vertices B, C and D.

As diagonals $\overline{AC}$ and $\overline{BD}$ bisect each other. Therefore:

$$OA = OB = OC = OD$$

Furthermore, there is only one centre O of the circle, therefore we can say that only one circle can be drawn through vertices of the square.

Now in right triangle ABC:

$(AC)^2 = (AB)^2 + (BC)^2$

$(AC)^2 = (6)^2 + (6)^2 = 36 + 36 = 72$

Taking square root on both sides, we have:

$AC = \sqrt{72} = 6\sqrt{2}$ cm

Radius $= OA =$ half of $AC = 3\sqrt{2}$ cm

Key Fact:

If two chords of a circle do not pass through the centre, they cannot bisect each other.

Theorem 9.2

Statement: A straight line drawn from the centre of a circle to bisect the chord is perpendicular to the chord

Class Activity:

(i) Draw a chord AB of a circle with centre O.

(ii) Take a point C on the chord such that AC = BC.

(iii) Join O to C and measure ∠OCA and ∠OCB.

Are the angles equal? Yes! they are.

∴ ∠OCA = ∠OCB i.e., $\overline{OC} \perp \overline{AB}$

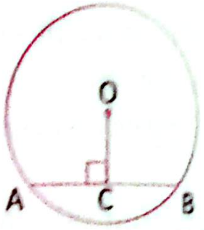
Corollaries:

(i) If a diameter or radial segment of a circle bisects the chord, it is perpendicular to the chord.

(ii) The diameter or radial segment of a circle passes through the mid points of two paralle chords of a circle.

Theorem 9.3

Statement: Perpendicular drawn from the centre of a circle on a chord, bisects it.



Class Activity:

(i) Draw a chord AB of a circle with centre O.

(ii) Take a point C on the chord such that ∠OCA = 90°.

(iii) Measure AC and BC.

Are they equal?

Yes! they are. i.e., AC = BC

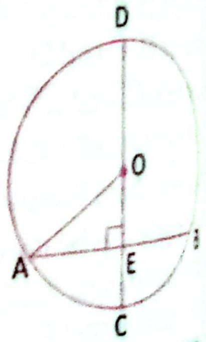
Corollaries:

(i) If a diameter of a circle is perpendicular to the chord, it bisects the chord.

(ii) Perpendicular (right) bisector of a chord of a circle passes through the centre of a circle.

Example:

Diameter and chord of a circle are 20 cm and 16 cm long respectively and are perpendicular to each other. Calculate the distance of chord form the centre.



Solution:

Diameter = CD = 20 cm

Radius = OA = 10 cm

Chord = AB = 16 cm

As the diameter and chord are perpendicular to each other.

Therefore, diameter bisects the chord.

Hence, AE = BE = 8 cm

Now in right triangle AEO:

$(AE)^2 + (OE)^2 = (OA)^2 \Rightarrow (8)^2 + (OE)^2 = (10)^2$

$\Rightarrow (OE)^2 = (10)^2 - (8)^2 = 100 - 64 = 36$

$\Rightarrow OE = \sqrt{36} = 6$ cm

∴ The distance of chord from the centre = 6 cm

Key Fact:

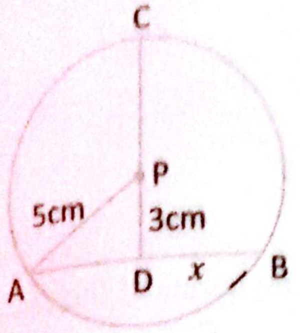
Diameters of a circle bisect each other.

Exercise 9.1

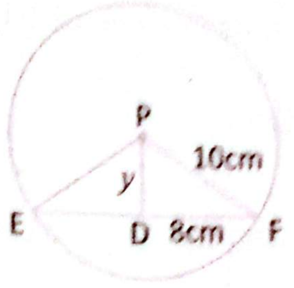
1. Construct an equilateral triangle with side 5 cm long. Show that one and only one circle can be drawn through the vertices of triangle. What is the radius of circle drawn?

2. Given that P is the centre of each of the circles. Find the values of unknown if line segment drawn from the centre of each circle is perpendicular to the chord.

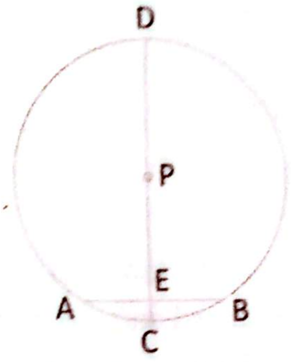
(i)



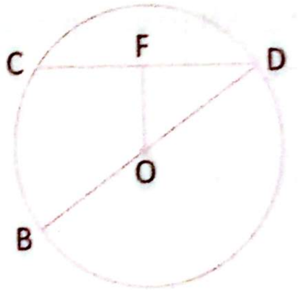
(ii)



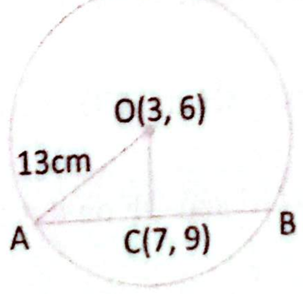
3. Find the length of diameter CD of the circle when AB = 10 cm, PE = 12 cm. Also find CE.



4. In the following diagram, OB = 15 cm, OF = 9 cm. Find the length of chord CD given that $\overline{OF} \perp \overline{CD}$.



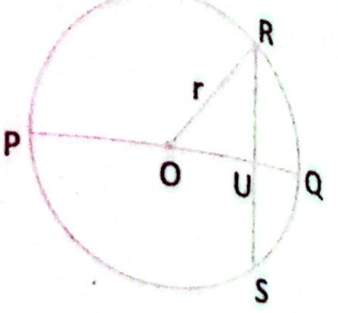
5. Calculate the length of chord AB in the following diagram if $\overline{OC} \perp \overline{AB}$.



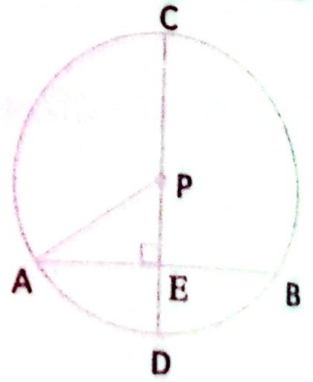
6. Construct a rectangle EFGH such that EF = 6 cm and FG = 4 cm. Draw a circle passing through its vertices and prove that it is the only circle that can be drawn through vertices.

7. Take four non collinear points A, B, C and D such that AB = BC = DB = 4cm. Draw all possible circles that can pass through A, C and D.

8. In the diagram, PU = 16 cm and RS = 10 cm.
   (i) Express OU in terms of radius r.
   (ii) Form an equation in r and solve it to find radius r.
   Hint: 16 – OU = r

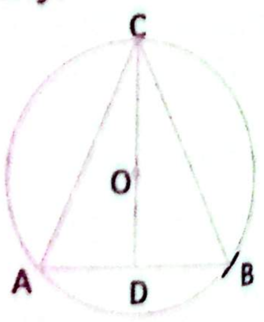


9. In the following diagram, AB = 16 cm, DE 4= cm. Find the radius of the circle.



10. Given that diameter and chord of a circle are 10 cm and 8 cm long respectively. The diameter bisects the chord and the distance between chord and centre of the circle is 3cm. Show that the diameter bisects the chord perpendicularly.

11. In the figure O is centre of the circle. AB is the chord and CD ⊥ AB passes through centre. Show that AC = BC.



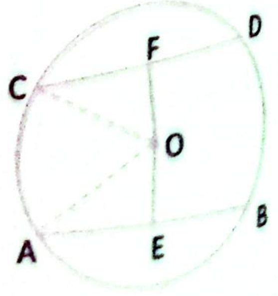
Theorem 9.4

Statement: Two congruent chords of a circle are equidistant from the centre.

Class Activity:

(i) Draw two equal chords AB and CD of a circle as shown in the adjoining figure.

(ii) Draw perpendiculars OE and OF on the chords AB and CD respectively.



(iii) Measure OE and OF.
Are they equal? Yes! they are equal.

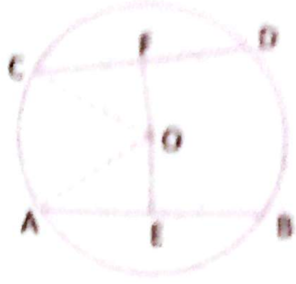
Theorem 9.5

Statement:
Two chords of a circle equidistant from the centre, are congruent.

Class Activity:

(i) Draw two chords AB and CD of a circle such that they are equidistant from the centre as shown in the adjoining figure.

(ii) Measure chords AB and CD.
You will see that they are equal in length.



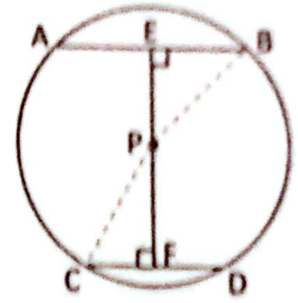
Corollary:

(i) If two chords of a circle are unequal, then longer chord is nearer to the centre and vice versa.

(ii) If two chords of two congruent circles are congruent, then they are equidistant from their respective centres and vice versa.

Example:

In the adjoining figure, $\overline{AB} \parallel \overline{CD}$ such that AB = 24 cm and CD = 10 cm. Find the radius of the circle if distance between chords is 17 cm.



Solution:

Let the radius of circle be r cm with centre P.

Draw $\overline{PE} \perp \overline{AB}$ and $\overline{PF} \perp \overline{CD}$

$\therefore BE = \frac{1}{2}AB \quad \text{and} \quad CF = \frac{1}{2}CD$

As the chords AB and CD are parallel, therefore points E, P and F are collinear and EF = 17 cm

Let PE = x, then PF = $17 - x$

Join P to B and C, then PB = PC = r

Now, PBE and PCF are right angled triangled. Therefore:

$(PB)^2 = (PE)^2 + (EB)^2$ and $(PC)^2 = (PF)^2 + (FC)^2$

$r^2 = x^2 + 12^2$ ......(i) and $r^2 = (17 - x)^2 + 5^2$ ......(ii)

$\Rightarrow x^2 + 12^2 = (17 - x)^2 + 5^2$ ..... [comparing (i) and (ii)]

$\Rightarrow x^2 + 144 = 289 + x^2 - 34x + 25$

$\Rightarrow 34x = 170$

$\Rightarrow x = 5$

Now from (i), we have:

Key Facts:
If AB is chord and diameter CD is right bisector of AB, then:
AC = BC and AD = BD

$r^2 = 5^2 + 12^2 = 25 + 144 = 169$

Taking square roots on both sides, we get r = 13.

∴ Radius of circle = r = 13 cm

Example:

Two parallel chords AB and CD of lengths 16 cm and 12 cm respectively are drawn on the opposite sides of centre O of the circle. Find the distance between the chords if the radius of the circle is 10 cm.

Solution:

Radius of circle = OA = OC = 10 cm

AB = 16 cm, CD = 12 cm

Draw $\overline{OE} \perp \overline{AB}$ and $\overline{OF} \perp \overline{CD}$

$\Rightarrow AE = \frac{1}{2}AB = 8$ cm and $CF = \frac{1}{2}CD = 6$ cm

As $\overline{AB} \parallel \overline{CD}$, therefore points E, O and F are collinear.

Now, OAE and OCF are right angled triangles. Therefore;

$(OE)^2 + (AE)^2 = (OA)^2$ and $(OF)^2 + (CF)^2 = (OC)^2$

$\Rightarrow (OE)^2 + (8)^2 = (10)^2$ and $(OF)^2 + (6)^2 = (10)^2$

$\Rightarrow (OE)^2 + 64 = 100$ and $(OF)^2 + 36 = 100$

$\Rightarrow (OE)^2 = 100 - 64 = 36$ and $(OF)^2 = 100 - 36 = 64$

Taking square roots, we get:

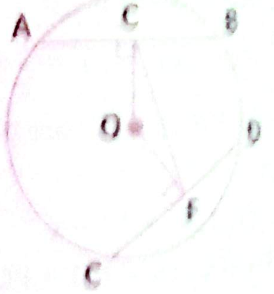
OE = 6 cm and OF = 8 cm

∴ Distance between chords = OE + OF = 6 + 8 = 14 cm

Exercise 9.2

1. In the figure, O is centre of the circle and ∠COF = 150° and $\overline{AB} = \overline{CD}$.

   Find the values of ∠OCF and ∠CFD.

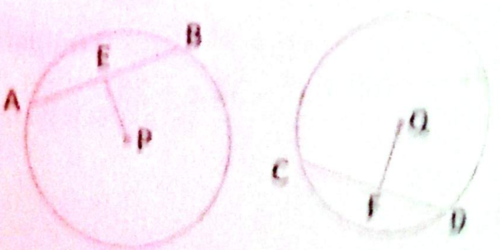


2. Two circles of radius 3 cm each are drawn with centres P and Q. AB and CD are their chords such that AB = CD = 4 cm.

   $\overline{PE} \perp \overline{AB}$ and

   $\overline{QF} \perp \overline{CD}$

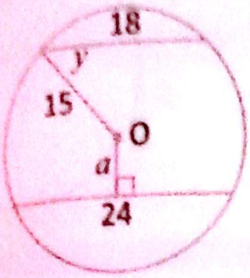
   Find $\overline{PE}$ and $\overline{QF}$. Are they equal?



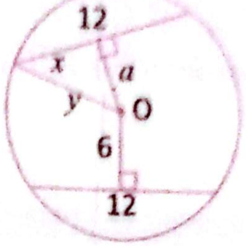
Find the distance between two chords PQ and RS of circle shown in the adjoining figure if: PQ = 6 cm, RS = 8 cm and radius = 5 cm

Given that O is the centre of each of the circles. Find the values of unknown. All dimensions are in cm.

(i)

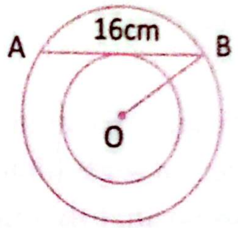


(ii)



Two parallel chords of lengths 24 cm and 12 cm are drawn on the opposite sides of a circle of radius 13 cm. Find the distance between the chords.

AB is chord of a bigger circle of radius 10 cm centered at O. Find the radius of the smaller circle passing through midpoint of $\overline{AB}$. Also find the difference of radii of both circles.



Two parallel chords of lengths 18 cm and 80 cm are drawn on the same side of a circle of radius 41 cm. Find the distance between the chords.

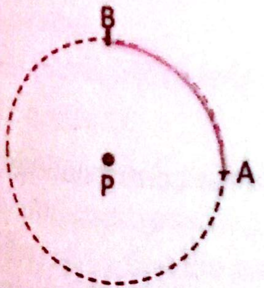
Two parallel chords AB and CD are 4 cm apart and lie on the opposite sides of the centre of a circle. If AB = 2 cm and CD = 6 cm, find the radius of the circle.

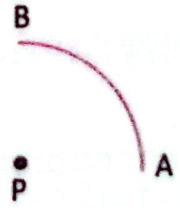
PQ and RS are two parallel chords lying on the same side of the centre of a circle and are 2 cm apart. If PQ = 8 cm and RS = 10 cm, find the radius of the circle.

Arcs of Circles

Arc of a Circle

An arc is a part of a circle. In the figure, AB is an arc of the circle P.
Arc AB is denoted by $\widehat{AB}$.





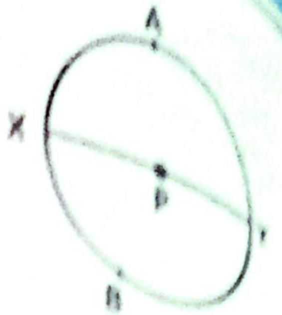
Key Facts

There is a difference between a circle and a circular region. A circular region is union of circle and its interior.

Semicircle

A diameter divides a circle into two equal parts. Each part is called semicircle or half circle. We use three letters to name a semicircle. The first and third letters are end points of diameter.

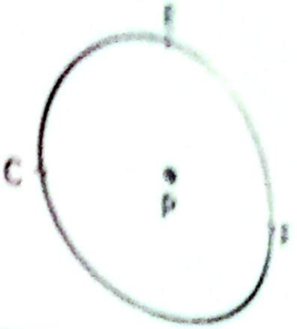
In the figure, $\overset{\frown}{XAY}$ is semicircle. Can you name the other semicircle?



Minor and Major arcs

Minor arc is the arc included in a semicircle while the arc which includes a semicircle is called major arc.

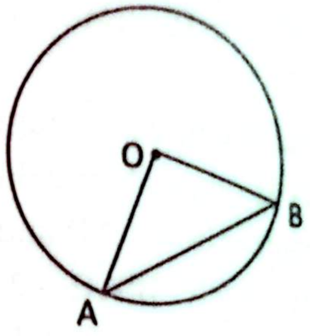
In the Fig. $\overset{\frown}{EF}$ is minor arc while $\overset{\frown}{ECF}$ is major arc.



(i) Major arc of a circle and semicircle are denoted by three letters.
(ii) Minor arc is less than and major arc is greater than the semicircle.

Central Angle

Angle subtended by an arc at the centre of a circle is called central angle of the arc. An angle whose vertex is the centre of circle and whose arms pass through end points of arc is known as central angle of arc.



If an arc is included in a semi-circle, then its central angle is less than 180°. Central angle of semi-circle is straight angle (180°). Central angle of major arc of the circle is greater than 180°.

In the figure ∠AOB is central angle.

Congruent Arcs

Two arcs of a circle (or of two congruent circles) are congruent if their central angles are congruent.

Theorem 9.6

Statement:

If two arcs of a circle (or of congruent circles) are congruent, then the corresponding chords are equal.

Class Activity:

(i) Take two congruent arcs AB and CD of a circle with centre O.

(ii) Join C to D and A to B.

(iii) Measure chords AB and CD.

You will see that AB = CD

Key Fact:

If two arcs of two congruent circles are congruent, then the corresponding chords of both circles are equal.

Theorem 9.7

Statement:

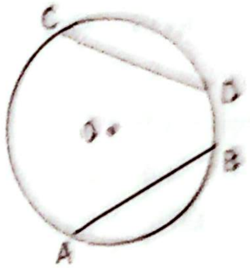
If two chords of a circle (or of congruent circles) are equal, then their corresponding arcs (minor, major or semicircle) are congruent.

Class Activity:

(i) Draw two equal chords AB and CD of a circle with centre O.

(ii) Measure corresponding arcs AB and CD.

They will be congruent.



Corollary:

(i) If two chords of a circle (or of congruent circles) are equal, then their corresponding minor or major arcs are congruent.

(ii) If two chords of a circle (or of congruent circles) are equal, then their corresponding semicircles are congruent.

Theorem 9.8

Statement:

Equal chords of a circle (or of congruent circles) subtend equal angles at the centre (at the corresponding centres).

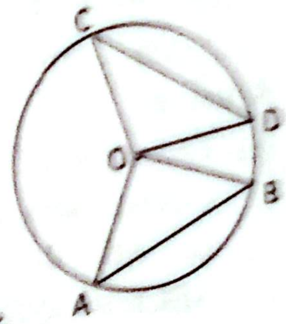
Class Activity:

(i) Draw two equal chords AB and CD of a circle with centre O.

(ii) Join O to A, B, C and D.

(iii) Measure corresponding angles AOB and COD.

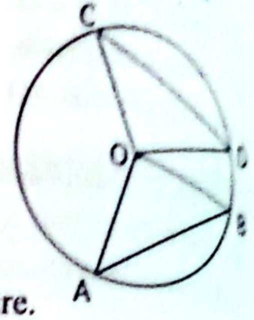
You will observe that the corresponding angles are equal in measure.



Theorem 9.9

Statement:

If the angles subtended by two chords of a circle (or congruent circles) at the centre (corresponding circles) are equal, the chords are equal.



Class Activity:

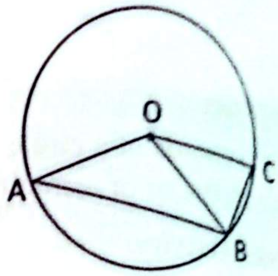
(i) Draw two chords AB and CD of a circle such that
$\angle AOB = \angle COD$

(ii) Measure corresponding chords AB and CD.
What do you notice?

You will observe that the corresponding chords are equal in measure.

Example:

In the figure, $\widehat{AB} = 4\,\widehat{BC}$ and $\angle AOB = 120^\circ$, find:



(a) $\angle BOC$ (b) $\angle OBC$ (c) $\angle OBA$

(d) $\angle OAB + \angle ABC + \angle BCO + \angle COA$

Solution:

(a) Let $\angle BOC = x$, then $\angle AOB = 4x$ ... $(\widehat{AB} = 4\,\widehat{BC})$

But $\angle AOB = 120^\circ$ .... (Given)

$\therefore\ 4x = 120^\circ \Rightarrow x = 30^\circ$

Hence, $\angle BOC = 30^\circ$

(b) ΔOBC is an isosceles triangle. (OB = OC = radii)

$\Rightarrow \angle OBC = \angle OCB$ (opposite angles of congruent sides of triangle)

$\therefore\ \angle OBC = \frac{1}{2}(180^\circ - 30^\circ) = 75^\circ$

(c) ΔOAB is an isosceles triangle. (OA = OB = radii)

$\Rightarrow \angle OAB = \angle OBA$ (opposite angles of congruent sides of triangle)

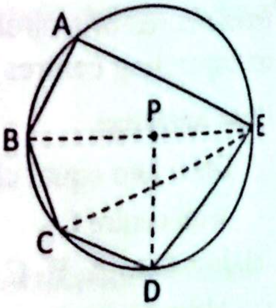
$\therefore\ \angle OBA = \frac{1}{2}(180^\circ - 120^\circ) = 30^\circ$

(d) $\angle OAB + \angle ABC + \angle BCO + \angle COA = 30^\circ + (30^\circ + 75^\circ) + 75^\circ + (30^\circ + 120^\circ)$
$= 30^\circ + 105^\circ + 75^\circ + 150^\circ = 360^\circ$

Example:

Diagram shows a pentagon inscribed in a circle with centre P.

If $\widehat{AB} = \widehat{BC} = \widehat{CD}$ and $\angle ABC = 140^\circ$, then find:



(a) $\angle AEB$ (b) $\angle AED$ (c) $\angle BPD$

Solution:

(a) Given that, $\widehat{AB} = \widehat{BC} = \widehat{CD}$, then:

Corresponding chords are equal in length.

i.e., AB = BC = CD

ABCE is a cyclic quadrilateral.

$\therefore\ \angle ABC + \angle AEC = 180^\circ$

⇒ $\angle AEC = 180^\circ - \angle ABC = 180^\circ - 140^\circ = 40^\circ$

⇒ $\angle AEB + \angle BEC = 40^\circ$ (angle sum postulate)

⇒ $2\angle AEB = 40^\circ$ $(\angle AEB = \angle BEC)$

⇒ $\angle AEB = 20^\circ$

(b) As, $AB = BC = CD$ (given)

$\angle AEB = \angle BEC = \angle CED = 20^\circ$

(Equal chords subtend equal angle at a point on the circumference.)

Now, $\angle AED = \angle AEB + \angle BEC + \angle CED = 20^\circ + 20^\circ + 20^\circ = 60^\circ$

If two arcs of a circle are unequal, then chord and angle corresponding to greater arc are greater.

(c) $\angle BPD = 2\angle BED$ (Angle at the centre is double the angle at a point on circumference.)

$= 2 \times 40^\circ = 80^\circ$

Example:

In the figure, chords $\overline{EF} \parallel \overline{GH}$. Prove that:

$\widehat{EG} = \widehat{FH}$

Solution:

Draw $\overline{EF}$ and $\overline{FG}$ passing through centre of circle P.

$\angle EPG = 2\angle EFG$ (Angle at the centre is double the angle at a point on the circumference.)

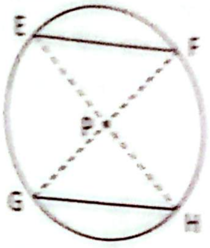
$\therefore$ $\angle EFG = \frac{1}{2}\angle EPG$ (i)

Similarly, $\angle FGH = \frac{1}{2}\angle FPH$ (ii)

But, $\angle EFG = \angle FGH$ (alternate angles)

$\therefore$ $\angle EPG = \angle FPH$ [from (i) and (ii)]

Hence, $\widehat{EG} = \widehat{FH}$ (Arcs subtending equal angles at the centre of circle are equal.)

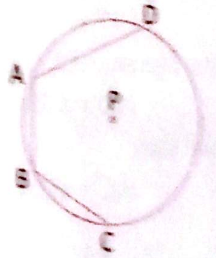


Check Point:

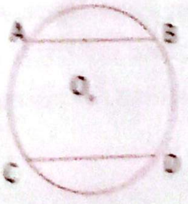
Prove that non-parallel sides of a cyclic trapezium are equal.

Exercise 9.3

1. In the figure, chord AB = chord BC.
   (i) Is $\widehat{AB} = \widehat{BC}$?
   (ii) Is $\angle APB = \angle BPC$?
   (iii) If $\widehat{AB} < \widehat{AD}$, then what is relation between corresponding chords?



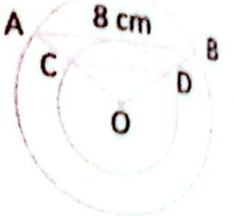
2. In the figure,
   $\widehat{AC} = \widehat{BD}$
   Prove that: $\overline{AB} \parallel \overline{CD}$



3. In the given figure, chords $\overline{AB}$ and $\overline{CD}$ intersect at P.
If AB = CD, then prove that:
$\widehat{AB} = \widehat{CD}$

4. Consider two circles of radii 2 cm and 4 cm centered at P and Q respectively. $\widehat{AB}$ and $\widehat{CD}$ are their arcs such that ∠APB = ∠CQD.
   - i. Are $\widehat{AB}$ and $\widehat{CD}$ congruent?
   - ii. What is relation between the lengths of $\widehat{AB}$ and $\widehat{CD}$?
   - iii. Show that ΔAPB ~ ΔCQD

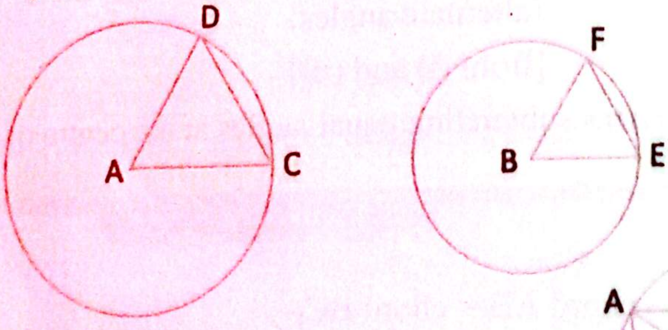
5. O is centre of two concentric circles of radii 3 cm and 6 cm. OA and OB are radii of larger circle and they meet smaller circle at C and D respectively.
   - i. Are $\overline{AB}$ and $\overline{CD}$ parallel?
   - ii. If AB = 8 cm, then find CD.
   - iii. Find distance between $\overline{AB}$ and $\overline{CD}$.



6. The length of an arc of a circle is 4 cm and its central angle is of measure 60°. Find central angle of the arc whose length is 8 cm.

7. Given that M is a point on a circle centered at P. If M is equidistant from the two radii PE and PF, show that $\widehat{ME} = \widehat{MF}$.

8. Consider two circles centered at A and B respectively. $\overline{CD}$ and $\overline{EF}$ are their chords such that CD = 12 cm, EF = 8 cm and ∠CAD = ∠EBF. Find radius of circle A if radius of circle B is 4 cm.

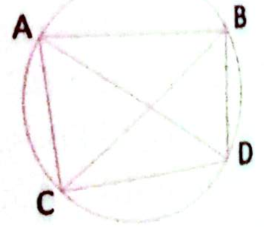


9. In the figure, ABDC is cyclic quadrilateral such that:

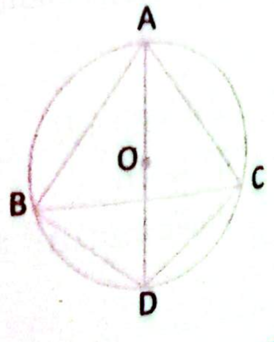
AB = CD.

Prove that:

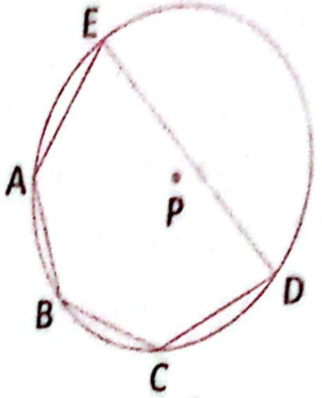
AC = BD



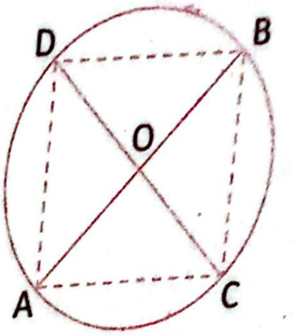
10. In the given figure, ABC is an isosceles triangle inscribed in a circle centered at O.
Prove that $\overline{AD}$ bisects ∠BDC.



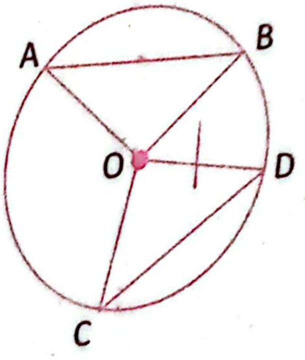
11. ABCDE is a pentagon inscribed in a circle.
If AB = BC = CD, ∠BCD = 130° and ∠BAE = 120°
Find:
(a) ∠ABC (b) ∠CDE
(c) ∠AED (d) ∠EAD



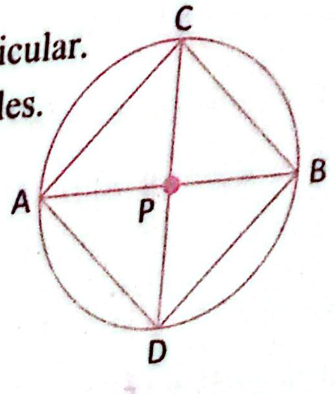
12. Given that O is the centre of a circle. Diameters $\overline{AB}$ and $\overline{CD}$ are perpendicular to each other. Prove that the line segments joining in order the end points of diameters form a square.



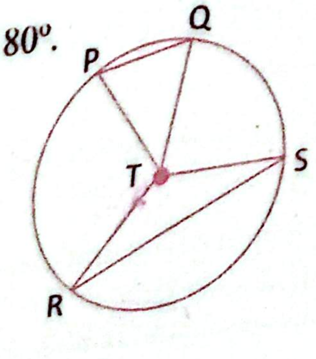
13. In the figure, O is centre of the circle.
AB = CD and ∠OAB = 40°.
Find ∠AOB and ∠COD.
Are the angles equal? If yes, then why?



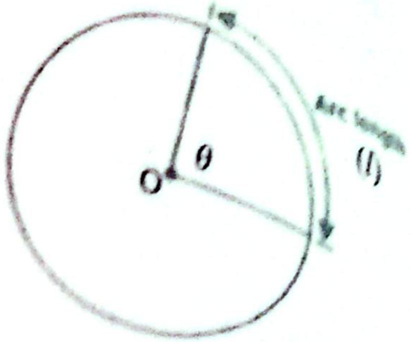
14. In the figure, diameters AB and CD of circle are perpendicular.
(i) Prove that all the four chords make equal central angles.
(ii) Find ∠PCB and ∠ACB.
(iii) Find other angles that are equal to ∠ACB.
(iv) What is the name of inscribed polygon?
(v) How many circles can be drawn through the vertices of polygon formed?



15. In the adjoining figure, RS = 3PQ and ∠PTQ + ∠RTS = 180°.
(i) Find ∠PTQ and ∠RTS.
(ii) What is the ratio between measures of ∠PTQ and ∠RTS?



Applications of Chords and Arcs



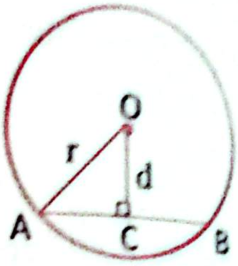
Arc Length

The length of an arc of a circle of radius '*r*' that subtends an angle θ at the center is calculated by the following formula:

Arc length $= l = \frac{\theta}{360^\circ} \times$ circumference of circle

$= \frac{\theta}{360^\circ} \times 2\pi r$

Where 360° is the interior angle subtended by a circle.



Chord Length

Length of a chord of a circle of radius '*r*' lying at a distance '*d*' from the centre is:

In the adjoining figure, length of chord AB is:

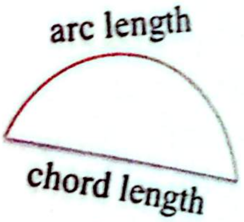
$AB = 2\sqrt{r^2 - d^2}$

Note that AC is half of AB.

Perimeter of a of Semicircle

The perimeter of a semicircle is given by the following formula.

$P = 2 \times$ radius + arc length $= 2r + \pi r$



Perimeter of a Segment of a Circle

Perimeter of segment of a circle is defined as:

P = arc length + chord length

$= \frac{\theta}{360^\circ} \times 2\pi r + 2\sqrt{r^2 - d^2}$

Area of Sector of Circle

Area of the sector of a circle having radius r and central angle θ is:

$A = \frac{\theta}{360^\circ} \times \pi r^2$

Where the angle θ is measured in degrees.

Example:

Find the length of arc and area of sector a circle with central angle of 60° and radius 14cm

Solution: Given that:

$\theta = 60^\circ$ and $r = 14\text{cm}$

Arc length $= \frac{\theta}{360^\circ} \times 2\pi r$

$= \frac{60^\circ}{360^\circ} \times 2 \times \frac{22}{7} \times 14 = 14.67$ cm

Area $= \frac{\theta}{360^\circ} \times \pi r^2 = \frac{60^\circ}{360^\circ} \times \frac{22}{7} \times (14)^2 = 102.67\ \text{cm}^2$

Example:

A circular decoration piece consists of two equal parts as shown in the figure. Find the length of one part of the piece. What is the cost of decoration piece @ Rs.8 per centimetre?

Solution:

Given that $r = 56\text{cm}$, $\theta = 90°$, then:

$$l = \frac{\theta}{360°} \times 2\pi r = \frac{90°}{360°} \times 2 \times \frac{22}{7} \times 56$$

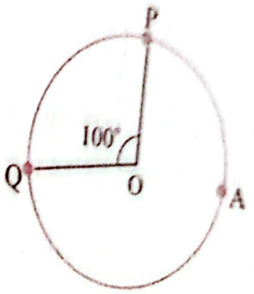
$l = 88$ cm

Length of one part of the decoration piece is $= 2 \times 88 = 176$ cm

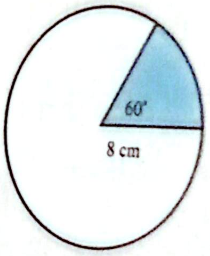
Cost of decoration piece @ Rs.8 per cm $=$ Rs. $2 \times 176 \times 8 =$ Rs. 2,816

Exercise 9.4

1. In the figure, radius of circle is 7cm. Find:
   (i) the length of minor arc $\widehat{PQ}$ and major arc $\widehat{PAQ}$.
   (ii) the circumference of circle.
   Is the sum of lengths of minor arc and major arc equal to circumference of circle?

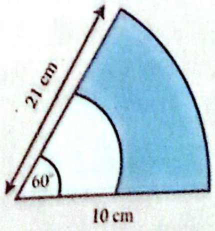


2. Given the radius of circle is 8cm. Find:
   (i) the area of minor sector and major sector.
   (iii) the area of circle.
   Is the sum of areas of minor and major sectors equal to area of circle?



3. Find the distance covered by the tip of hour hand of a clock in 5 hours if the length of hour hand is 10cm.

4. Given is the sector of circle. Find:
   (i) the length of outer minor arc.
   (ii) the length of inner minor arc.
   (iii) the difference of lengths of both arcs.
   (iv) the area of minor sector of bigger circle.
   (v) the area of minor sector of smaller circle.
   (vi) the area of shaded part of sector.



5. Radius of the given circle is 14cm. Find:

(i) length of LM

(ii) length of LM.

(iii) perimeter of segment along chord LM.

(iv) area of triangle OLM.

(v) area of sector OLM.

(vi) area of segment along chord LM.

6. In a circle whose diameter is 12 cm, there is a central angle whose measure is 120. A chord joins the endpoints of the arc cut off by the angle. Find the length of arc along the chord

7. The diameter of a circle is 10 cm long and a chord parallel to it is 6cm long. Find the distance between the chord and the diameter of the circle

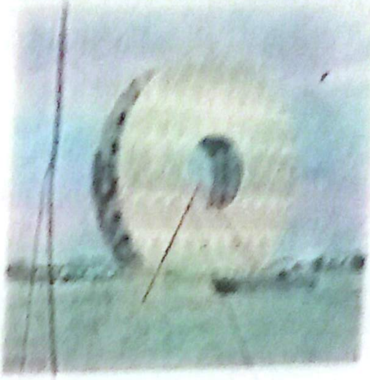
8. Find the area and perimeter of semicircular window if its radius is 2.8feet.

9. The building shown resembles a flying saucer that has landed on its four legs. Find the length of supporting beam AB if it makes an angle of 98° with center and the radius of the beam is 21 m.

10. Figure shows a bridge of semi-circular shape with arc length of 44 m.
Find the length of road constructed below.

11. Find the perimeter of lower portion (below the line PQ) of circular window if:
radius of window = 2.1 feet,
∠POQ = 105°,
and shortest distance between centre O and $\overline{PQ}$ = 1.4 feet.
Also find the area of upper portion (above line PQ) of window.

A circular roller coaster is shown in the figure. Find the distance covered by driver of train between two points making an angle of 80° with the centre of roller coaster if its radius is 21ft.



building shown in the figure is located in Guangzhou province, China. The building's height is 138 m with 33 stories. It has an empty circular core almost 59 m in diameter.

(i) What is the covered area of the building?

(ii) Find the circular length of the part of building that touches ground if it makes an angle of 54° with the centre.

I have Learnt

Solving the problems by using the properties of a circle that:

- One and only one circle can pass through three non collinear points.
- A straight line drawn from the centre of a circle to bisect the chord is perpendicular to the chord and vice versa.
- Two congruent chords of a circle are equidistant from the centre and vice versa.
- If two arcs of a circle (or of congruent circles) are congruent, then the corresponding chords are equal and vice versa.
- Equal chords of a circle (or of congruent circles) subtend equal angles at the centre (at the corresponding centres) and vice versa.
- Applying concepts of chords and arcs to real life word problems such as decorative features, rainbow, bridges and roller coaster track.

MISCELLANEOUS EXERCISE

1. Encircle the correct option in the following.

i. One and only one circle can pass through ...... non collinear points.

(a) 2 (b) 3 (c) 4 (d) 5

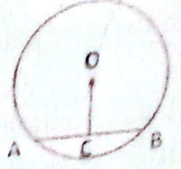
ii. .... number of circles can pass through a point.

(a) 1 (b) 2 (c) 3 (d) infinite

iii. Diameter of circle which bisects the chord is .... to the chord.

(a) collinear (b) parallel (c) perpendicular (d) equal

iv. In the figure, OC = 3 cm, AB = 8 cm.
The radius of circle is:



(a) 4 cm (b) 4.5 cm (c) 5 cm (d) 10 cm

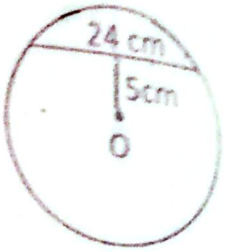
v. Diameter of circle perpendicular to the chord .... the chord.
(a) intersects (b) bisects (c) trisects (d) touches

vi. Two chords which are equidistant from the .... are congruent.
(a) centre (b) diameter (c) circle (d) chord

vii. Two .... which are equidistant from the centre are congruent.
(a) circles (b) diameters (c) segments (d) chords

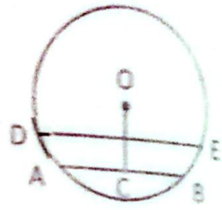
viii. Length of chord of a circle of radius 5 cm is 8 cm. The distance of chord from centre is:
(a) 3cm (b) 4cm (c) 5cm (d) 6cm

ix. What is the radius of circle in the figure?



(a) 26cm (b) 10cm
(c) 5cm (d) 13cm

x. In the figure, AB = 4cm, OC = 3cm. Chord DE passes through mid of OC and is parallel to AB. What is the length of DE?



(a) 4.5 cm (b) 5.6 cm
(c) 6.6 cm (d) 3.3 cm

xi. An angle whose vertex is centre of circle and whose arms pass through end points of an arc is known as_____________ angle.
(a) inscribed (b) central (c) interior (d) exterior

xii. Corresponding arcs of two congruent chords of a circle are ___________.
(a) unequal (b) major (c) congruent (d) minor

xiii. Length of two chords of a circle are in the ratio 1 : 3, and central angle of one arc is 60°, the second arc is ___________.
(a) minor (b) major (c) semicircle (d) circle

xiv. The central angle of quadrant of a circle is ___________.
(a) 30° (b) 45° (c) 60° (d) 90°

xv. If a circle is divided into ten equal arcs, then central angle of each arc is _____.
(a) 10° (b) 36° (c) 60° (d) 90°

xvi. How many central angles of an arc can be drawn?
(a) one (b) two (c) finite (d) infinite

xvii. Two congruent chords of two congruent circles have .... central angles.
(a) different (b) same (c) proportional (d) acute

xviii. Central angle of an arc which includes a semicircle in it is ___________.
(a) < 90° (b) > 90° (c) < 180° (d) > 180°

10

Tangents and Angles of a Circle

After studying this unit students will be able to:

- If a line is drawn perpendicular to a radial segment of a circle at its outer end point, it is tangent to the circle.
- The tangent to a circle and the radial segment joining the point of contact and the centre are perpendicular to each other.
- The two tangents drawn to a circle from a point outside the circle are equal in length.
- If two circles touch externally (internally), the distance between their centres is equal to the sum (difference) of their radii.
- The measure of a central angle of a minor arc of a circle, is double that of the angle subtended by the corresponding major arc.
- Any two angles in the same segment of a circle are equal.
- The angle in a semi-circle is a right angle.
- The angle in a segment greater than semi-circle is less than a right angle.
- The angle in a segment less than semi-circle is greater than a right angle.
- The opposite angles of any quadrilateral inscribed in a circle are supplementary.
- Apply concepts of tangents and angles to real life world problems such as architecture, monuments and pyramids etc.

Tangents have several important applications across various fields, particularly in mathematics, physics, engineering, and geometry. Some applications of tangents are:

- Tangents play an important role in the study of curves. They are lines that touch a curve at a single point, perpendicular to the curve's radius at that point. Tangents are used to define geometric properties such as the radius, diameter, chord, and arc length of circles.
- Tangents are used to design curves in buildings, such as arches and domes.
- Tangents are used to design curves in roads, bridges, and other structures. By using tangents, engineers can create curves that are safe and efficient for travel.
- Tangents are used to analyse the motion of objects. For example, when a car or bike travels down a road, the road becomes tangent at each location where the wheels roll.

Tangent Line

Tangent to a Circle

Tangent Line

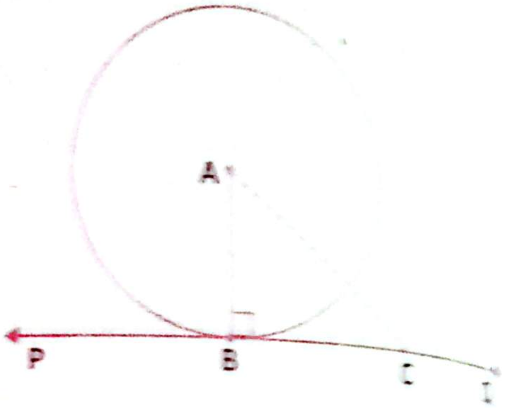
A line which touches the circle at only one point is called tangent line. A tangent line is perpendicular to the radial segment. More over the distance of tangent line from the centre is equal to radius of the circle.

Theorem 10.1

Statement:

If a line is drawn perpendicular to a radial segment of a circle at its outer end point, it is tangent to the circle.

Class Activity:



(i) Draw a radial segment AB ($\overline{AB}$) of a circle as shown in the adjoining figure.

(ii) Draw a perpendicular line PQ ($\overleftrightarrow{PQ}$) at the outer end of AB.

(iii) Check whether $\overleftrightarrow{PQ}$ touches the circle at only one point or not.
Is $\overleftrightarrow{PQ}$ tangent line?

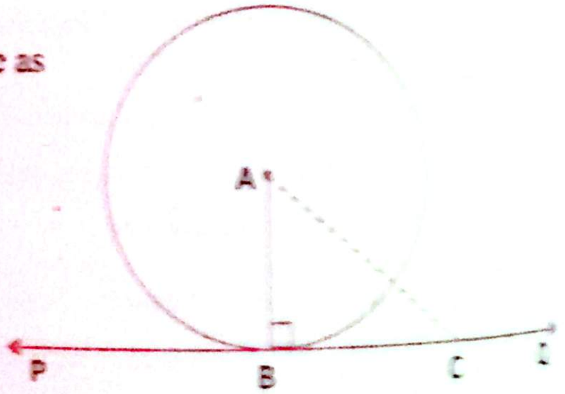
(iv) Take any point C on $\overleftrightarrow{PQ}$ and join C to A.
Why $\overline{AC}$ is not the radial segment?

Theorem 10.2

Statement:

The tangent to a circle and the radial segment joining the point of contact and the centre are perpendicular to each other.

Class Activity:



(i) Draw a radial segment AB ($\overline{AB}$) of a circle as shown in the adjoining figure.

(ii) Draw a tangent $\overleftrightarrow{PQ}$ at point B.

(iii) Measure the angel between $\overline{AB}$ and tangent $\overleftrightarrow{PQ}$. i.e. $\angle ABC$
What is the measure of $\angle ABC$?

(v) Take any point C on $\overleftrightarrow{PQ}$ and join C to A. Which is shortest $\overline{AB}$ or $\overline{AC}$? Why $\angle ABQ = 90^\circ$ and $\angle ACB \neq 90^\circ$?

Corollaries:

(i) The line perpendicular to the tangent at point of contact passes through the centre of circle.

(ii) The lines perpendicular to the diameter of a circle at its end points are tangents to the circle.

he line perpendicular to the tangent of circle from centre passes through point of contact.

e tangents at end points of diameter of a circle are parallel.

Theorem 10.3

ment:

wo tangents drawn to a circle from a point outside the circle are equal in length.

Activity:

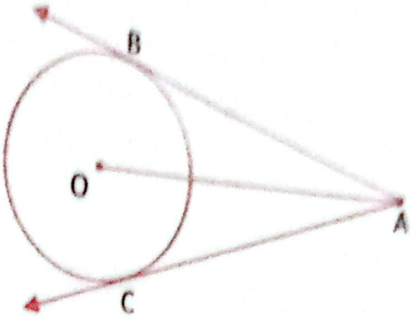
raw a circle with centre O.

ake a point A outside the circle and join with O.

raw two tangents $\overrightarrow{AB}$ and $\overrightarrow{AC}$, where B and C are points on the circle.

easure $\overline{AB}$ and $\overline{AC}$.

re they equal?



llary:

wo tangents drawn to a circle from an outer point make equal angles with the line ent joining the centre and outer point.

Theorem 10.4

ement:

circles touch externally, the distance between their centres is equal to the sum of their ii.

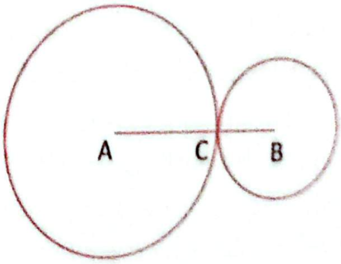
Activity:

raw two circles of radii 2 cm and 1 cm with centres A and B respectively such that the circles touch externally as shown in the figure.

in the centres of both circles by drawing AB.

Measure AB. It will be 3 cm.

cm equal to the sum of radii of both the circles? Can we write AB = AC + BC ?



ollary:

congruent circles touch externally, the distance between their centres is equal to eter of each circle.

Theorem 10.5

ement:

circles touch internally, the distance between their centres is equal to the difference of r radii.

Class Activity:

(i) Draw two circles of radii 3.5 cm and 1 cm with centres A and C respectively such that the circles touch internally as shown in the figure.

(ii) Draw AB.

(iii) Measure AC. It will be 2.5 cm.

Is 2.5 cm equal to the difference of radii of both the circles?

Can we write $AC = AB - BC$?

where AB and BC are radii of both circles.

Corollary:

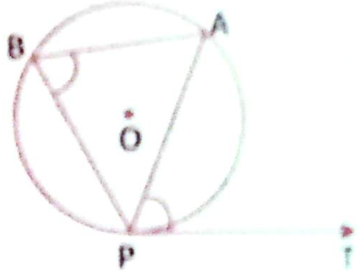
If two congruent circles touch internally, the distance between their centres is equal to zero.

Theorem 10.6 (Alternate Segment Theorem)

Statement:

An angle between a tangent and chord of a circle through the point of contact is equal to the angle in the alternate segment.

Class Activity:



(i) Draw a tangent $\overrightarrow{PT}$ and a chord $\overline{PA}$ at the point of contact T of a circle centered at O.

(ii) Measure $\angle APT$ and note it.

(iii) Draw angle $\angle ABP$ in the alternate segment and measure it.

You will see that $\angle APT = \angle ABP$

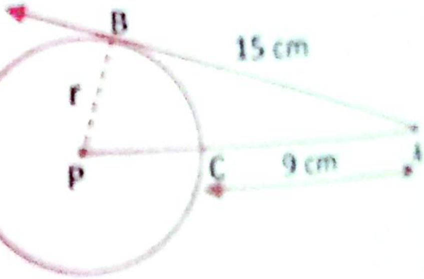
Example:

In the figure, $\overleftrightarrow{AB}$ is tangent to the circle of centre P.

(i) Express the length AP in terms of r.

(ii) What is measure of $\angle ABP$? and why?

(iii) What is the radius of the circle?



Solution:

(i) $AP = AC + PC = (9 + r)$ cm

(ii) $\angle ABP = 90°$ because tangent line and radial segment are perpendicular to each other

(iii) ABP is right angled triangle, therefore using Pythagoras theorem, we have

$$(AP)^2 = (AB)^2 + (BP)^2$$

$\Rightarrow \quad (9 + r)^2 = (15)^2 + r^2 \qquad \Rightarrow \quad 81 + r^2 + 18r = 225 + r^2$

$\Rightarrow \quad 18r = 225 - 81 = 144 \qquad \Rightarrow \quad r = 144 \div 18 = 8$

$\therefore$ Radius of the circle = 8 cm

Example:

In the figure, $\overline{OP}$ and $\overline{OQ}$ are tangents to a circle centered at S. If $\angle POQ = 40°$, then find:

(ii) ∠SPQ (iii) ∠OPQ

right angled triangle, therefore

= 90°

= 30° (Half of 60°, OP is angle bisector of ∠RPQ)

= 90° − 20° = 70°

∠PSO + ∠QSO = 70° + 70° = 140° (∠PSO = ∠QSO)

PS = QS (Radii of same circle)

∠SPQ = ∠SQP (Opposite angles of congruent sides of triangle)

∠SPQ = ∠SQP = x

x + x + ∠PSQ = 180° ⇒ 2x + 140° = 180°

2x = 180° − 140° = 40° ⇒ x = 20°

∠SPQ = ∠SQP = 20°

∠OPQ = ∠OPS − ∠SPQ = 90° − 20° = 70°

Find the measures of angles a and b in the following figure.

and QR are tangents, therefore:

= ∠OQP = 90°

ORPQ is a kite.

45° + 90° + a + 90° = 360°

a = 360° − 90° − 90° − 45° = 135°

is a straight line and P lies on it.

+ ∠QPS = 180° ⇒ 135° + ∠QPS = 180°

∠QPS = 180° − 135° = 45°

PQ = PS (Radii of same circle)

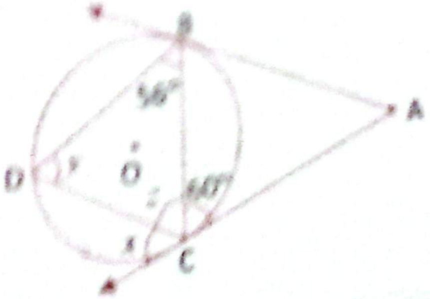
∠PQS = ∠PSQ (Opposite angles of congruent sides of triangle)

b + b + 45° = 180° ⇒ 2b + 45° = 180°

2b = 180° − 45° = 135° ⇒ b = 67.5°

The distance of a point P from the centre O of a circle of radius 6 cm, is 10 cm. Find the length of tangents to the circle from P.

Find the measures of angles x, y and z in following figure.



Solution

We know that

= ∠BCA (Theorem 10.6)

y = 60°

In triangle BCD

60° + 56° + x = 180°

+ 60° + z = 180°

$\Rightarrow z = 180° - 50° - 60°$

Now,

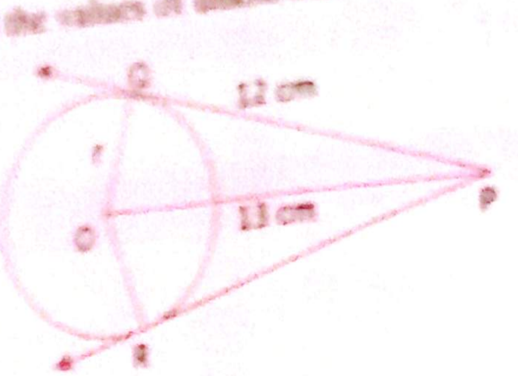
$y = \angle CBD$ (Theorem 10.8)

$\Rightarrow y = 50°$

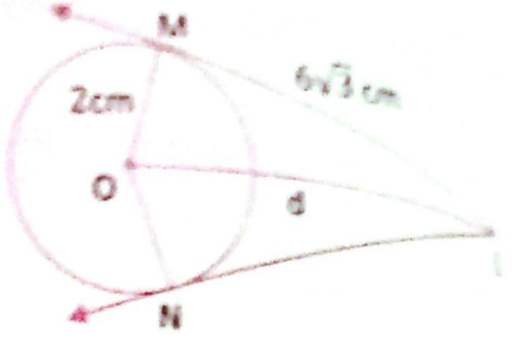
Exercise 10.1

1. Find the values of unknown in the following figures.

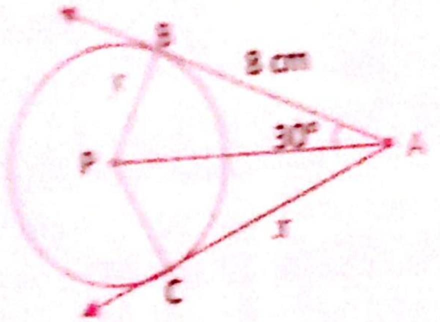
(i)



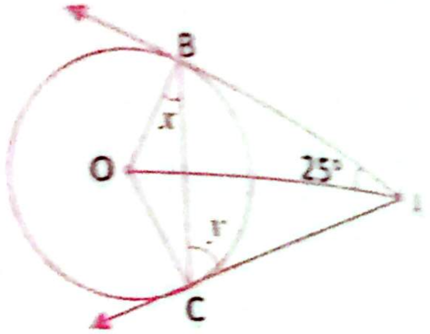
(ii)



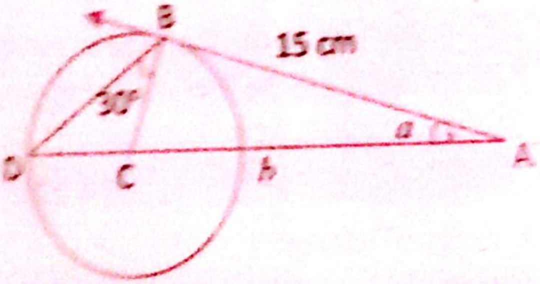
(iii)



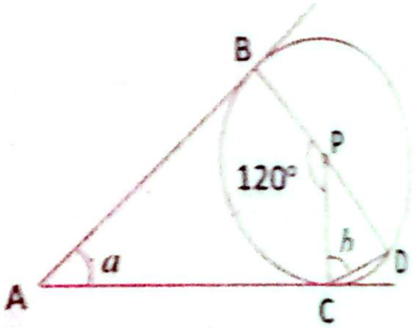
(iv)



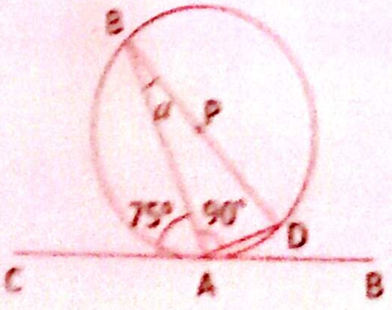
(v)



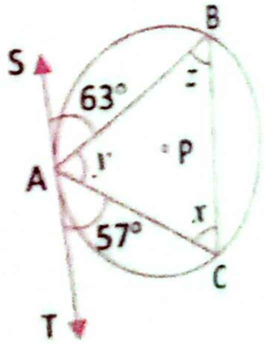
(vi)



(vii)



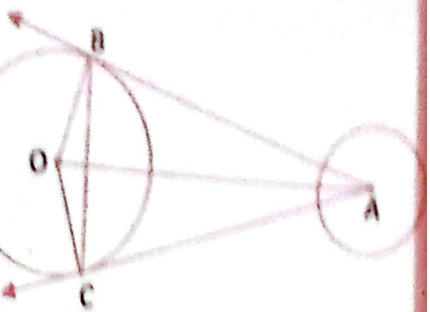
(viii)



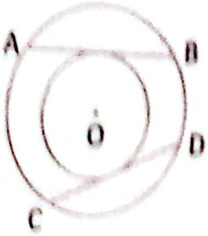
2. In the figure AB and AC are two tangents of a circle with centre O at point B and C respectively. If ∠BOC = 120°, then find:

(i) ∠OBA (ii) ∠OCB

(iii) ∠BAC (iv) ∠ABC

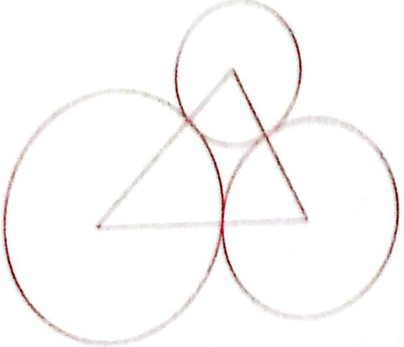


3. O is centre of two concentric circles. AB and CD are chords of outer circle tangent to inner circle. Prove that AB = CD.

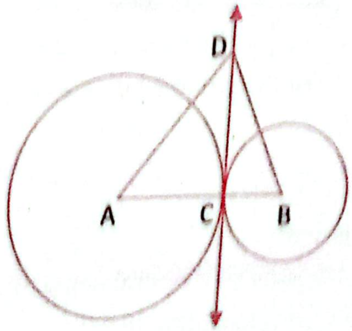


4. Ali is whirling a stone tied with a string of length 3 feet. All of a sudden string is broken and the stone moves along tangent direction of circular path and hits a point 5 feet away from Ali. Find the distance covered by the stone.

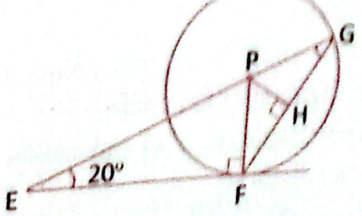
5. Three circles touch externally.
The ratio of distances between centers is 2 : 3 : 4. The perimeter of triangle joining the centers is 36 cm.
Find radius of each circle.



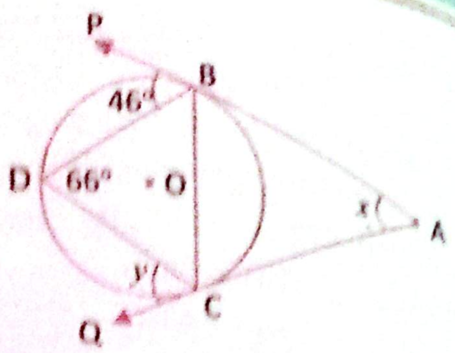
6. In the figure, AD = 5 cm, CD = 4 cm and BD = 4.5 cm. Find the distance between centres of circles.



7. In the adjoining figure, ∠PEF = 20°. H is mid-point of $\overline{FG}$. Find ∠PGH.



8. In the figure, triangle ABC, AB = AC.
Find the values of angles x and y.



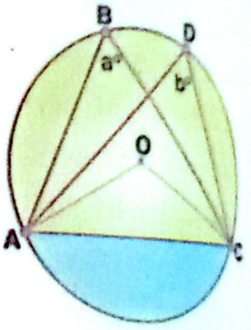
Angle in a Segment of a Circle

Segment of the Circle

A chord of a circle divides the circular region into two parts, called segment of the circle.
The region between chord and minor arc is called minor segment and the region between chord and major arc is called major segment. In the figure, the shaded region is minor segment and remaining region is major segment.

Angle in a Segment

An angle whose vertex is a point on the arc of a circle and whose arms pass through end points of chord, is known as inscribed angle of the arc.
In the adjoining figure, angles a and b are examples of angles in the segment.



Key Fact:

Major segment is greater than semi-circular region and always contains the centre of the circle while minor segment is smaller than the semi-circular region.

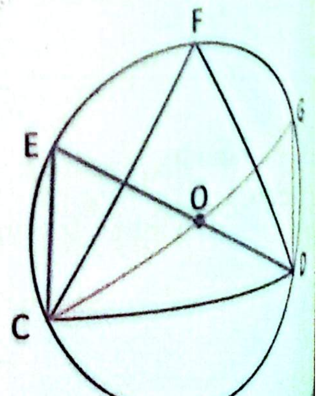
Theorem 10.7

Statement:

The measure of a central angle of a minor arc of a circle, is double that of the angle subtend by the corresponding major arc.

Class Activity:

(i) Draw the circle O and take points E, F and G on the same segment of the circle.
(ii) Measure angles E, F, G and central angle COD.
(iii) Check whether:
$2\angle E = 2\angle F = 2\angle G = \angle COD$



Prove theorem 10.7 by taking central angle ∠COD = 180° and ∠COD > 180° (reflex angle).

Theorem 10.8

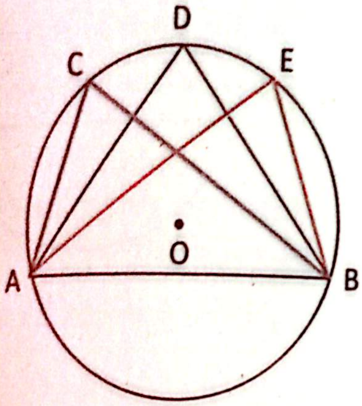
Statement: Any two angles in the same segment of a circle are equal.

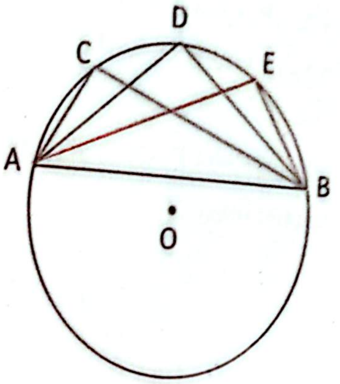
Class Activity:

(i) Draw two circles of different radii.

(ii) Draw a chord AB in both the circles.

(iii) Take three points C, D and E on the same segment and join them with end points of chord AB.





(iv) Complete the following table.

S. No	∠C	∠D	∠E	Result
1.				
2.				

Are ∠C=∠D=∠E ?

Result:

Any number of angles in the same segment of a circle are equal.

Theorem 10.9

Statement:

The angle in a semi-circle is a right angle.

Class Activity:

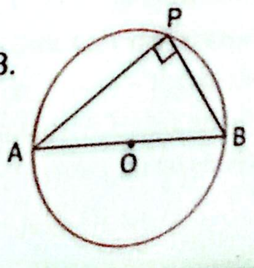
(i) Draw a semicircle O.

(ii) Take a point P on the semi-circle and draw AP and PB.

(iii) Measure angle APB.

What do you notice?

You will see that angle APB is equal to 90°.



Corolary:

If an arc of a circle subtends a right angle at any point of the remaining part of the circle, then the arc is a semi-circle.

Theorem 10.10

Statement:

The angle in a segment greater than semi-circle is less than a right angle.

Class Activity:

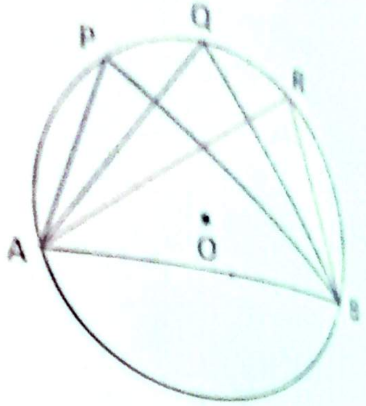
(i) Draw a circle O.

(ii) Draw a chord AB such that the centre lies in the major segment.

(iii) Take three points P, Q and R on the major segment and join them with end points of chord AB.

(iv) Measure ∠P=∠Q=∠R

You will see that angles P, Q, R are less than right angles (acute angles).



Theorem 10.11

Statement:

The angle in a segment less than semi-circle is greater than a right angle.

Class Activity:

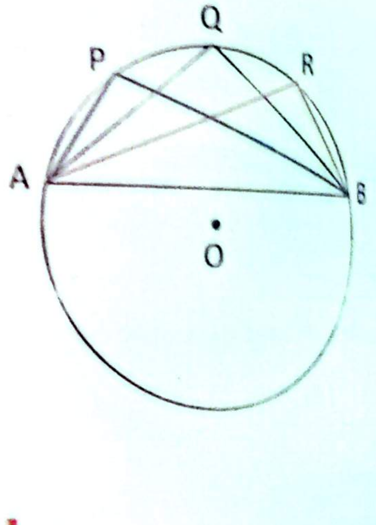
(i) Draw a circle O.

(ii) Draw a chord AB such that the centre lies in the major segment.

(iii) Take three points P, Q and R on the minor segment and join them with end points of chord AB.

(iv) Measure ∠P=∠Q=∠R

You will see that the angles P, Q, R are greater than right angle (obtuse angle)

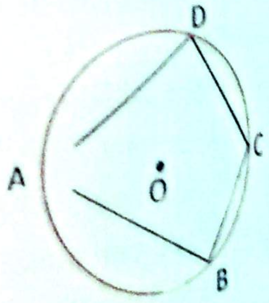


Cyclic Quadrilateral

A quadrilateral inscribed in a circle is called cyclic quadrilateral. Or a quadrilateral having vartices on the circle is called a cyclic quadrilateral.

In the adjoining figure: ABCD is cyclic quadrilateral.

The four points on the circle are called concyclic.



Theorem 10.12

ment:

posite angles of any quadrilateral inscribed in a circle are supplementary. Or sum of opposite angles in a cyclic quadrilateral is 180°.

Activity:

Draw a circle O and inscribe a quadrilateral ABCD in the circle as shown in the adjoining figure.

Measure interior angles A, B, C and D of quadrilateral.

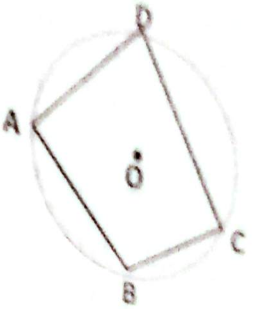
Add ∠A and ∠C.

What is the sum of these angles? Yes! it is 180°.

Again add ∠B and ∠D.

What is the sum of these angles? Yes! it is 180°.

at is sum of opposite interior angles of cyclic quadrilateral?

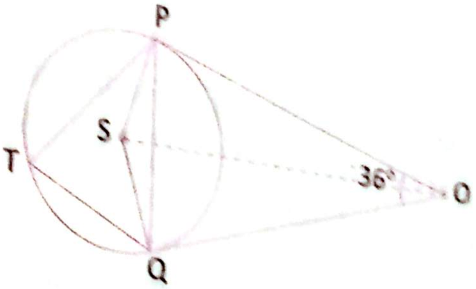


mple:

the figure, $\overline{OP}$ and $\overline{OQ}$ are tangents to a circle ered at S. If ∠POQ = 36°, then find:

∠PSQ (ii) ∠SQP (iii) ∠PTQ

ution:



Join O and S.

As OPS is right angled triangle, therfore:

∠OPS = 90°

Also ∠POS = 18° (Half of 36°. $\overline{OS}$ is angle bisector of ∠POQ.)

∴ ∠PSO = 90° − 18° = 72°

Hence, ∠PSQ = ∠PSO + ∠QSO = 72° + 72° = 144° (∠PSO = ∠QSO)

In ΔPQS, PS = QS (Radii of same circle)

∴ ∠SPQ = ∠SQP (Opposite angles of congruent sides of triangles)

Let ∠SPQ = ∠SQP = x

Then, $x + x$ + ∠PSQ = 180° ⇒ $2x$ + 144° = 180°

⇒ $2x$ = 180° − 144° = 36° ⇒ x = 18°

Hence, ∠SPQ = ∠SQP = 18°

As, the measure of a central angle of a minor arc of a circle, is double that of the angle tended by the corresponding major arc.

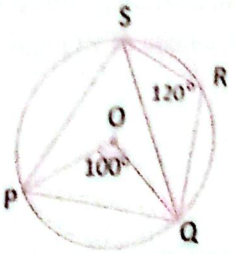
∴ ∠PTQ = $\frac{1}{2}$(∠PSQ) = $\frac{1}{2}$ × 144° = 72°

mple:

the figure, A, B, C and D are points on a circle.

∠POQ = 100°, ∠QRS = 120°, then find:

∠PSQ (ii) ∠OPQ (iii) ∠SPO



Solution:

(i) ∠POQ = 100° (Central angle)

As, 2∠PSQ is angle subtended by the corresponding major arc.

∴ 2∠PSQ = ∠POQ = 100°

⇒ ∠PSQ = 50°

(ii) ΔPQS is isosceles with OP = OQ = radii. Therefore:

∠OPQ + ∠OQP + ∠POQ = 180°

⇒ 2∠OPQ + 100° = 180° (∠OPQ = ∠OQP)

⇒ 2∠OPQ = 180° – 100° = 80°

⇒ ∠OPQ = 80° ÷ 2 = 40°

(iii) PQRS is a cyclic quadrilateral. Therefore:

∠SPQ + ∠SRQ = 180° (Opposite angles in cyclic quadrilateral PQRS.)

⇒ ∠SPQ + 120° = 180° ⇒ ∠SPQ = 180° – 120° = 60°

Now, ∠SPO + ∠OPQ = ∠SPQ ⇒ ∠SPO + 40° = 60°

⇒ ∠SPO = 60° – 40° = 20°

Example:

In the figure, A, B, C, D and E lie on the circumference of a circle. If ∠ABD = 46° and ∠ACB = 70° and ∠EAB = 50°, then find:

(i) ∠DCA (ii) ∠AEB

Solution:

(i) ∠ADB = ∠ACB = 70° (angles in same segments are equal.)

In ΔABD:

∠ABD + ∠ADB + ∠BAD = 180°

⇒ 46° + 70° + ∠BAD = 180° ⇒ ∠BAD = 180° – 46° – 70° = 64°

∠BAD + ∠BCD = 180° (Opposite angles in cyclic quadrilateral ABCD.)

64° + ∠BCD = 180° ⇒ 64° + (70° + ∠DCA) = 180°

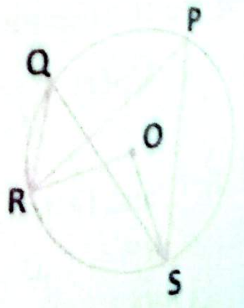
⇒ ∠DCA = 180° – 64° – 70° = 46°

(ii) ∠AEB + ∠ACB = 180° (Opposite angles in cyclic quadrilateral AEBC.)

∠AEB + 70° = 180° ⇒ ∠AEB = 180° – 70° = 110°

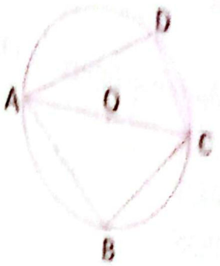
Exercise 10.2

1. In the adjoining figure ∠P = 30°. Find values of ∠Q and ∠ROS.

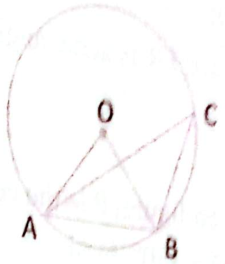


In the adjoining figure ∠BAO = 30°.
Find ∠ABO, ∠AOB, ∠ACB.

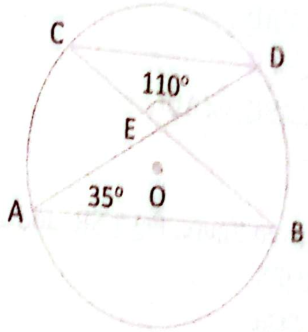
In the adjoining figure ∠BAC = 40° and ∠ACD = 60°. Find ∠ABC, ∠BCA, ∠CAD.



In the figure, AOB is equilateral triangle.
Find the measures of ∠O and ∠C.

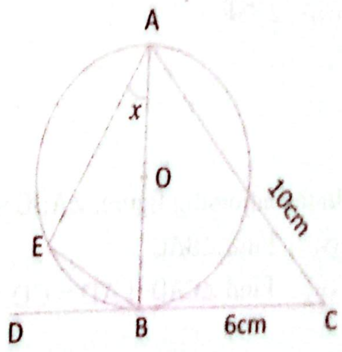


In the figure, $\overline{AB}$ is parallel to $\overline{CD}$.
Find the values of ∠AEB, ∠B, ∠C and ∠D.

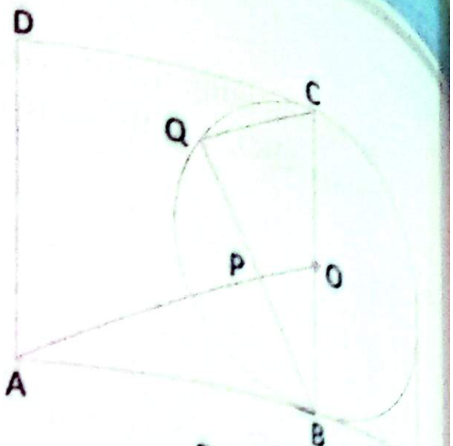


In the figure, $\overline{AB}$ is diameter and $\overline{CD}$ is tangent touching the circle at B. If $x = 30°$, then find:

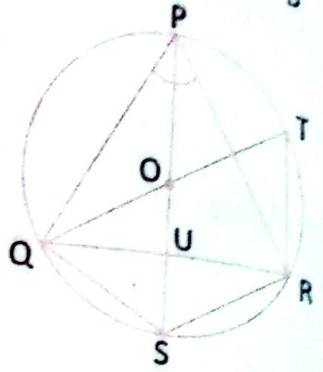
(i) radius of circle
(ii) ∠ABE
(iii) ∠ABC
(iv) ∠EBD



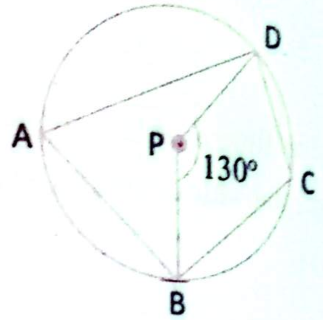
7. In the figure, ABCD is a square having side 10 cm. Triangle BOP is enlarged as triangle BCQ. Find:
(i) Scale factor of enlargement if OP = 2 cm.
(ii) CQ
(iii) ∠BQC and ∠BPO
(iv) AO
(v) AR



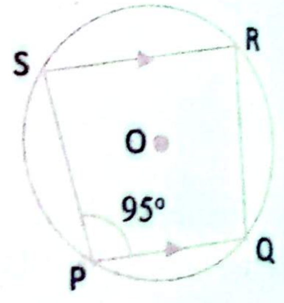
8. In the adjoining figure, ∠QPR = 60°. Find:
(i) ∠QSR
(ii) ∠PRS
(iii) ∠PQS
(iv) ∠QTR
(v) Why ∠QTR is acute angle?



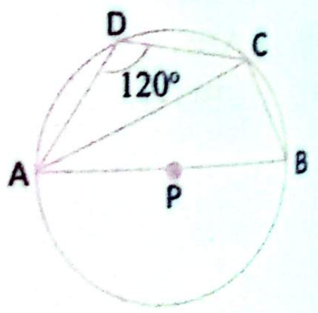
9. In the given figure, P is the centre of the circle and ∠BPD = 130°. Find:
(i) ∠BAD
(ii) ∠BCD
(iii) ∠ABC + ∠ADC



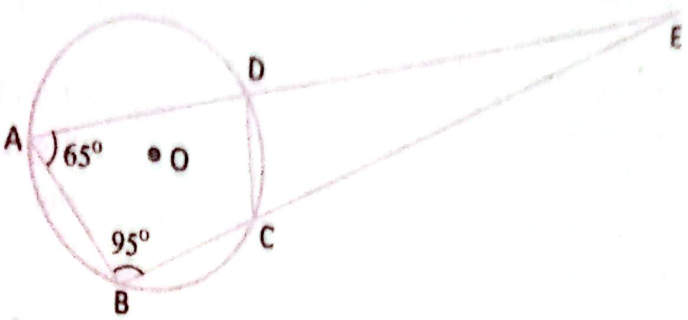
10. In the given figure, $\overline{PQ} \parallel \overline{SR}$ and ∠QPS = 95°. Find:
(i) ∠QRS
(ii) ∠PQR
(iii) ∠PSR



11. In the adjoining figure, ∠ADC = 120°.
(i) Find ∠BAC.
(ii) Find ∠CAD if AD = CD.



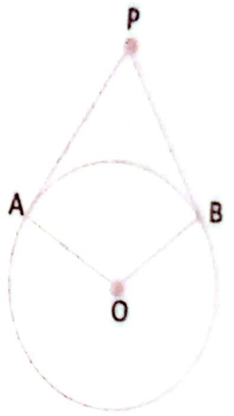
In the figure, ∠ABE = 95° and ∠BAE = 65°.
Find (i) ∠ECD (ii) ∠CDE (iii) ∠CED



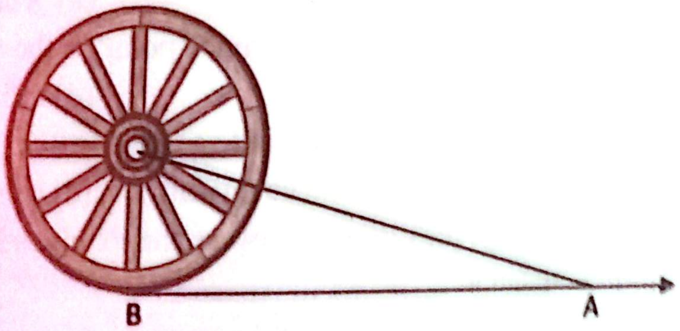
A circular wall clock of radius 1ft is hung by a 4 feet long string with a nail at P.

(i) Find distance between centre of clock and nail.

(ii) How far above the clock is the nail?



A wooden wheel moving on the ground is 99cm away from a point A. Find diameter of the wheel if the distance between centre of wheel and point A is 101 cm. How much distance does the wheel cover in one round?



I have Learnt

- If a line is drawn perpendicular to a radial segment of a circle at its outer end point, it is tangent to the circle.
- The tangent to a circle and the radial segment joining the point of contact and the centre are perpendicular to each other.
- The two tangents drawn to a circle from a point outside the circle are equal in length.
- If two circles touch externally (internally), the distance between their centres is equal to the sum (difference) of their radii.

- The measure of a central angle of a minor arc of a circle, is double that of the angle subtended by the corresponding major arc.
- Any two angles in the same segment of a circle are equal.
- The angle in a semi-circle is a right angle.
- The angle in a segment greater than semi-circle is less than a right angle.
- The angle in a segment less than semi-circle is greater than a right angle.
- The opposite angles of any quadrilateral inscribed in a circle are supplementary.

MISCELLANEOUS EXERCISE-10

1. Encircle the correct option in the following.

i. A tangent line touches the circle at ………. point(s).

(a) 1 (b) 2 (c) 3 (d) 4

ii. A tangent line is …………… to radial segment.

(a) parallel (b) equal (c) perpendicular (d) similar

iii. How many tangents can be drawn on the circle from a point outside the circle?

(a) 1 (b) 2 (c) 3 (d) infinite

iv. If two tangents are drawn on both ends of diameter of a circle, they are:

(a) perpendicular (b) parallel (c) intersecting (d) none

v. Given that radius of a circle is 4 cm. The distance between two tangents drawn at the outer ends of diameter is:

(a) 2 cm (b) 4 cm (c) 6 cm (d) 8 cm

vi. How many tangents can be drawn on the circle from a point on the circle?

(a) 1 (b) 2 (c) none (d) infinite

vii. Two tangents drawn from a point outside the circle are:

(a) perpendicular (b) parallel (c) not congruent (d) congruent

viii. If two circles touch externally, the distance between their centres is equal to sum of ________ of both circle.

(a) radii (b) diameters (c) circumferences (d) area

ix. If two congruent circles touch externally, the distance between their centres is equal to ________ of a circle.

(a) radius (b) diameter (c) chord (d) sector

x. If two circles of radii 1.4cm and 2.5 cm touch internally, the distance between their centres is equal to:

(a) 1.4 cm (b) 2.5 cm (c) 1.1 cm (d) 3.9 cm

xi. Angle subtended by an arc at centre of circle is called ………… angle.

(a) reflex (b) inscribed (c) straight (d) central

xii. An angle inscribed in a half circle is:

(a) 0° (b) 45° (c) 90° (d) 180°

xiii. If central angle of minor arc of a circle is 100°, angle inscribed in corresponding major arc is:

(a) 200° (b) 100° (c) 75° (d) 50°

xiv. Central angle of minor arc of a circle is:

(a) less than 360° (b) less than 180°

(c) greater than 360° (d) greater than 180°

xv. Central angle of major arc of a circle is:

(a) less than 90° (b) less than 180°

(c) greater than 90° (d) greater than 180°

xvi. All angles in a same segment of a circle are:

(a) equal (b) acute (c) obtuse (d) supplementary

xvii. If ABCD is a cyclic quadrilateral and ∠A = 60°, then ∠C = ...

(a) 180° (b) 150° (c) 120° (d) 60°

xviii. An exterior angle of a cyclic quadrilateral is .... the opposite interior angle.

(a) greater than (b) less than (c) equal to (d) supplement of

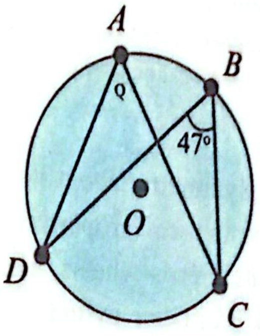
xix. Inscribed angle of a quadrant of a circle is:

(a) 45° (b) 90° (c) 145° (d) 130°

xx. In the figure, angle θ is equal to:

(a) 45° (b) 94°

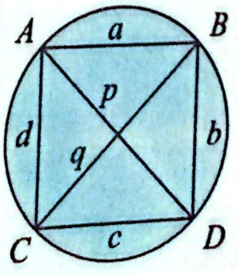
(c) 23.5° (d) 47°



In a circle whose diameter is 12 cm, there is a central angle whose measure is 90°. A chord joins the endpoints of the arc cut off by the angle. Find the length of the chord.

The diameter of a circle is 20 cm long and a chord parallel to it is 12cm long. Find the distance between the chord and the center of the circle.

In a cyclic quadrilateral ABCD with sides $a = AB$, $b = BC$, $c = CD$, and $d = DA$, and diagonals $p = AC$ and $q = BD$. We can express diagonals in terms of the sides as:
$p \times q = (a \times c) + (b \times d)$.
Find q if $a = 4$cm, $b = 5$cm, $c = 6$ cm, $d = 3$ cm and $p = 8$ cm.



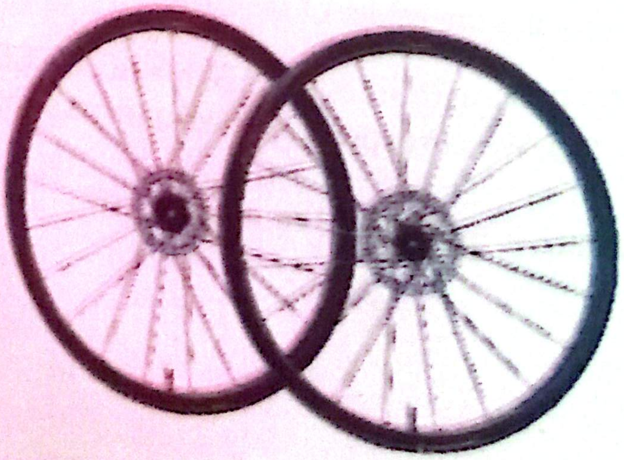
11

Practical Geometry of Circles

After studying this unit students will be able to:

- Locate the centre of a given circle.
- Draw a circle passing through three given non-collinear points.
- Complete the circle (i) by finding its centre, (ii) without finding its centre, when a part of its circumference is given by
- Draw a tangent to a given arc, without using the centre, through a given point P when:
  (i) P is middle point of the arc (ii) P is at the end of arc (iii) P is outside the arc.
- Draw a tangent to a given circle from a point P when:
  (i) P is on the circumference (ii) P is outside the circle.
- Draw two tangents to a circle meeting each other at a given angle.
- Draw (i) direct common tangents or external tangents (ii) transverse common tangents or internal tangents, to two equal/unequal circles.
- Draw a tangent to two unequal (i) touching circles (ii) intersecting circles.
- Apply concepts of practical geometry of circles to real life word problems such as athletic tracks, recreational parks, Ferris wheels and mechanical machines.

A circle is symmetric about its centre and any of its diameters. Architects often use the circle's symmetrical properties when designing athletic tracks, recreational parks, buildings, roundabouts, Ferris-wheels, etc. Artists and painters find the circle almost indispensable in their work. Circular cylinders are used to print newspapers. Engineers exploit the circle's symmetrical properties as seen by the use of the circle in making watches, clocks, bicycles, cars, trains, ships, aero-planes, radios, telephones, trolleys, wheel-barrows, air-conditioners, rockets etc. Civilizations have progressed dramatically because of the invention of the circular wheel.



Construction of Circles

Locate the Centre of a Circle

en: A circle without centre.

quired: To locate the center of circle.

ps of Construction:

Take three points A, B and C on a circle.
Join these points to obtain a triangle ABC.
Draw right bisectors of sides AB, BC and AC which meet each other at point O.
O is required center of circle.

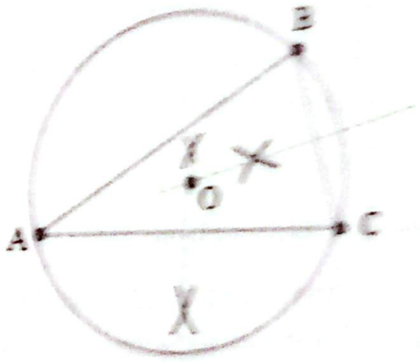
Draw a Circle Passing Through Three Non-Collinear Points

ven: Three non-collinear points A, B and C.

quired: To draw a circle passing through these points.

ps of Construction:

Take three non-collinear points A, B and C on a paper.
Join these points to obtain a triangle ABC.
Draw right bisectors of sides BC and AC which meet each other at point O.
With center O, construct a circle of radius OA or OB.



Complete the Circle when a Part of its Circumference is Given:

By finding the Centre (b) Without finding the Centre

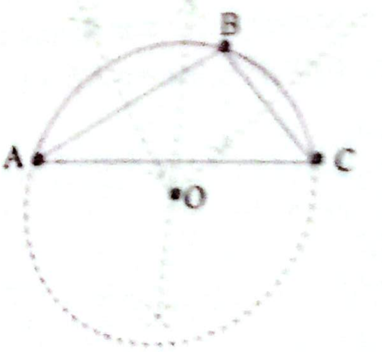
To Complete a Circle by Finding its Centre when Part of Circumference is Given

en: A part of circle i.e. an arc.

ired: To complete the circle by finding its center.

of Construction:

Take three points A, B and C on the given arc.
Join these points to obtain a triangle ABC.
Draw right bisectors of sides AB, BC and AC which meet each other at point O.
With center O, construct a circle of radius OA or OB of which the given arc is a part.



To Complete a Circle without Finding its Centre when Part of Circumference is Given

Given: A part of circle i.e. an arc.

Required: To complete the circle without finding its center.

First Method

Steps of Construction:

(i) Take three points A, B and C on the given arc such that AB = BC.
(We can also AB ≠ BC.)

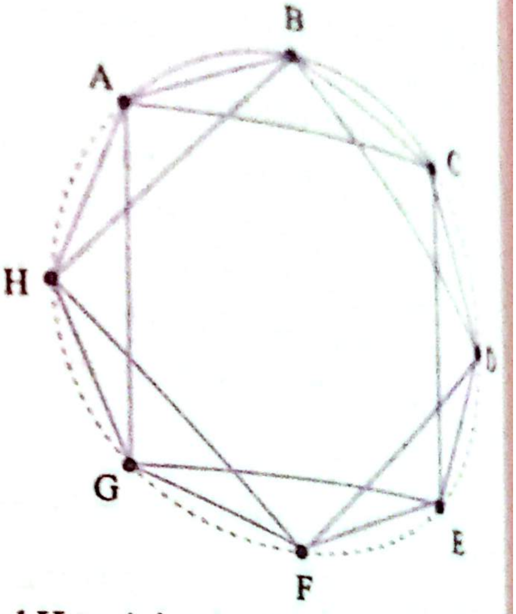
(ii) Join these points to obtain a triangle ABC.

(iii) Draw BD = AC and CD = BC to obtain a triangle BCD.

(iv) Again draw CE = BD and DE = CD to obtain a triangle CDE.

(v) Continue in this way and construct triangles DEF, EFG, FGH and GHA.

(vi) Join points C to D, D to E, E to F, F to G, G to H and H to A in circular path to obtain required circle.



Second Method

Steps of Construction:

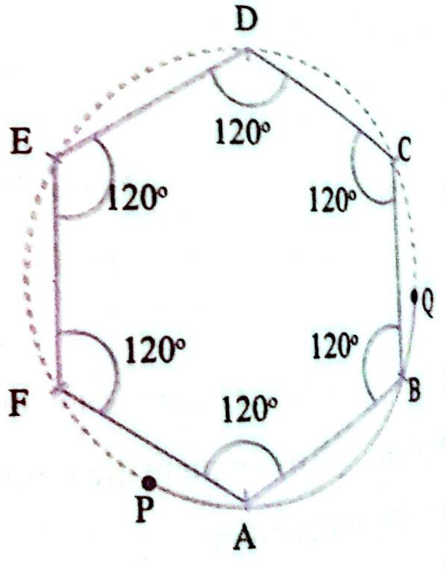
(i) Draw a chord AB of suitable length on an arc PQ.

(ii) Construct an internal angle of measure 120° at point B and draw BC equal to AB.

(iii) Again, construct an internal angle of measure 120° at point C and draw CD equal to BC.

(iv) Continue this procedure to obtain points E and F.

(v) Draw arcs BC, CD, DE, EF and FA carefully such that length of these arcs is equal to the arc AB.



Exercise 11.1

...a circle, locate its centre.

...points X, Y and Z are located on an arc. Find the centre of that arc.

...non-collinear points A, B and C are such that AB = BC = 3cm and ∠B = 120°. Construct a circle passing through A, B and C.

...and QR are perpendicular segments in a plane such that PQ = 4cm and QR = 3cm. Construct a circle passing through P, Q and R. What is the radius of that circle?

...a circle with the help of any round object, then find its centre.

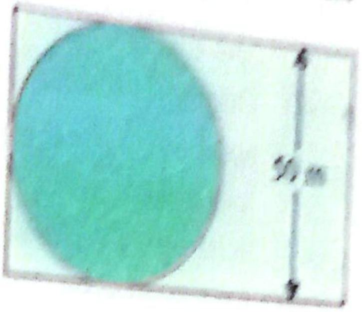
...the centre of an arc DEF, such that chord DE = 3.5cm, chord EF = 4.2cm and angle between both segments is 60°.

...an arc of any length. Complete the circle without finding its centre of which the given arc is the part.

...engineer is constructing a circular path in a recreational park as shown below. He wants to plant a pine tree in the centre of the circular path.

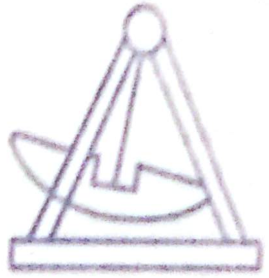
(i) How will he locate the centre of the path?

(ii) What is the radius of the circular path?



A factory produces arc shaped tiles of different colors each of inner radius 7 inches. Arc length of each tile is 2 inches. ...mza wants to purchase these tiles for making a circular design on the wall. How many tiles are required?

In the figure a boat swing is shown. Trace the circular path of swing during a complete round.



Tangent to the Circle

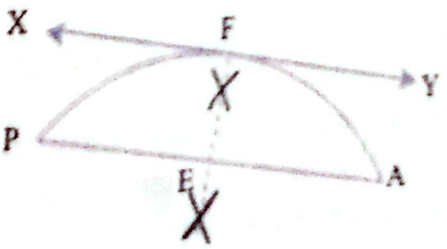
...a Tangent to a Given Arc without using the Centre, through a Given Point P

...P is:

(a) **The middle point of the arc.**

(b) **At the end of the arc.**

(c) **Outside the arc.**



...ven: An arc PA without centre.

...quired: To draw tangent at mid-point of arc.

Steps of Construction:

(i) Draw an arc PA.

(ii) Joint P to A.

(iii) Draw right bisector EF of PA where E is mid-point of $\overline{PA}$ and F is a point on given arc.

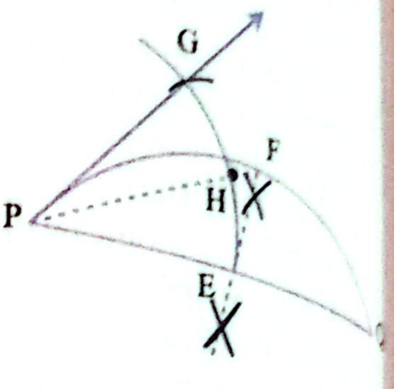
(iv) Draw a line XY passing through F parallel to $\overline{PA}$ using set square or by any other method.

$\overleftrightarrow{XY}$ is required tangent.

(b) Given: An arc PQ without centre.

Required: To draw a tangent at end point P of the arc.

Steps of Construction:



(i) Draw an arc PQ.

(ii) Join P to Q.

(iii) Draw right bisector EF of PQ where E is mid-point of $\overline{PQ}$ and F is a point on given arc.

(iv) Join point P to F.

(v) With centre P, draw an arc of radius PE intersecting PF at point H.

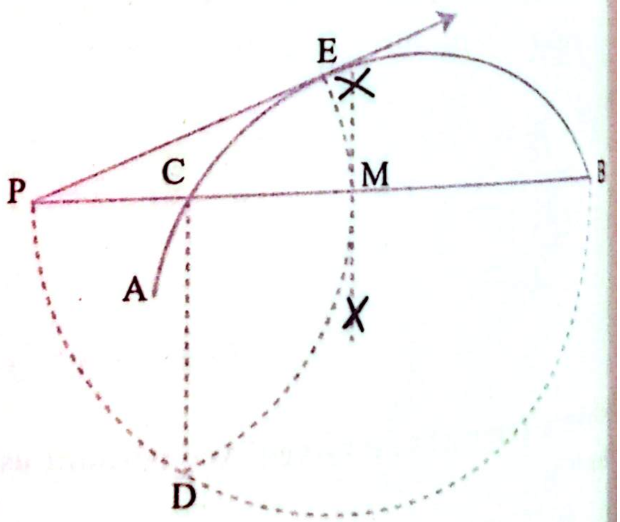
(vi) Take point G on this arc such that EH = HG. Now join P to G.

$\overrightarrow{PG}$ is required tangent.

(c) Given: An arc AB without knowing its centre.

Required: To draw a tangent from a point P where P is outside the arc.

Steps of Construction:



(i) Draw an arc AB.

(ii) Draw PB intersecting the arc AB at point C.

(iii) Bisect PB at point M.

(iv) With centre M, draw a semi-circle of radius MP or MB.

(v) Draw CD perpendicular to PB intersecting semicircle at point D.

(vi) With centre P, draw another arc of radius PD intersecting given arc AB at point E.

(vii) Joint P to E. $\overrightarrow{PE}$ is required tangent.

To draw a Tangent to a Given Circle from a Point P when:

(a) P lies on the Circ[illegible]ference.

(b) P lies outside the Circle.

Given: A circle of radius 3 cm.

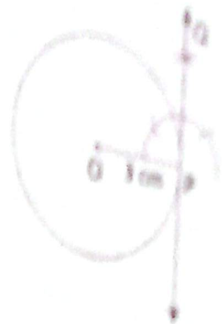
Required: To draw a tangent from a point P lying on the circle.

Steps of Construction:

(i) Construct a circle of radius 3 cm by taking O as centre.

(ii) Take a point P on the circle and join point O to point P.

(iii) Construct an angle of 90° at point P and draw a line PQ. The line PQ is required tangent.



Given: A circle of radius 3 cm and a point P 8 cm away from the centre of circle.

Required: To draw a tangent from a point P at a distance of 8 cm from the center of the circle.

Steps of Construction:

(i) Construct a circle of radius 3 cm by taking O as centre.

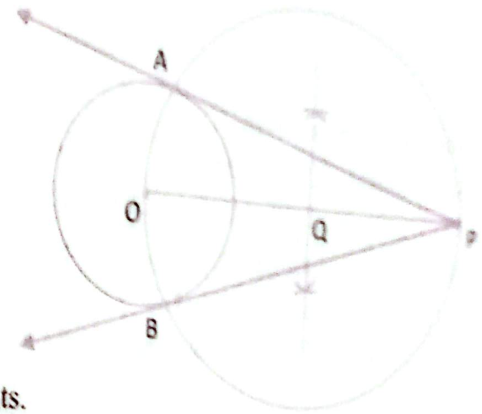
(ii) Take a point P at a distance of 8 cm from the centre.

(iii) Bisect $\overline{OP}$ at point Q.

(iv) Draw a dotted circle of radius OQ or PQ.

(v) The two circles intersect each other at points A and B.

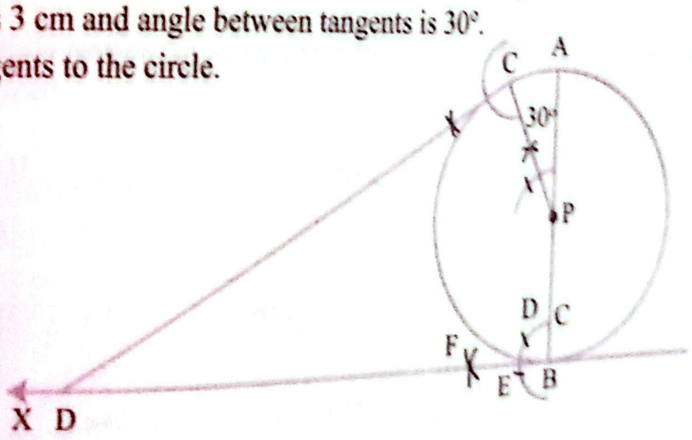
(vi) Draw $\overrightarrow{PA}$ and $\overrightarrow{PB}$ which are required tangents.



Draw two Tangents to a Circle Meeting each other at a Given Angle

Given: A circle of radius 3 cm and angle between tangents is 30°.

Required: To draw two tangents to the circle.



Steps of Construction:

Construct a circle of radius 3 cm at point P.

Draw a diameter AB.

Draw a tangent BX to the circle at point B.

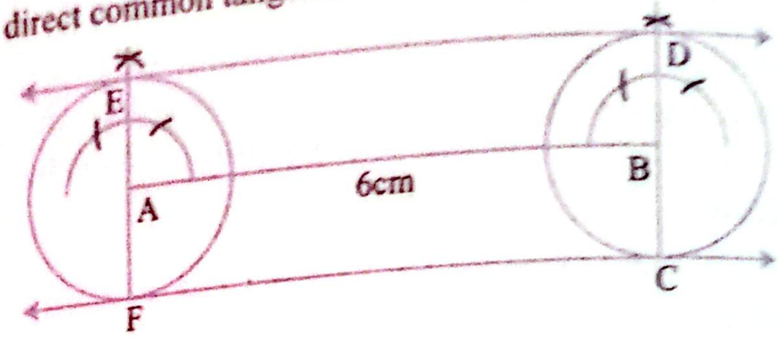
(iv) Construct $\angle APC = 30^\circ$ and draw another tangent $\overline{CD}$ at point C intersecting the firs tangent at point D.

(v) Measure $\angle BDC$. It will be 30°.

To draw Direct Common Tangents (External Tangents) to two Equal Circles

Given: Two equal circles each of radius 2.5 cm and distance between centers of circles i 6 cm.

Required: To draw direct common tangents or external tangents.



Steps of Construction:

(i) Draw $\overline{AB} = 6$ cm.

(ii) Construct two circles each of radius 2.5cm at points A and B.

(iii) Draw diameters CD at point B and EF at point A, perpendicular to $\overline{AB}$.

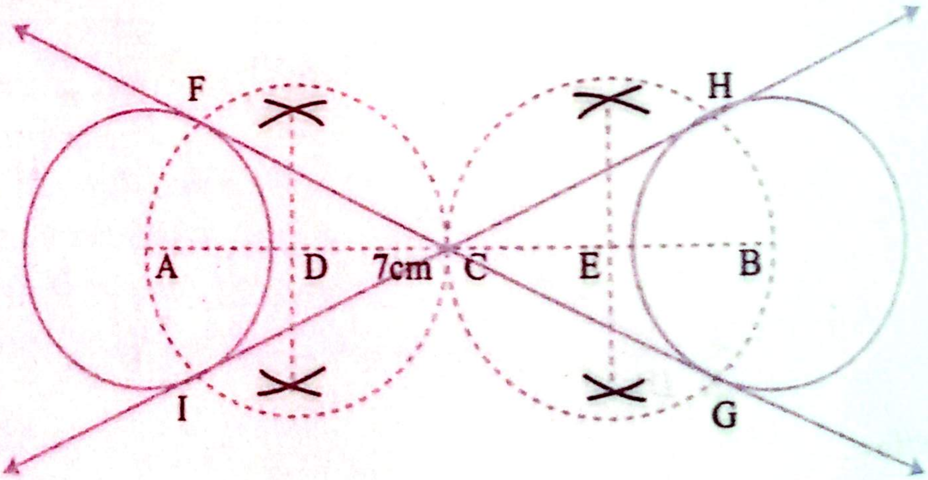
(iv) Join E to D and F to C.

Lines ED and FC are required direct common tangents.

To draw Transverse Common Tangents (Internal Tangents) to two Equal Circles

Given: Two equal circles each of radius 3 *cm* and distance between centers of circles i 7 *cm*.

Required: To draw transverse common tangents or internal tangents.



Steps of Construction:

(i) Draw $\overline{AB} = 7cm$.

(ii) Construct two circles each of radius 3 *cm* at points A and B.

(iii) Bisect $\overline{AB}$ at point C.

Further bisect $\overline{AC}$ at point D and $\overline{BC}$ at point E.

With centre D, construct a dotted circle of radius AD, intersecting the circle with centre A at points F and I.

With centre E, construct another dotted circle of radius BE, intersecting the circle with centre B at points H and G.

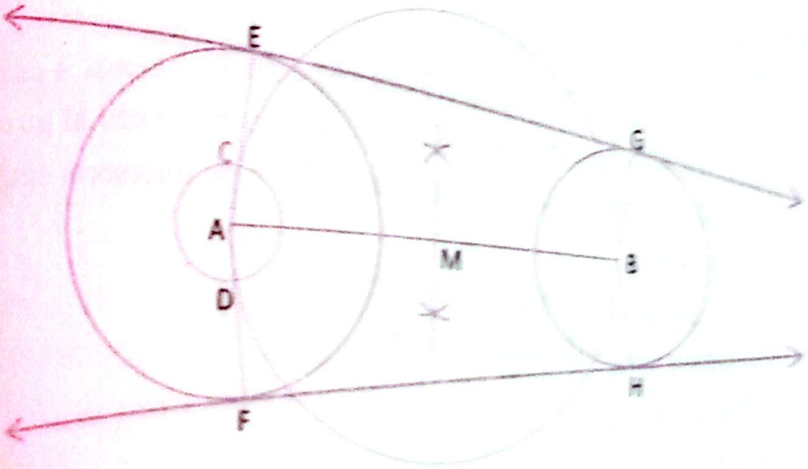
Join H to I and F to G.

$\overleftrightarrow{FG}$ and $\overleftrightarrow{HI}$ are the required transverse common tangents.

draw Direct Common Tangents to two Unequal Circles

Given: Two circles of radii 2 cm and 1.2 cm respectively and distance between centers of circles is 5 cm.

Required: To draw direct common tangents or external tangents.



of Construction:

Draw AB = 5cm.

Construct two circles of radii 2 cm and 1.2 cm respectively at ends A and B.

With center as A, construct a circle of radius 0.8 cm (difference of radii of given circles).

Bisect $\overline{AB}$ at M and construct a circle of radius AM which intersects circle of radius 0.8cm at points C and D.

Produce $\overline{AC}$ to point E and $\overline{AD}$ to point F.

Draw $\overline{BG} \parallel \overline{AE}$ and $\overline{BH} \parallel \overline{AF}$.

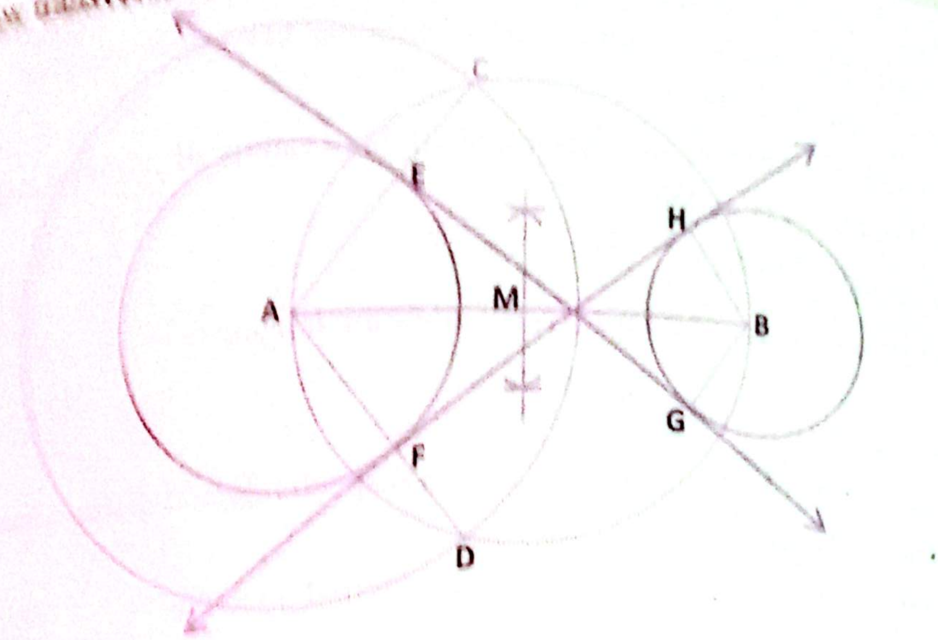
Join E to G and F to H.

$\overline{EG}$ and $\overline{FH}$ are required direct common tangents.

draw Transverse Common Tangents to two Unequal Circles

Given: Two circles of radii 2 cm and 1.3 cm respectively and distance between centers of circles is 5 cm.

Problem: To draw transverse common tangents



Steps of Construction:

(i) Draw $\overline{AB}$ = 5cm.

(ii) Construct two circles of radii 2 cm and 1.3 cm respectively at ends A and B.

(iii) With centre as A, construct a circle of radius 3.3 cm (sum of radii of given circles).

(iv) Bisect $\overline{AB}$ at M and construct a circle of radius AM which intersects circle of radius 3.3cm at points C and D.

(v) Draw $\overline{AC}$ intersecting circle of radius 2cm at E.

(vi) Draw $\overline{AD}$ intersecting circle of radius 2cm at F.

(vii) Draw $\overline{BG} \parallel \overline{AE}$ and $\overline{BH} \parallel \overline{AF}$.

(viii) Join E to G and F to H.

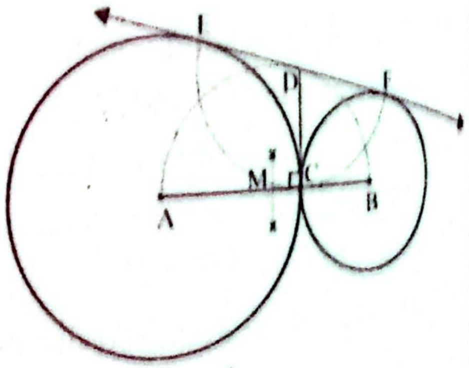
$\overline{EG}$ and $\overline{FH}$ are required transverse common tangents.

To draw a Tangent to two Unequal Touching Circles

Given: Two touching circles of radius 2.5 cm and 2 cm.

Required: To draw a tangent to the circles.

Steps of Construction:



(i) Construct a circle of radius 2.5 cm at point A and a circle of radius 2 *cm* at point B in such a way that both circles touch each other externally at point C.

(ii) Join A to B.

(iii) Bisect $\overline{AB}$ at point M.

(iv) With centre M, draw a dotted semi-circle of radius AM or BM.

(v) Draw a perpendicular $\overline{CD}$ on $\overline{AB}$ where D is a point on the semi-circle.

(vi) With centre D, draw an arc of radius CD intersecting the given circles at points E and F respectively.

(vii) Join E to F.

EF is the required tangent.

Class Activity:

Draw a common tangent to two circles that are:

(i) touching internally,

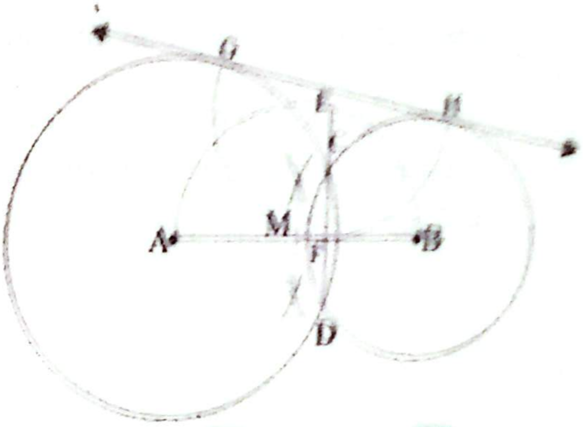
(ii) touching externally,

such that tangent drawn is perpendicular to the radial segments of both circles.

To draw a Tangent to two Unequal Intersecting Circles

Given: Two intersecting circles of radius 2.5 cm and 3 cm.

Required: To draw a tangent to the circles.



Steps of Construction:

(i) Construct a circle of radius 2.5 cm at point A. Construct another circle of radius 3 cm at point B in such a way that both circles intersect each other at points C and D.

(ii) Join A to B.

(iii) Bisect $\overline{AB}$ at point M. With centre M, draw a dotted semi-circle of radius AM or MB.

(iv) Join point D to C and produce it to point E lying on semi-circle. $\overline{DE}$ intersects $\overline{AB}$ at point F.

(v) With centre E, draw an arc of radius EF intersecting the given circles at points G and H respectively.

(vi) Join G to H. Line GH is the required tangent.

Exercise 11.2

1. Draw an arc and divide it into two equal parts.
2. Draw an arc of any length. Draw a tangent without using centre, through point A, when point A is: (i) in the middle of arc. (ii) at the right end of the arc.
3. Draw an arc ABC. Take a point X outside the arc and draw a tangent from X to the arc without using centre of the arc.
4. Construct a circle of radius 4cm. Mark a point P on the circumference and draw a tangent passing through P.
5. Construct a circle of radius 3cm. Draw two tangents from a point A, 5.8cm away from the centre of circle.
6. Construct a circle of radius 2.5cm. Draw two tangents making an angle of:
   (i) 45° (ii) 90° with each other.

7. Construct two equal circles each of radius 3cm. Centres of the circles are 6cm apart. Draw their:

(i) direct common tangents. (ii) transverse common tangents.

8. Construct two circles of radii 2cm and 3.5cm. Draw:

(ii) direct common tangents (ii) transverse common tangents to the circles, if the distances between centres of circles is 6.4cm.

9. Construct two touching circles of radii 2.6cm and 3.6 cm. Draw:

(i) a common tangent that passes through the point of contact of both circles.

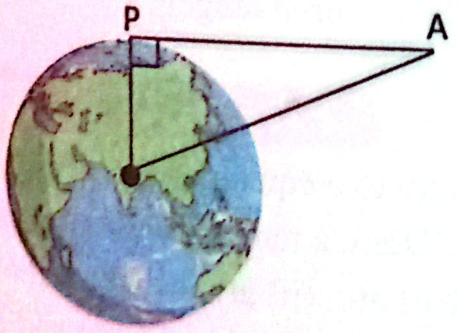
(ii) a common tangent that does not pass through the point of contact of both circles.

(iii) two common tangents.

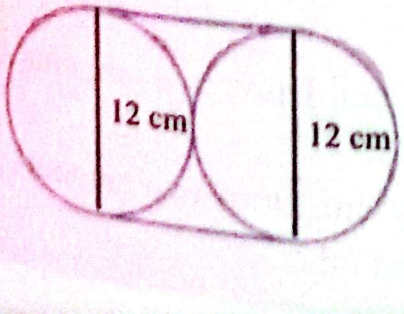
10. Construct two intersecting circles of radii 2.5cm and 3cm whose centres are 4.8cm apart. Draw two common tangents to the circles.

11. In the figure, two pullies of radii 7 inches and 14 inches respectively are shown. Centres of pullies are 28 inches apart. Find the approximate length of rope moving on pullies.

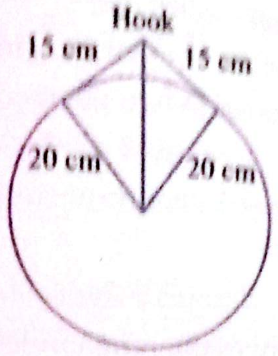
12. A satellite A is moving around Earth at an altitude of 7600 km from the surface. The radius of Earth is 6400 km. How far is the satellite from the city P?



13. Two cylindrical rods are bound with a strap. Each rod has diameter 12 cm. How long is the strap?



A circular mirror with radius 20 cm hangs by a wire from a hook. The wire is 30 cm long and is a tangent to the mirror in two places. How far above the top of the mirror is the hook?



I have Learnt

- Locating the centre of a given circle.
- Drawing a circle passing through three given non-collinear points.
- Completing the circle (i) by finding its centre, (ii) without finding its centre, when a part of its circumference is given by
- Drawing a tangent to a given arc, without using the centre, through a given point P when: (i) P is middle point of the arc (ii) P is at the end of arc (iii) P is outside the arc.
- Drawing a tangent to a given circle from a point P when: (i) P is on the circumference (ii) P is outside the arc.
- Drawing two tangents to a circle meeting each other at a given angle.
- Drawing (i) direct common tangents or external tangents (ii) transverse common tangents or internal tangents, to two equal/unequal circles.
- Drawing a tangent to two unequal (i) touching circles (ii) intersecting circles.
- Applying concepts of practical geometry of circles to real life word problems such as athletic tracks, recreational parks, Ferris wheels and mechanical machines.

MISCELLANEOUS EXERCISE-II

1. Encircle the correct option in the following.

i. Angle between tangent and radial segment of a circle is:
(a) 30° (b) 45° (c) 60° (d) 90°

ii. Direct common tangents of equal circles are:
(a) parallel (b) intersecting (c) converging (d) not parallel

iii. How many tangents can be drawn to two intersecting circles?
(a) 1 (b) 2 (c) 3 (d) infinite

iv. Two circles having radii 3cm and 3.2cm respectively, touch externally. The distance between centres of the circles is:
(a) 6cm (b) 0.2cm (c) 6.2cm (d) 5.3cm

v. Centres and point of contact of two touching circles are: ________.
(a) collinear (b) non-collinear (c) converging (d) coincident

vi. In which type of triangle, incentre and circumcenter are coincident?
(a) scalene (b) isosceles (c) equilateral (d) right triangle

vii. Two circles with radii 4cm and 4.8cm respectively, touch internally. The distance between centres of the circles is:
(a) 4cm (b) 8 cm (c) 8.8cm (d) 0.8cm

viii. Two tangents are drawn at the ends of diameter of a circle of radius 3.5cm. The distance between tangents is:
(a) 3.5cm (b) 7cm (c) 5cm (d) 10.5cm

ix. Transverse common tangents intersect each other at:
(a) 1 point (b) 2 points (c) 3 points (d) 4 points

x. A circle is inscribed in a square having length of side 6cm. What is the radius
(a) 24 (b) 12 (c) 6 (d) 3

xi. A square is inscribed in a circle of radius 5cm. What is the length of diagonal of the square?
(a) 2.5cm (b) 5cm (c) 10cm (d) 20cm

xii. Perpendicular bisectors of ………… always pass through centre of circle.
(a) tangents (b) secants (c) radial segments (d) chords

xiii. How many circles can be drawn through three non-collinear points?
(a) infinite (b) 1 (c) 2 (d) 3

xiv. A circle have …… pair(s) of parallel tangents.
(a) 1 (b) 2 (c) 3 (d) infinite

12

Basic Statistics

fter studying this unit students will be able to:

- Construct cumulative frequency table, cumulative frequency polygon and cumulative frequency curve i.e. ogive
- Interpret the quartiles, deciles, percentiles and inter quartile range(IQR) from cumulative frequency curve
- Interpret and analyze box and whisker plot correlation
- Construct and interpret data from scatter diagram and also draw lines of best fit
- Measure correlation using scatter diagram
- Calculate range, standard deviation and variance for grouped data
- Use the mean and standard deviation to compare 2 sets of data
- Solve real life problems involving variance and SD of grouped data
- Apply concepts from measures of dispersion to solve real life problems
- Calculate the probability of combined events using (where appropriate):Sample space diagram, possibility diagram, tree diagram & venn diagram
- Apply addition law of Probability to solve problems involving mutually exclusive events
- Apply Multiplication law of Probability to solve problems involving independent and dependent events

mi and Haider participated in a Qiraat petition, with 8 other equally competent testants. Judges have to decide about top rformers. Can you guess the chance of ery contestant to be the top first? After cision of First, can you guess the chance of ery remaining contestant to be top second? ter decision of Second, can you guess the ance of every remaining contestant to be top ird?

ntestants	Sami	Haider	Ibraheem	Wali	Ali	Hashir	Qasim	Hassan	Muhammad	Ahmad
nce of ing	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$

Frequency Distribution

In statistics it is very important to represent a data in a manageable way so that it becomes easy to understand, analyze and to draw useful results. This useful presentation of data can be made by various methods. Grouped frequency distribution is one of the most important methods t represent the quantitative data. Data can be classified in many ways for efficient usage an productivity. In previous grades we have discussed two types of data

i. raw data (ungrouped data) ii. grouped data (frequency distribution)

Here we will study the construction of cumulative frequency distribution by revising frequency polygon.

Frequency Polygon

Polygon is a many sided closed figure. To draw a frequency polygon mid points (class mark are marked on the horizontal axis and frequencies are marked on the vertical axis. Points ar plotted with class frequencies and their corresponding class marks. These points are joined b straight line segments. To complete a closed polygon we add extra classes at both ends with zer frequencies. In this way two extra points on x-axis are obtained on both ends. The points ar joined by their nearer plotted points. Finally a frequency polygon is obtained.

Cumulative Frequency Distribution

A table in which cumulative frequencies of classes are written against classes is called cumulative frequency table "The number of values of a data less than an upper-class boundary i called its cumulative frequency".

The cumulative frequency table is another way to analyze the frequency distribution. Th frequency distribution tells us how many values of the data are present within each class interva while a cumulative frequency tells us how many values of a data are less than or within eac class interval.

In the following table cumulative frequencies are found and interval of 0 frequency is added the top. Column of upper class boundary is also included which are necessary for construction c cumulative frequency polygon or ogive.

Marks	Marks(upper class boundaries)	No. of students	Cumulative frequency	c.f
0	0.5	0	0	0
1 – 10	10.5	2	2	2
11 – 20	20.5	4	4+2=6	2 + 4 = 6
21 – 30	30.5	9	9+4+2=15	6 + 9 = 15
31 – 40	40.5	6	6+9+4+2=21	15 + 6 = 21
41 – 50	50.5	7	7+6+9+4+2=28	21 + 7 = 28
51 – 60	60.5	3	3+7+6+9+4+2=31	28 + 3 = 31
61 – 70	70.5	4	4+3+7+6+9+4+2=35	31 + 4 = 35

lative Frequency Polygon and Cumulative Frequency Curve

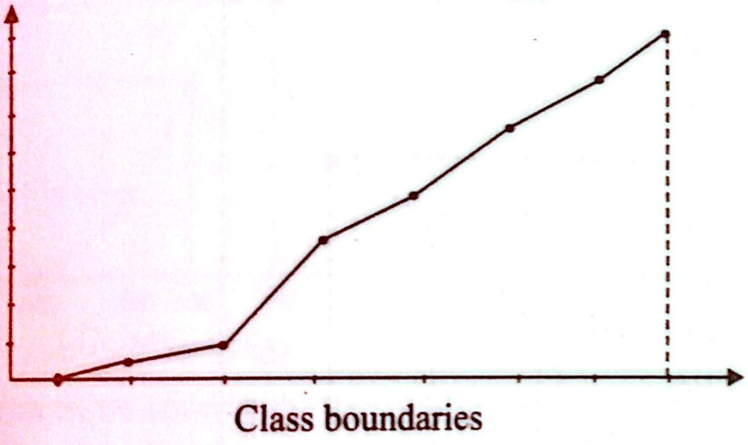
ve)

ncy distribution can also be represented by a cumulative frequency polygon or Ogive. ve and a cumulative frequency polygon are related but distinct concepts in statistics and alization.

ive Frequency Polygon

ncy polygon made by taking cumulative frequency on y axis is called cumulative cy polygon. It is made to visualize the cumulative distribution of a variable. w a cumulative frequency polygon, the class boundaries are plotted on x-axis and ive frequencies on y-axis. The points are marked on the graph using the upper class ies and the respective cumulative frequencies. These points are joined by straight line ts and the polygon is completed by drawing a vertical line segment to join the last marked with x-axis. Following is a cumulative frequency polygon representing the data given

Title: A cumulative frequency polygon showing the marks of 35 students in a test.



le: Late arrival fine imposed to a player is shown below against the terms of 4 weeks:

eeks	1- 4	5-8	9-12	13-16
nt of fine (Rs)	2000	1000	500	1500

onstruct cumulative frequency table including
- i. Column of upper class boundaries
- ii. Interval for 0 frequency
- iii. Column of cumulative frequencies

ow much fine is to be paid after 3 terms?

onstruct cumulative frequency polygon of the above data.

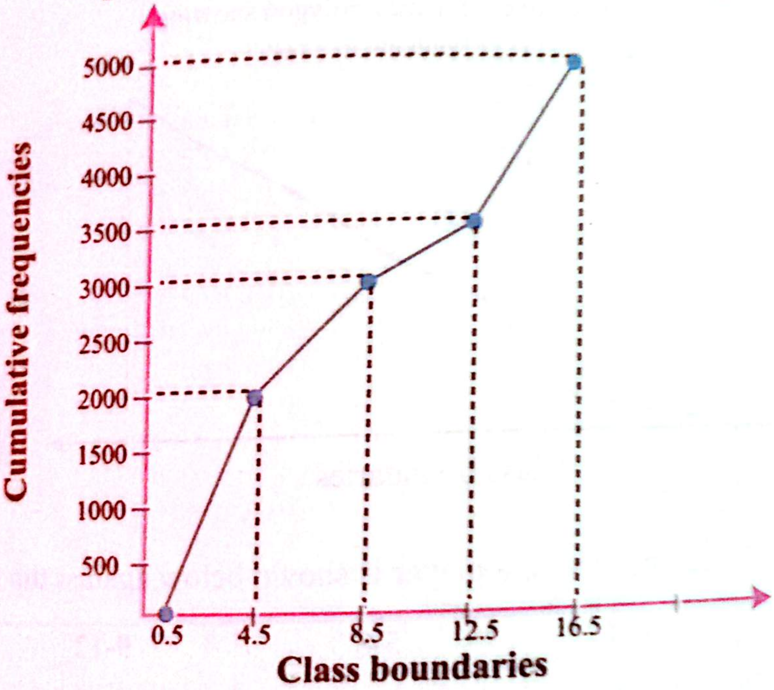
Solution: (a) Cumulative frequency table

weeks	Upper class boundaries	Amount of fine (Rs)	Cumulative frequencies
0	Less than 0.5	0	0
1-4	Less than 4.5	2000	2000
5-8	Less than 8.5	1000	3000
9-12	Less than 12.5	500	3500
13-16	Less than 16.5	1500	5000

b) After 3 terms, The player has to pay a total amount of Rs. 3500.

c) Construction of cumulative frequency polygon is given below:

Title: Cumutative freq polygon showing fine imposed on a player against terms of 4 weeks



Cumulative Frequency Curve (Ogive)

A graph showing the cumulative frequency (or cumulative relative frequency) of a variable as a curve, with the x-axis showing the values of the variable and the y-axis showing the respective cumulative frequency is called Cumulative Frequency Curve (Ogive). It is also known as a cumulative distribution function (CDF)

Example:

The table below shows the cumulative frequency distribution of masses of cakes in Coo made at a festival:

of cakes (grams)	x ≤ 400	x ≤ 500	x ≤ 600	x ≤ 700	x ≤ 800	x ≤ 900
of cakes	0	20	50	70	80	100

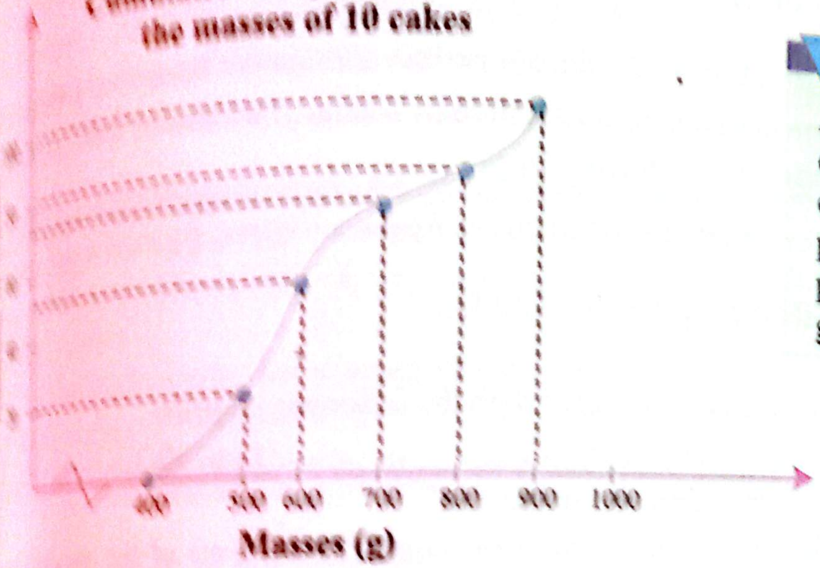
a) Draw cumulative frequency curve.

b) Estimate from the curve, the number of cakes having masses 750 g or less

c) Taking class intervals 400< x ≤ 500, 500 < x ≤ 600 …construct a cumulative frequency distribution .

Solution (a) Cumulative frequency curve (Ogive)

Cumulative frequency curve of the masses of 10 cakes



Key Facts

An ogive is a smooth curve, while a cumulative frequency polygon is a polygon-shaped graph

From the curve, we observe that the number of cakes having masses 750 g or less is 75.

Construction of the frequency cumulative distribution.

Mass of cakes (x g)	Cumulative frequency	Mass of cakes (x g)	frequency
x ≤ 500	20	400 ≤ x ≤ 500	20
x ≤ 600	50	500 ≤ x ≤ 600	50 – 20 = 30
x ≤ 700	70	600 ≤ x ≤ 700	70 – 50 = 20
x ≤ 800	80	700 ≤ x ≤ 800	80 – 70 = 10
x ≤ 900	100	800 ≤ x ≤ 900	100 – 80 = 20

Measures of Distribution

measures are used to understand the spread and distribution of the data, and are often used easures of central tendency to get a better understanding of the dataset. They include:

3. Quartiles

The values which divide the whole data into four equal parts are called quartiles. Quartiles are usually denoted by Q_1, Q_2 and Q_3. The lower quartile Q_1 divides the data in 1 : 3. The middle quartile Q_2 divides the data in 1 : 1. It is infect nothing different from median of data. The upper quartile Q_3 divides the data in 3 : 1.

Q_1 = [illegible] th value of data & $Q2 = (\frac{2n}{4} = \frac{n}{2})^{th}$ value of data = median

When data is arranged in some order, it can be represented by a single value that lies exactly at the centre of data. "The central or middle most value of an arranged data is called its median".

Q_3 = [illegible] th value of data

The quartiles can also be estimated graphically from a cumulative frequency polygon.

- To estimate Q_1 (lower quartile), mark $\frac{n}{4}$ on y-axis.
- Draw a line parallel to x-axis from that point to intersect cumulative frequency polygon.
- Drop a perpendicular from this point of intersection to x-axis. The value of data where the perpendicular meets x-axis is lower quartile Q_1.

Similarly, Q_2 and Q_3 can be estimated from cumulative frequency curve, by marking $\frac{n}{2}$ and $\frac{3n}{4}$ on y-axis and following the same procedure used for Q_1.

Example:

The scores on a test were compiled into a dataset with the following quartiles:

Q1 = 40, Q2 or median = 70, Q3 = 90

What percentage of students scored above 90 on the test?

Solution: From the quartiles it is obvious that one fourth (25 percent) of the students scored above 90.

ii. Deciles

Deciles divide a dataset or a distribution into 10 equal parts, each representing 10% of the data. By looking at the deciles, we can get a sense of the shape of the distribution, including where the majority of the data points are concentrated and where the outliers are lying.

Ten Deciles of a dataset are described as follows:

D1: lowest 10% of the data
D2: lowest 20% of the data
D3: lowest 30% of the data
……………………………………………
D10: highest decile under which100% of the data lies.

Key Fact:

Deciles are often used in statistics, economics, and social sciences to understand the distribution of a variable, such as income, wealth or exam scores.

Example: A company wants to analyze the salaries of its employees. The related department compiles a dataset and calculates the deciles. The deciles are as follows:

s-40,000 • D5 or median = Rs.60,000 • D10 = Rs.80,000

rcentage of employees earn a salary above Rs.60,000 but below Rs.80,000?

ion: From the above data, percentage of employees earning salaries above 60,000 and w Rs. 80,000 are between D5(median) and D10 i.e 90 % – 50% = 40 %

mber: D10 represents highest 10 % i.e Rs. 80,000 and D9 represents below Rs. 80,000)

Percentiles

ntiles divide a dataset or a distribution into 100 equal parts, each representing 1% of the a. It's a way to understand the distribution of a dataset or a variable by looking at the ntage of data points that are below a specific value.

e common percentiles include:

- 25th percentile (Q_1): 25% of data points are below this value.
- 50th percentile (Q_2 or median): 50% of data is below Q_2.
- 75th percentile (Q_3): 75% of data points are below this value.
- 90th percentile: 90% of data points are below this value.
- 99th percentile: 99% of data points are below this value

Maths Play Ground

Help students to design quiz wheel or quiz machine for the reinforcement of measures of distribution.

Percentiles are often used to:

- nderstand the distribution of a variable (e.g., income, scores, masses).
- dentify outliers.
- ompare individual values to the overall distribution.
- et benchmarks (e.g., top 10 %).
- valuate performance (e.g., moving up or down in percentiles).

xample: Uzair scored in the 75th percentile on a test. Find how much percentage of the dents scored below him and how much percentage of students scored above him.

lution: From the above information it is clear that the student's score is higher than 75% of e total students. It means that

- 75% of the students scored below him
- 25% scored above him

Inter Quartile Range

IQR = Q3 – Q1

e IQR represents the range of values within which the middle 50% of the data points fall. In er words, it's the spread of the data within the middle half of the distribution.

QR can describe:

- Spread: A small IQR indicates less scattered, while a large IQR indicates more scattered data.
- Variability: The IQR is a more efficient measure of variability than the Range.
- Outliers: The IQR can help identify outliers. (data points beyond 1.5 times the IQR from the rtiles are considered outliers).
- Comparison: IQR is useful for comparing the spread of different datasets.

Example:
A teacher wants to analyze the test scores of her students. She compiles a dataset as: first quartile Q1 = 60 and third quartile Q3 = 95

a) What is the Interquartile Range (IQR) of the test scores?

b) What percentage of students scored between 60 and 95 (inclusive)?

Solution:

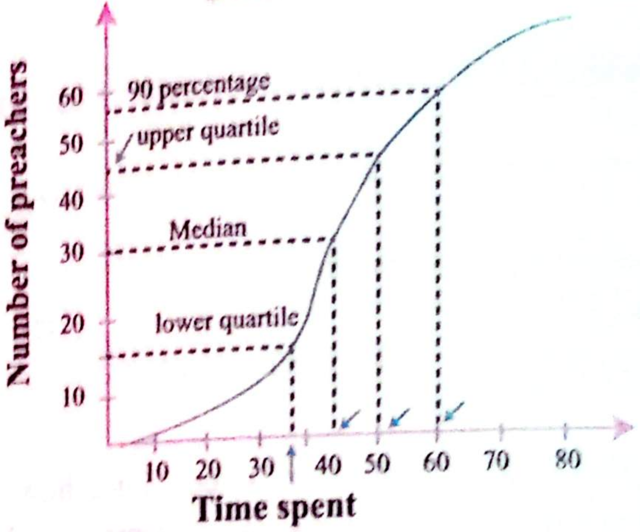
a) The Interquartile Range (IQR) of the test scores is given by the relation:

IQR = Q3 – Q1 = 95 – 60 = 35

b) As Q3 – Q1 represents spread of the middle half of a data set so 50% of students sco between 60 and 95 (inclusive)

Example:
The adjoining figure shows an ogive representing the distribution of time in minutes spent by 6 preachers of SEED to deliver single Seerah lecture. Find Q1, Q2, Q3 and inter quartile range

Ogive of time spent to deliver a seerah lesson



Solution:
To estimate the median time spent from ogive, we observe that 50 % (half of) preachers spe less than or equal to the median time to deliver the lecture.

Median corresponds to the 5th decile and 50th percentile, i.e.

$$\text{Median} = Q2 = D5 = P50 = \frac{1}{2} \times 60 = 30$$

From 30 on y-axis, draw a horizontal line to meet the curve and then draw a vertical line to me x-axis as in above figure.

From the figure, we observe that median time to deliver a lecture is 43.5 minutes.

Lower quartile corresponds to 25th percentile i.e.

Q1 (the lower quartile) = P25 = one quarter (25% of) total frequency $= \frac{1}{4} \times 60 = 15$

From the curve, the lower quartile = 37 minutes.

Upper quartile corresponds to 75th percentile i.e.

Q3 (the upper quartile) = P75 = three quarters (75% of) total frequency $= \frac{3}{4} \times 60 = 45$

From the curve, the upper quartile = 50.5 minutes.

The inter Quartile Range = Q3 – Q1 = 50.5 – 37 = 13.5

Exercise 12.1

...cumulative frequency column for following frequency table and give answers.

ass Interval	1 – 10	11 – 20	21 – 30	31 – 40	41 – 50	51 – 60
equency	3	4	7	9	5	2

What is the total number of items in the data?
In which class interval 8th item of the data lies?
What is number of items having worth less than 21?
Which group contains highest number of items?
What is the lower boundary of the last class?

...ass gained by a newly born baby is given below:

Age in months	1	2	3	4	5	6
No. of kg gained	4	3	2	3	2	1

...w a cumulative frequency polygon (not ogive) to represent the data. Also find from the ...ph, how much he/she weighed at the age of 4 and half months?

...e ages of refugees in Ghaza camp were recorded as below:

Ages in years	20 – 24	25 – 29	30 – 34	35 – 39	40 – 44	45 – 49
No. of refugees	5	16	12	10	8	4

...w an ogive (not polygon) to represent the data and Find

...First quartile. b) 7th decile. c) 60th percentile.

...res of a Hifz e Quran test were grouped into the following frequency distribution.

scores	41-50	51 – 60	61 – 70	71 – 80	81 – 90	91 – 100
f	3	17	20	30	18	2

...w a cumulative frequency table and cumulative frequency curve (ogive) and tell:

How many boys appeared in the test?
What is median score?
How many boys scored under median?
How many boys fell in first decile?
How many boys fell in 30th percentile?
If excellence award is to be given to boys falling in 100th percentile, how many boys get excellence award?

...masses measured to the nearest kg of 50 cubs are shown below:

Mass (kg)	60 – 64	65 – 69	70 – 74	75 – 79	80 – 84	85 – 89
f	2	6	12	14	10	6

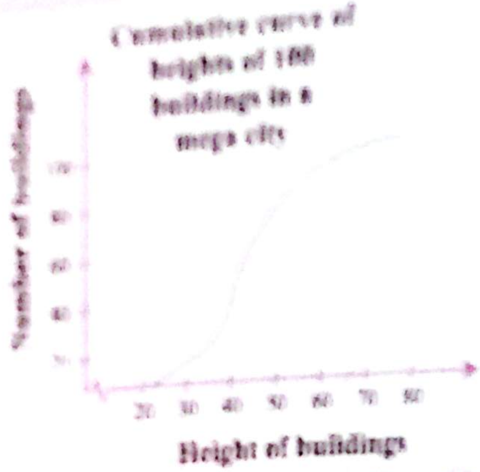
(i) Construct a cumulative frequency table

(ii) Construct a cumulative frequency polygon and estimate number of cubs above

(iii) Construct a cumulative frequency curve and estimate number of cubs above

(iv) Compare the results of (ii) & (iii)

6. Study the adjoining curve showing heights (m) of 100 buildings in a megacity carefully, and find



a) Number of buildings with heights less than 50 m.

b) % age of buildings higher than 40 m.

c) Ratio of the number of buildings less than 50m and more than 50m

d) No. of Buildings falling in first quartile

e) Maximum height of buildings falling under 8^{th} decile.

f) Maximum height of buildings not falling under 30^{th} percentile.

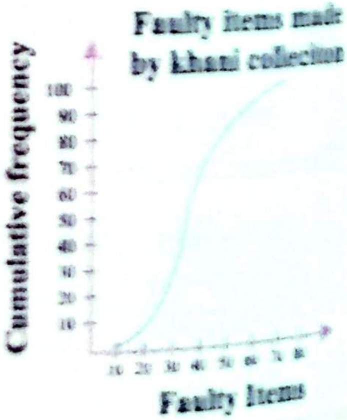
7. Mr. Rashid wants to analyze the salaries of his employees. The first quartile (Q1) is of Rs.50,000 a second quartile (Q2) of Rs.70,000, a third quartile (Q3) of Rs.90,000 and maximum and minimum salaries are of Rs.25,000 & Rs.150,000 respectively. Find range and inter quartile range of the data.

8. Khadeeja scored 85 on a test, and the teacher reported that her score was in the 75th percentile. What does this information indicate about status of Khadeeja in her class?

9. Uzair wants to price his house based on the prices of similar houses in the area. The prices have a Q1 of Rs. 20,000,000, Q2 of Rs. 25,000,000, Q3 of Rs. 30,000,000, and a maximum price of Rs. 35,000,000. What is the possible price of the house if it is in the third quartile? Also find IQR of the data.

10. The adjoining cumulative frequency curve shows the number of faulty items in a shipment of 100 items produced by Khani's Collection. Estimate

i. The median
ii. Lower quartile
iii. Upper quartile
iv. IQR
v. 7th decile
vi. 30th percentile



The Box and Whisker Plot Correlation

Whisker Plot (or simply box plot)

plot typically consists of the following components:

rectangular box that represents the interquartile range (IQR). The box is divided into two by a vertical line, which represents the median (Q2) of the data.

Two horizontal lines that extend from the box to the minimum and maximum values dataset, excluding outliers. The whiskers represent the range of the data.

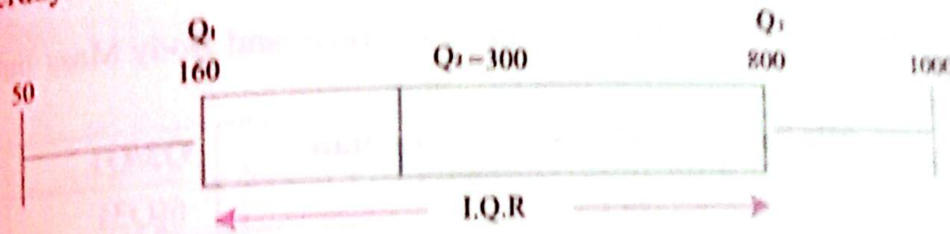
Data points that fall beyond 1.5 times the IQR from the quartiles are considered outliers plotted individually.

The vertical line inside the box represents the median (Q2) of the data.

The edges of box represent 25th percentile (Q1) and the 75th percentile (Q3).

plots are useful for comparing datasets, identifying outliers, visualizing distribution and summarizing data

mple: The given box and whisker plot shows the prices of stationery items, Hashir purchased yesterday.



a) What is the median price of an item?

b) what are the min & max prices of the items purchsed by Hashir?

c) What is IQR of the data? d) what %age of items is above Rs. 800?

lution:

a) From the box plot, it is clearly seen that median price of an item is Rs. 300

b) From the box plot, it is observed that min price of any item is Rs. 50 & max price of any item is Rs. 1000

c) IQR = Q3 – Q1 = 800 – 160 = 640

d) As items between Q3 and Q4 are 25 % so %age of items above Rs. 800(Q3) is 25%.

and Whisker Plot Correlation

e box and whisker plot correlation means the patterns or relationships between the tributions of two or more variables. Box and whisker plot correlations include:

sitive correlation:

the boxes and whiskers of both variables tend to move in the same direction (e.g., both crease or decrease together), it may indicate a positive correlation.

gative correlation:

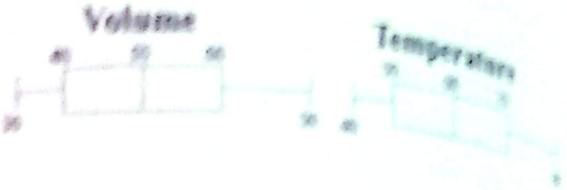
the boxes and whiskers of both variables tend to move in opposite directions (e.g., one creases while the other decreases), it may indicate a negative correlation.

3. **No correlation:**
If the boxes and whiskers of both variables appear unrelated or don't show a clear pattern, it may indicate no correlation.

REMEMBER Box and whisker plots only provide a visual representation and don't quantify the correlation.

Example shows volume of a gas is shown in first box and whisker plot and increase in temperature is shown in second box and whisker plot in the adjoining figures.

In this example, we observe that the boxes and whiskers of both variables tend to move in the same direction (both increasing together), suggesting a positive correlation.



Example:
The following datasets show 2 Variables. Hours of Exercise and Body Mass Index, (BMI) of gym members.

variables	Min/max	Q1/Q3	median	Q3/Q1	Max/min
Hours	0(min)	2(Q1)	4	6(Q3)	8(max)
BMI	30(max)	28(Q3)	25	22(Q1)	20(min)

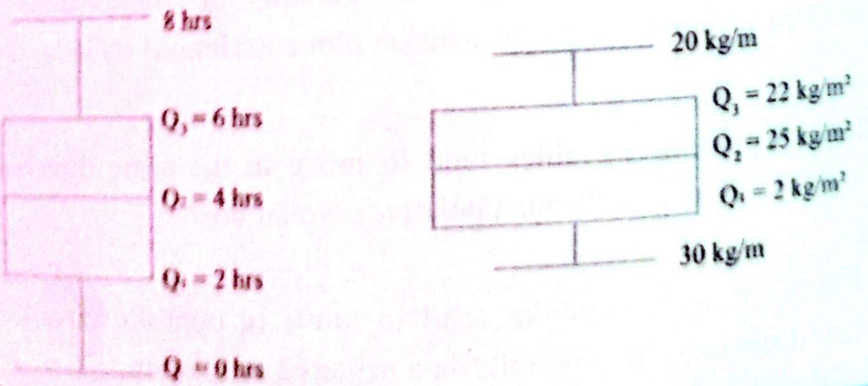
Draw box and whisker plots.
a) Find type of correlation b) Comment on type of correlation

Solution:

a) As the hours of exercise (Variable A) increase, the BMI (Variable B) tends to decrease. The boxes and whiskers of both variables move in opposite directions, indicating a negative correlation.

b) The negative correlation suggests that exercise may lead to weight loss, as higher exercise hours are associated with lower BMI values.

Box-and-whisker plosts
Time (hrs) of exercise & BMI (kg/m)



Construction of Scatter Diagram, Lines of Best Fit and Interpretation of Data

Construction of a scatter diagram consists of:

i. Plotting the data points on a coordinate plane.

ii. Representing each data point by a mark (e.g., dot, circle, or square).

We solve an example to understand the concept.

Example: Table below shows the masses and heights of 11 boys.

Heights(cm)	167	165	184	153	180	170	179	174	160	176	157	177
Mass(kg)	60	54	69	48	64	58	68	64	51	67	46	63

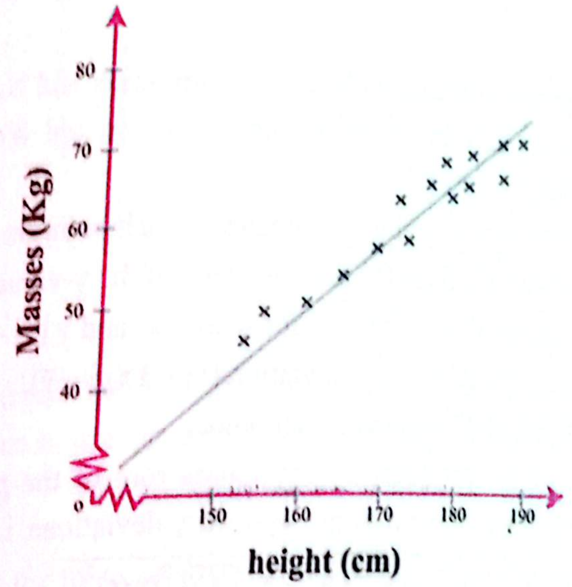
a) Draw a scatter diagram for the above data.

b) What type of correlation is observed in the scatter diagram?

c) Draw a line of best fit on the scatter diagram.

d) Use this line of best fit to estimate mass of a boy whose height is 150cm.

Solution (a):



(b) From the graph, it is observed that there is a strong positive correlation between masses and heights of boys. More height, more the mass of boys.

(c) Line of best fit is drawn having almost half the data points on either sides of it.

(d) From scatter diagram, the boy with height 150cm should have a mass of 39 kg app.

Interpretation of a Scatter Diagram

Pattern: There are several types of patterns:

- Positive correlation: As x increases, y also increases.
- Negative correlation: As x increases, y decreases.
- No correlation: No apparent relationship between x and y.

Strength: Observe the strength of the relationship. A stronger relationship means the points are closer to a straight line.

Outliers: Identify data points that don't fit the pattern.

Drawing Lines of Best Fit

A line of best fit helps to predict the value of a variable y for a given x. It is a line that appears to fit the pattern. It may or may not pass through any data points however almost equal number of points should lie on either side of it. The line of best fit can be represented by

$y = mx + c$.

Interpretation of a Line of Best Fit includes following attributes:

Slope (m): The rate of change of y with respect to x.

Intercept (c): The value of y when x is 0.

Prediction: Use the equation to predict the value of y for a given x.

A line of
best fit is not always
perfect fit and it's
essential to consider
strength of the relation
and the presence of
outliers when
interpreting the resul

Measuring Correlation using Scatter Diagram

Measuring correlation using a scatter diagram involves analyzing the pattern and strength of the relationship between two variables.

It includes following steps:

1. Plot the data
2. Identify the pattern (Positive correlation, Negative correlation and No correlation)
3. Assess the strength (Strong correlation, Moderate correlation and Weak correlation)
4. Calculate the correlation coefficient r by:
   - Identifying the data points (x, y) by coordinates of each point on the scatter plot.
   - Finding the mean of the x-values ($\bar{x}$) and the mean of the y-values ($\bar{y}$).
   - Calculating the deviation from the mean for x ($xi - \bar{x}$) and y ($yi - \bar{y}$).
   - Multiplying the x-deviation by the y-deviation $(xi - \bar{x})(yi - \bar{y})$.
   - Adding up the multiplied deviations for all points.
   - Dividing the sum of the products by the square root of the product of the sum of the squared x-deviations and the sum of the squared y-deviations. i.e.

$$r = \sum\{(xi - \bar{x}) \times (yi - \bar{y})\} / \sqrt{(\sum(xi - \bar{x})^2 \times \sum(yi - \bar{y})^2)}$$

Where r is the correlation coefficient, ranging from −1 (perfect negative correlation) to 1 (perfect positive correlation).

5. Interpret the results high r value i.e. close to 1 or −1 means Strong correlation, Low r value i.e. close to 0 means weak correlation & sign of r indicates the direction of the correlation i.e. positive or negative.

mple:

table below shows the reading time of 10 children aged between 11 and 16 years to win the pages reading activity in 10 minutes.

Age (yrs)	13.2	14.9	13.3	15.3	13.6	15.5	14.0	14.3	12.8	14.8
Time (min)	13.3	12.1	13.9	12.3	13.2	11.6	13.1	12.3	13.9	13.1

a) Draw a scatter diagram for the above data.

b) What type of correlation is observed in the scatter diagram?

c) Draw a line of best fit on the scatter diagram.

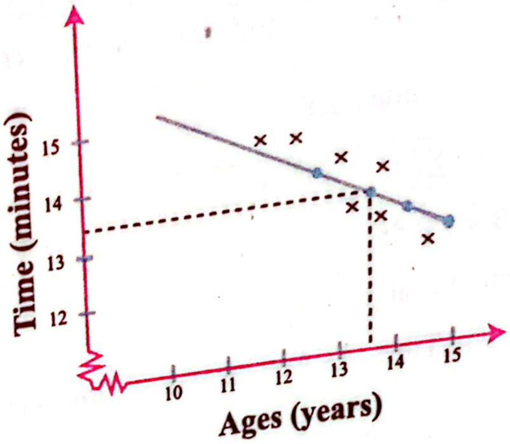
d) Use your line to estimate the time that a participant aged 14.5 years is expected to complete 50 pages reading activity.

e) Measure the correlation coefficient.

f) Is your line of best fit reliable to estimate the time taken by a child of age 1 year?

Solution:

a)



b) From the data it is observed that older the age of child corresponds to greater speed in reading activity. The line is sloping down from left to right showing negative correlation (and neg. slope as well)

c) .... drawn on the graph

d) From the graph reading for 14.5 years age, expected time to complete 50 pages reading activity is 12.7min.

e) For measuring correlation coefficient, let's calculate the means:

Age ($\bar{x}$): 14.2 Time ($\bar{y}$): 12.9

Next, we calculate the deviations, their products and squared deviations.

age	time	$(x_i-\bar{x})$	$(y_i-\bar{y})$	$(x_i-\bar{x})(y_i-\bar{y})$	$(x_i-\bar{x})^2$	$(y_i-\bar{y})^2$
13.2	13.3	−1.0	0.4	−0.4	1	0.16
14.9	12.1	0.7	−0.8	−0.56	0.49	0.64
13.3	13.9	−0.9	1.0	−0.9	0.81	1.0
15.3	12.3	1.1	−0.6	−0.66	1.21	0.36
13.6	13.2	−0.6	0.3	−0.18	0.36	0.09
15.5	11.6	1.3	−1.3	−1.69	1.69	1.69
14.0	13.1	−0.2	0.2	−0.04	0.04	0.04
14.3	12.3	0.1	−0.6	−0.06	0.01	0.36
12.8	13.9	−1.4	1.0	−1.4	1.96	1.0
14.8	13.1	0.6	0.2	0.12	0.36	0.04
		$\sum(x_i-\bar{x})$	$\sum(y_i-\bar{y})$	$\sum[(x_i-\bar{x})\times(y_i-\bar{y})]$ $= -5.77$	$\sum(x_i-\bar{x})^2$ $=7.93$	$\sum(y_i-\bar{y})^2$ $=5.38$

Finally, we calculate the correlation coefficient (r):

$$r = \sum[(x_i-\bar{x}) \times (y_i-\bar{y})] \,/\, \sqrt{\left(\sum(x_i-\bar{x})^2 \times \sum(y_i-\bar{y})^2\right)}$$

$$r = (-5.77) \,/\, \sqrt{7.93 \times 5.38} \approx -0.88$$

The correlation coefficient (r) is approximately -0.88, indicating a strong negative linear relationship between Age and Time. As Age increases, Time tends to decrease, and vice versa.

f) No, it is highly unreliable as it lies outside the range as there are no children with age less than 10 years.

Exercise 12.2

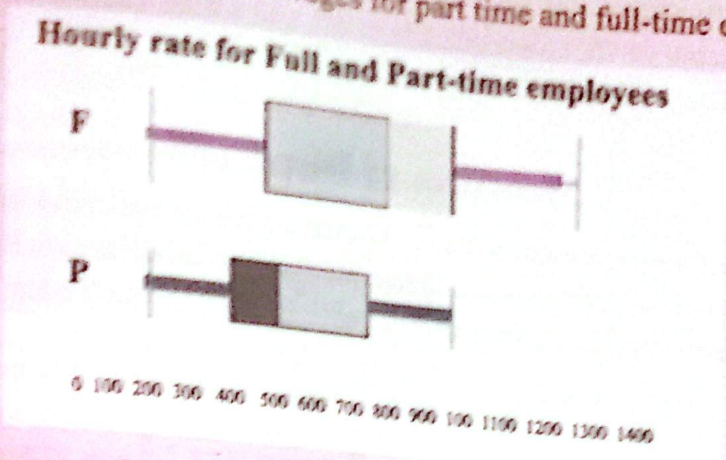
1. A mechanical engineer wants to analyze the relationship between the mass of a car and fuel efficiency. Show the data by box and whisker plots.

variables	Min/Max	Q1	median	Q3	Min/Max
Mass (kg)	1100	1200	1500	1700	2000
Fuel efficiency (Km/l)	24	20	14	12	10

a. Find IQR b. What is type of correlation?

c. What conclusion can be drawn from this?

2. Adjoining box & whisker plot shows wages for part time and full-time employees:



a) What is the amount of median wages paid to full time employees?
b) What are minimum and maximum wages paid to part time employees?
c) What is IQR for the dataset of fulltime employees?
d) What is IQR for the dataset of part time employees?
e) What type of correlation is observed in both datasets?

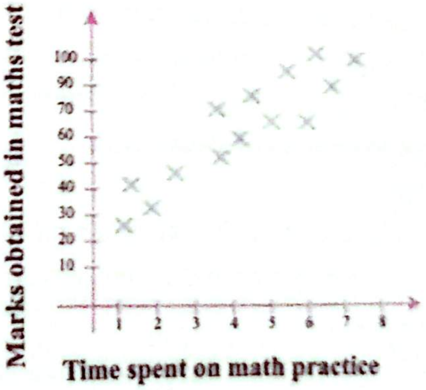
3. Maryam wants to investigate the relationship between the amount of rainfall and the number of flowers blooming in her garden. Draw the scatter diagram.

Rain (inch)	2	4	6	8	10
No. of flowers	10	22	55	70	80

a. Draw a line of best fit. b. Comment on type of correlation

4. Given scatter diagram is showing relation between practice time and marks obtained in math's test.

a. Find type of correlation.
b. Draw line of best fit.



5. Draw the scatter diagram for the data showing the relationship between amount of pollution (measured in Air Quality Index (AQI)) and number of sick days reported by a locality:

AQI (Pollution Level)	20	40	60	80	100	120	10	30	50	70
No. of Sick Days	5	18	25	40	60	80	3	15	20	27

Find correlation and what does this suggest?

6. A teacher wants to analyze the test scores of her students. Draw box and whisker plot to show a median score of 75, a first quartile of 60, and a third quartile of 85. Find:
   a. What percentage of students scored above 85? b. IQR

Measures of Dispersion

The single value which can represent a data is known as average or measure of central tendency. In grade 9, different types of averages e.g. mean, median and mode are studied.

Let us find the arithmetic mean of scores of five participants of a math quiz:

V (Bilal's score) $= 5, 5, 5, 5, 5 \Rightarrow \bar{V} = \frac{\Sigma V}{n} = \frac{25}{5} = 5$

W (Umer's score) $= 1, 3, 5, 8, 8 \Rightarrow \bar{W} = \frac{\Sigma W}{n} = \frac{25}{5} = 5$

X (Usman's score) $= 2, 4, 5, 6, 8 \Rightarrow \bar{X} = \frac{\Sigma X}{n} = \frac{25}{5} = 5$

Y (Hassan's score) $= -10, 0, 10, 12, 13 \Rightarrow \bar{Y} = \frac{\Sigma Y}{n} = \frac{25}{5} = 5$

Z (Abdullah's score) $= -15, -5, 10, 15, 20 \Rightarrow \bar{Z} = \frac{\Sigma Z}{n} = \frac{25}{5} = 5$

We have seen that all of the above five data have same measure of central tendency (arithmetic mean) but they vary in different ways. The values of above data are scattered away from their average in different ways. It is a confusing situation because we are unable to compare the data on the basis of their average. We can say that the average of a data is not more significant unless the scatterings of values around the average are measured.

The techniques or formulas which are used to measure, how the values of a data are scattered around its central value, are called measures of dispersion.

Types of Measures of Dispersion

(a) Range

Range is the difference between the maximum and the minimum values of the data. i.e. $R = x_m - x_o$, where x_m = maximum value and x_o = Minimum value

(b) Variance

Variance is the mean of squared deviation taken from mean.

Usually it is denoted by S^2. The formula for variance of grouped data:

$$S^2 = \frac{\Sigma f(xi - \bar{x})^2}{\Sigma f} \quad \text{(Proper mean formula)}$$

or $$S^2 = \frac{\Sigma fx^2}{\Sigma f} - \left(\frac{\Sigma fx}{\Sigma f}\right)^2 \quad \text{(Direct formula)}$$

(c) Standard Deviation

Standard deviation is a positive square root of variance. It is denoted by S and given by following formulae:

$$S = \sqrt{\frac{\Sigma f(xi - \bar{x})^2}{\Sigma f}} \quad \text{(Proper mean formula)}$$

$$\text{or} \quad S = \sqrt{\frac{\Sigma f x^2}{\Sigma f} - \left(\frac{\Sigma f x}{\Sigma f}\right)^2} \quad \text{(Direct formula)}$$

ample: Data shows scores of Huffaz in a Hifz Quran test:

scores	41-50	51 – 60	61 – 70	71 – 80	81 – 90	91 – 100
f	3	17	20	30	18	2

Find Range, variance and standard deviation of the data.

lution: Range = $x_m - x_o$ = 100 – 41 = 59

For the variance and SD, we need to construct the following table:

scores	x	x^2	f	fx	fx^2
41-50	45.5	2070.25	3	136.5	6210.75
51-60	55.5	3080.25	17	943.5	52364.25
61-70	65.5	4290.25	20	1310	85805
71-80	75.5	5700.25	30	2265	171007.5
81-90	85.5	7310.25	18	1539	131584.5
91-100	95.5	9120.25	2	191	18240.5
			$\sum f = 90$	$\sum f x = 6385$	$\sum f x^2 = 465212.5$

$$\text{Variance: } S^2 = \frac{\Sigma f x^2}{\Sigma f} - \left(\frac{\Sigma f x}{\Sigma f}\right)^2 = \frac{465212.5}{90} - \left(\frac{6385}{90}\right)^2 = 135.9 \text{ app.}$$

$$\text{Standard deviation: } S = \sqrt{\frac{\Sigma f x^2}{\Sigma f} - \left(\frac{\Sigma f x}{\Sigma f}\right)^2} = \sqrt{\frac{465212.5}{90} - \left(\frac{6385}{90}\right)^2}$$

$$= \sqrt{135.9} = 11.6$$

Coefficient of Variation (C.V.)

The percentage ratio between the standard deviation and mean of a data is called coefficient of variation i.e.

$$C.V. = \frac{SD}{Mean} \times 100$$

In above example, $CV = \frac{11.6}{70.5} \times 100 = 16.4$ app.

C.V. is used to compare the dispersion of different data which differ in their means and units. It is also used as a criterion for consistent performance. The data having smaller C.V is more consistent than the data having larger C.V.

Comparison of 2 Data Sets using Mean and Standard Deviation

To compare two data sets using mean and standard deviation, follow these steps:

1. Calculate the arithmetic mean of each data set.
2. Calculate the standard deviation of each data set.
3. Compare the means which indicates a difference in the central tendency of the two data sets.
4. Compare the standard deviations which indicate a difference in the variability or spread of the two data sets.

Example: Following are scores of 2 girl sections orchid and teal in Math quiz.

Score chart for Teal:

scores	41-50	51-60	61-70	71-80	81-90	91-100
No. of students	3	8	6	7	4	2

Score chart for orchid:

scores	41-50	51-60	61-70	71-80	81-90	91-100
No. of students	3	6	4	9	1	7

i. Calculate the arithmetic mean of each data set.
ii. Calculate the standard deviation of each data set.
iii. Compare the means. What does this indicate?
iv. Compare the standard deviations. What does this indicate?
v. Calculate CV for both data sets.

Solution:

Teal section				
scores	x	f	fx	$f(x-\bar{x})^2$
41-49	45.0	4	180	2116
50-60	55.0	6	330	1014
61-70	65.5	6	393	37.5
71-80	75.5	8	604	450
81-90	85.5	4	342	1225
91-100	95.5	2	191	1512.5
		$\sum f$ = 30	$\sum fx$ = 2040	$\sum f(x-\bar{x})^2$ = 6355

Orchid section				
scores	y	f	fy	$f(y-\bar{y})^2$
41-49	45.0	3	135	2187
50-60	55.0	6	330	1734
61-70	65.5	4	262	169
71-80	75.5	9	679.5	110.25
81-90	85.5	1	85.5	182.25
91-100	95.5	7	668.5	3865.75
		$\sum f$ = 30	$\sum fy$ = 2160.5	$\sum f(y-\bar{y})^2$ = 8248.25

i) $\bar{x} = \frac{\sum fx}{\sum f} = \frac{2040}{30} = 68$

$\bar{y} = \frac{\sum fy}{\sum f} = \frac{2160.5}{30} = 72.0$

ii) SD teal :

$S_{teal} = \sqrt{\frac{\sum f(x-x)^2}{\sum f}} = \sqrt{\frac{6355}{30}} = 14.6$

SD orchid:

$S_{orchid} = \sqrt{\frac{\sum f(y-\bar{y})^2}{\sum f}} = \sqrt{\frac{8248.25}{30}} = 16.6$

mparison:

The mean score of Orchid section (72.0) is higher than the mean score of Teal section (68.0), indicating that the average performance of Orchid is better than Teal.

The standard deviation of Orchid (16.6) is greater than the standard deviation of Teal (14.6), indicating that the performance of students in teal is more consistent and less spread out than Orchid. Such performance is more reliable.

$CV_{teal} = \frac{14.6}{68} \times 100 = 21.47$ $\qquad CV_{orchid} = \frac{16.6}{72} \times 100 = 24.41$

Real Life Application of Variance and Standard Deviation

Variance and standard deviation have numerous real-life uses in various fields, e.g Finance, Quality Control, Healthcare, Sports, Education, Environmental Science & Engineering

Exercise 12.3

1. What is dispersion? Write names and formulas of 3 measures of dispersion.
2. The energy consumption for a month is given below for Rabee's home.

Units consumed	101-200	201-300	301-400	401-500	501-600	601-700
No. of months	2	1	3	1	3	2

Find Range and variance of the energy consumption.

3. Maria wants to analyze the water supply in her area, given in the data as shown:

	Water supply			
Water supply (gallons)	501-700	701-900	901-1100	1101-1300
No. of hours	4	5	2	1

Find i. mean supply of water in the area. ii. Variance of the supply of water.
iii. SD of the supply of water. Iv. CV of the supply of water.

4. An organization wants to analyze salaries, grouped by department as shown below:

Department	Salary Range	Number of Employees
Elementary	40,000 - 60,000	10
Secondary	60,000 - 80,000	12
Higher secondary	80,000 - 100,000	8

Find the range, variance and standard deviation of the salaries across all departments. Also calculate CV of the data.

5. A teacher wants to analyze the scores of students in different sections of class 10 as shown:

Class	Scores Range	Number of Students
10 Mauve	70 - 80	15
10 teal	60 - 70	12
10 hazel	50 - 60	10

Find the variance and standard deviation of the grades of all sections .

6. Mutahhir wants to analyze the purchases made by customers, grouped by age as shown.

Age Group	Purchase worth (in Rs.)	Number of Customers
kids	0 - 500	20
juniors	500 - 1000	30
seniors	1000 - 2000	25

Find the range, variance and standard deviation of the purchases across all age groups.

7. Namra wants to analyze blood pressure of patients in OT & OPD grouped by age as shown.

Patients in OT

BP	100-110	110-120	120-130	130-140	140-150	150-160
No. of patients	3	8	6	7	4	2

Patients in OPD

BP	100-110	110-120	120-130	130-140	140-150	150-160
No. of patients	9	1	5	7	6	2

Find the Range, Mean, variance and standard deviation of the blood pressure of both datasets. Also find CV and comment on results.

8. Khola wants to forecast the temperature for the next week for Jhelum and Indiana. She has the history of a week for both of the cities as given:

Jhelum

Temperature(F°)	96 - 100	101-105	106-110	111-115
No. of days	2	3	1	1

Indiana

Temperature(F°)	51 - 60	61-70	71-80	81-90
No. of days	3	1	2	1

Help her to find:

i. Mean of both data sets. ii. Variance of both datasets

Also comment on the possible predictions for next week temperature for both cities. (Hint: In weather forecasting, variance is used to measure the spread or uncertainty in temperature predictions. A higher variance indicates a wider range of possible temperatures making it more challenging to predict the exact temperature.)

Probability

Probability is a measure of the likelihood of an event occurring. It is a number between 0 and 1 both inclusive that represents the chance of an event happening. Here are some key concepts in probability:

Experiment: An action or process that can produce a set of outcomes.
Outcome: A specific result of an experiment.
Sample Space: The set of all possible outcomes of an experiment.
Event: A set of one or more outcomes of an experiment.
Probability of an Event: A number between 0 and 1 both inclusive that represents the likelihood of an event.

Types of Probability

There are several types of probability such as:

- Theoretical Probability
  The number of favorable outcomes divided by the total number of possible outcomes.
- Experimental Probability
  Based on the results of repeated trials of an experiment.
- Conditional Probability
  The probability of an event occurring given that another event has occurred.
- Dependent Probability
  The probability of two or more events occurring dependently on each other.
- Mutually Exclusive Probability
  The probability of two or more events that cannot occur simultaneously.
- Joint Probability
  The probability of two or more events occurring together.

Probability of Combined Events

The probability of combined events depends on whether the events are independent, dependent, or mutually exclusive.

Probability of Independent Events

The probability of two or more independent events occurring is the product of their individual probabilities.

i.e. $P(A \text{ and } B) = P(A) \times P(B)$

Probability of Dependent Events

The probability of two or more dependent events occurring is the product of their individual probabilities, considering the conditional probability of each event.

i.e. $P(A \text{ and } B) = P(A) \times P(B|A)$

Probability of Mutually Exclusive Events

The probability of two or more mutually exclusive events is the sum of their individual probabilities.

i.e. P(A or B) = P(A) + P(B)

Conditional Probability

Probability of an event occurring given that another event has occurred.

i.e. P(A|B) = P(A and B) / P(B)

For example:

1. Probability of drawing two Kings (with replacement):
   P(King) × P(King) = 1/13 × 1/13= 1/169 ≈ 0.006 (or 0.6%)
   Probability of drawing two Kings (without replacement):
   P(King) × P(King) = 4/52 × 3/51 = 3/663 ≈ 0.0045 (or 0.45%)
   P(King) × P(King) = 4/52 × 3/51 = 3/663 ≈ 0.0045 (or 0.45%)
2. Probability of rolling two 6s: P(6) × P(6) = 1/6 × 1/6 = 1/36
3. Probability of drawing a King and then a Queen:
   P(King) × P(Queen | King) = (4/52) × (4/51) = 16/2652

Remember to consider the type of events and their relationships when calculating the probability of combined events.

Probability of Combined Events using Sample Space Diagrams

A sample space diagram is a visual representation of all possible outcomes of an experiment. It can be used to calculate the probability of combined events by identifying the favorable outcomes and dividing them by the total number of possible outcomes.

To calculate the probability of combined events using a sample space diagram:

- Draw a sample space diagram showing all possible outcomes of the experiment.
- Identify the favorable outcomes that meet the conditions of the combined event.
- Count the number of favorable outcomes.
- Divide the number of favorable outcomes by the total number of possible outcomes in the sample space.

Example:

What is the probability that both coin A and coin B land heads up if both coins are flipped simultaneously?

Solution:

Favorable Outcomes: hH (both coins land heads up)

Number of Favorable Outcomes: 1

Total Number of Possible Outcomes: 4

Probability: 1/4 or 0.25

This means that the probability of both coins landing heads up is 1/4 or 25%.

Sample space diagram		Coin 1	
		H	T
Coin2	h	hH	hT
	t	tH	tT

Probability of Combined Events using Diagram

There are several types of sample space diagram, including:

Possibility Diagrams   Tree Diagrams   Venn Diagrams

By using an appropriate sample space diagram, we can easily visualize and calculate the probability of combined events. This method is especially helpful while dealing with complicated experiments and multiple events.

(a) Probability of Combined Events using Possibility Diagram

Possibility diagram is graphical representation of all possible outcomes of an experiment.

Example:

Raabi and Jerry are hoping on their play circles of four sectors each painted in red, green, yellow and blue colours.

Find the probability that they will jump on:

(a) At the same colour (b) At the different colours

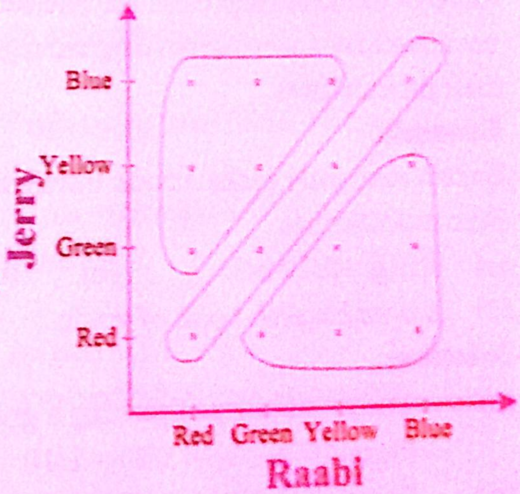
Solution: We can associate the ordered pair (x, y), where x represents hops of Raabi and y represents hops of Jerry to show an outcome in the sample space. Here (blue, green) means Raabi lands on blue while Jerry lands on green.

So the sample space is:

S = {(red, red), (red, green), (red, yellow), (red, blue), (green, red), (green, green), (green, yellow), (green, blue), (yellow, red), (yellow, green), (yellow, yellow), (yellow, blue), (blue, red), (blue, green), (blue, yellow), (blue, blue)}

Which seems complicated and time taking method of writing sample space for bigger problems.

We can express the same in more systematic and efficient way through possibility diagram.



There are 16 dots representing individual outcome, i.e. $n(S) = 16$

Let E_1 be the event that both girls jump on same colour

And E_2 be the event that they jump on different colours.

The loop on the diagonal enclosing four dots is associated With possible outcomes of E_1.

$\therefore n(E_1) = 4$ and $P(E_1) = 4/16 = ¼$

The triangles enclosing 6+6 dots not on the diagonal are Possible outcomes of E_2.

$\therefore n(E_2) = 12$ and $P(E_2) = 12/16 = 3/4$

By using a possibility diagram, we can easily visualize and calculate the probability of combined events. This method is especially helpful when dealing with sequential experiments and conditional probability.

(b) Probability of Combined Events using Tree Diagram

Tree diagram is a graphical representation of a sequence of events or decisions, showing the possible outcomes and their probabilities in the form of:

Root Node (starting point) Branches (representing possible outcomes or decisions)

Leaf Nodes (representing the final outcomes)

Tree diagrams are used to visualize probability problems, identify possible outcomes, calculate probabilities, analyze decision-making processes and show conditional probability.

Example:

Maria and Haleema toss 2 coins Haleema after Maria.

(a) Draw tree diagram of the experiment. Using tree diagram, find probabilities:

(b) Both tails up (c) Maria's coin head up and Haleema's coin tail up

Solution:

(a) Tree diagram

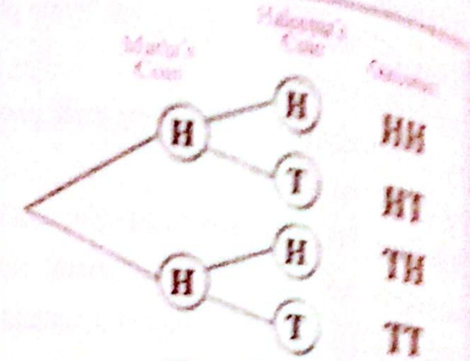
(b) Let E_1 be the probability of both heads up, then obviously there is only one such outcome among total of 4 outcomes. i.e. $E_1 = \{HH\}$

$\therefore P(E_1) = 1/4$

(c) Let E_2 be the probability of Maria's coin head up and Haleema's coin tail up, then obviously there is only one such outcome among total of 4 outcomes.

i.e. $E_2 = \{HT\}$

$\therefore P(E_2) \quad = 1/4$



Tree Diagram

(c) Probability of Combined Events using Venn Diagram

A Venn diagram is a pictorial representation of sets and their relationships, using overlapping disjoint circles to represent the intersection of sets. It can be used to calculate the probability of combined events by identifying the regions that represent the favorable outcomes and calculating their probabilities.

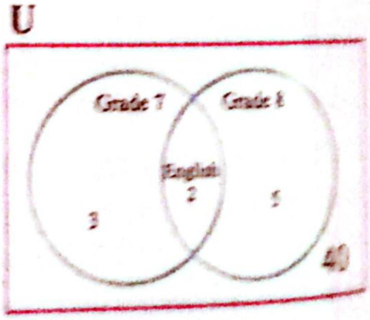
Example:

Zainab randomly picks a book from a library shelf containing 50 books, as shown in fig.

Find probabilities:

(a) It is grade7 book (b) It is an English book

(c) It is an English book of grade 7



Solution:

(a) The probability of picking a grade 7 book:

P(grade7 book) = 5/50 = 1/10

(b) The probability of picking an English book (given that it's a grade 7 book):

P(Englis book from grade 7 books) = 2/5

(c) To find the probability of picking a book that is both a grade 7 book and an English book, multiply the above probabilities:

$\therefore P(E) = (1/10) \times (2/5) = 1/25$

So, the probability of picking a book randomly from the 50 books that is a grade 7 English book is 1/25 or 4%.

By using a Venn diagram, we can easily visualize and calculate the probability of combined events. This method is especially helpful when dealing with multiple events and their intersections.

Probability Rules

Addition Rule

The probability of two mutually exclusive events is the sum of their individual probabilities.

Multiplication Rule

The probability of two or more events occurring independently is the product of their individual probabilities.

Complement Rule

The probability of an event not occurring is 1 minus the probability of the event occurring.

Addition Rule of Probability for Mutually Exclusive Events

The addition rule of probability states that the probability of two or more mutually exclusive events occurring is the sum of their individual probabilities.

$P(A \text{ or } B) = P(A) + P(B)$, where A and B are mutually exclusive events.

Applying addition rule of probability to problems of rolling even and odd number on a dice

The addition rule of probability can be applied to solve problems involving even and odd numbers rolled on a dice by treating the outcomes as mutually exclusive events

Example:

Hajrah rolled a fair cubical dice once. Find the probability of rolling an even or an odd number.

Solution:

On a fair cubical dice, there are 3 even numbers: 2, 4, 6 and 3 odd numbers: 1, 3, 5

Since the dice has 6 faces, the total number of possible outcomes is 6.

The probability of rolling an even number or an odd number is:

$P(\text{Even}) = 3/6 = 1/2$ and $P(\text{Odd}) = 3/6 = 1/2$

Since the events are mutually exclusive (a roll can't be both even and odd), we can add the probabilities:

$P(\text{Even or Odd}) = P(\text{Even}) + P(\text{Odd}) = 1/2 + 1/2 = 1$

Applying addition rule of probability to problems involving left and right hand turns

The addition rule of probability can be applied to solve problems involving taking left or right turns, by treating the outcomes as mutually exclusive events.

Example:

Mr. Bilal can make either a left turn (L) or a right turn (R) at an intersection on his journey to Tiri Mengal. The probabilities of making a left turn and a right turn are:

$P(L) = 0.3$ (30%) & $P(R) = 0.7$ (70%). Find the probability of making either a left or a right turn.

Solution:

Since making a left turn and making a right turn are mutually exclusive events (he can't make both at the same time), so

we can apply the addition rule to find the probability of making either a left turn or a right turn:

P(L or R) = P(L) + P(R) = 0.3 + 0.7 = 1

Applying addition rule of probability to problems involving tossing a coin

The addition rule of probability can be applied to solve problems involving tossing a coin by treating the outcomes as mutually exclusive events.

Example:

Sumyra tossed a fair coin with two outcomes: Heads (H) and Tails (T). The probabilities are P(H) = 0.5 (50% chance of getting Heads) and P(T) = 0.5 (50% chance of getting Tails). Find the probability of getting either Head or Tail.

Solution:

Since getting Head and getting Tail are mutually exclusive events (the coin can't land on both at the same time), so

we can apply the addition rule to find the probability of getting either Heads or Tails:

P(H or T) = P(H) + P(T) = 0.5 + 0.5 = 1

Applying addition rule of probability to problems involving winning or losing

The addition rule of probability can be applied to solve problems involving winning or losing by treating the outcomes as mutually exclusive events.

Example:

Affan played a game with two possible outcomes L and W such that P(W) = 0.4 (40%) and P(L) = 0.6 (60%). Find the probability of either winning or losing the game.

Solution:

Since winning and losing are mutually exclusive events (he can't both win and lose at the same time), we can apply the addition rule to find the probability of either winning or losing.

P(W or L) = P(W) + P(L) = 0.4 + 0.6 = 1

Multiplication Rule of Probability for Independent and Dependent Events

The multiplication rule of probability states that the probability of two or more events occurring is the product of their individual probabilities.

(i) For independent events:

P(A and B) = P(A) × P(B)

(ii) For dependent events:

P(A and B) = P(A) × P(B|A); where P(B|A) is the conditional probability of B given A.

Example: (Independent Events)

Haneen flipped a fair coin twice. What is the probability of getting heads both times?

Solution:

P(Heads on first flip) = ½ and P(Heads on second flip) = 1/2

Since the coin flips are independent, we can apply the multiplication rule:

∴ P (Heads on both flips) = P(Heads on first flip) × P(Heads on second flip) = 1/2 × 1/2 = 1/4

Example: (Dependent Events)

Zeemal drew 2 cards from a standard deck of cards. What is the probability of drawing two spades in a row?

Solution:

P(First card is spade) = 13/52 = 1/4

P(Second card is spade | First card is spade) = 12/51

Since the events are dependent, we can apply the multiplication rule with conditional probability:

P(Two spades in a row) = P(First card is spade) × P(Second card is spade | First card is spade)

= 1/4 × 12/51 = 3/51

Multiplication Rule of Probability Involving Trade

The multiplication rule of probability can be applied to solve problems involving trade by treating the outcomes as independent events.

Example:

Mahmil made 2 trades, as given:

Trade 1: Buying Mangoes (BM) and Trade 2: Selling Berries (SB)

With the probabilities:

P(BM) = 0.6 and P(SB) = 0.7

Find the probability of both trades being successful.

Solution:

Since the trades are independent (the outcome of one trade doesn't affect the other), we can apply the multiplication rule to find the probability of both trades being successful:

P(BM and SB) = P(BM) × P(SB) = 0.6 × 0.7 = 0.42

This means there is a 42% chance of both trades being successful.

Multiplication Rule of Probability Involving Flipping a Coin

The multiplication rule of probability can be applied to solve problems involving flipping a coin by treating the outcomes as independent events.

Example:

Azkaa flipped a biased coin twice with following outcomes:

Flip 1: Heads (H) or Tails (T) and Flip 2: Heads (H) or Tails (T)

With the probabilities:

P(H) = 0.7 nad P(T) = 0.3 Find the probability of:

i. Both heads

ii. First head then tail

iii. First tail then head

iv. Both tails

Solution:

Since the coin flips are independent (the outcome of one flip doesn't affect the other), we can apply the multiplication rule to find the probability of specific outcomes:

i. P(HH) = P(H) × P(H) = 0.7 × 0.7 = 0.49 (49% chance of getting Heads both times)

ii. P(HT) = P(H) × P(T) = 0.7 × 0.3 = 0.21 (21% chance of getting Heads then Tails)

iii. P(TH) = P(T) × P(H) = 0.3 × 0.7 = 0.21 (21% chance of getting Tails then Heads)

iv. P(TT) = P(T) × P(T) = 0.3 × 0.3 = 0.09 (9 % chance of getting Tails both times)

Multiplication Rule of Probability for Drawing Cards with Replacement

The multiplication rule of probability can be applied to solve problems involving drawing 2 cards with replacement by treating the outcomes as

Example:

Romana draws 2 cards from standard deck with replacement as given :

First draw → Any card (52 possibilities)

Second draw → Any card (52 possibilities, since the first card is replaced)

Find the probabilities of:

i. Both black cards ii. Both hearts

iii. First spade then heart iv. First Queen of Spades then King of Hearts

Solution:

Since the events are independent (the outcome of the second draw does not depend on the first draw), we can apply the multiplication rule to find the probability of specific outcomes:

i. P(Both black cards) = P(black) × P(black) = 26/52 × 26/52 = 1/4

ii. P(Both hearts) = P(heart) × P(heart) = 13/52 × 13/52 = 1/16

iii. P(spade then heart) = P(spade) × P(heart) = 13/52 × 13/52 = 1/16

iv. P(Queen of Spades then King of Hearts)

= P(Queen of Spades) × P(King of Hearts)

= 1/52 × 1/52 = 1/ 1704

Multiplication Rule of Probability for Drawing Cards without Replacement

The multiplication rule of probability can be applied to solve problems involving drawing 2 cards without replacement by treating the outcomes as dependent events.

Example:

Sundas draws 2 cards one after the other from a standard pack of playing cards, such that:

First draw → Any card (52 possibilities)

Second draw → Any card (51 possibilities, since one card has already been drawn)

then find the probabilities of:

i. Both black cards ii. Both hearts

iii. First spade then heart iv. First Queen of Spades then King of Hearts

Solution:

Since the events are dependent we can apply the multiplication rule to find the probability :

i. P(Both black cards) = P(black) × P(black/black) = 26/52 × 25/51 = 25/102

ii. P(Both hearts) = P(heart) × P(heart/heart) = 13/52 ×12/51 = 3/51

iii. P(spade then heart) = P(spade) × P(heart/spade) = 13/52 × 13/51 = 13/204

iv. P(first Queen of Spades then King of Hearts)

= P(Queen of Spades) × P(King of Hearts/ Queen of Spades) = 1/52 × 1/51 = 1/ 2652

Exercise 12.4

1. Musaab picked a card randomly from a box containing cards bearing numbers 1,2 and 3. He then replaced and picked another after well shuffling. He found product of two numbers he got in 2 trials. Draw the possibility diagram of this experiment and find probability of getting an even product.

2. Ashas flipped a coin thrice. Draw tree diagram to find the probability of getting three Tails?
3. A survey of 100 students' shows that 40 like pizza, 45 like burgers. Draw venn diagram and find probability of students like either pizza or burgers, assuming none like both?
4. What is the probability of a patient testing positive for a disease with either Test A or Test B, given that the probability of testing positive with Test A is 0.3 and the probability of testing positive with Test B is 0.4, and the tests are independent?
5. What is the probability of getting either Job A or Job B, given that the probability of getting Job A is 0.5 and the probability of getting Job B is 0.4 and the offers are independent?
6. What is the probability of taking either Volvo bus or Bedford bus to work, given that the probability of taking Volvo bus is 1/2 and the probability of taking Bedford bus is 1/3, and the buses are to be chosen independently?
7. What is the probability of either Team Sudaisee or Team Huzaifee winning the championship, given that the probability of Team Sudaisee winning is 12/27 and the probability of Huzaifee team winning is 13/27, and the teams play independently?
8. HJ produces party abayaas. In the products, probability of colour damage is 0.1 and the probability of bad stitching is 0.2. Find the probability of a product having either colour damage or bad stitching.
9. Fakeha is tested for a disease with Test A and Test B and the probability of testing positive with A is 0.9 and the probability of testing positive with B is 0.7. Find the probability of Fakehaa for testing positive with both Test A and Test B, when the tests are independent?
10. Cocobakes produces festive cakes. The quality control department performs 2 quality checks A and B on an item. Find the probability of a product passing both Quality Control Test A and Quality Control Test B, given that the probability of passing Test A is 0.9 and the probability of passing Test B is 0.8, and the tests are independent?

I have Learnt

- A table consisting of values of a data along with their frequencies is called a frequency table.
- In a group data, the table consisting of class intervals and their respective frequencies is called frequency table or a frequency distribution.
- The representation of data by a polygon in which class marks are plotted along x-axis and frequencies along y-axis is called a frequency polygon.
- In a group data, number of values less than or within the limits of a class is called cumulative frequency of the respective class.
- A polygon made by joining the points whose x-coordinates are class marks and y-coordinates are their cumulative frequencies is called a cumulative frequency polygon.

- An ogive is a smooth curve, while a cumulative frequency polygon is a polygon-shaped graph made by straight segments.
- The three values which divide the whole data into four equal parts are called quartiles.
- Deciles divide a dataset or a distribution into 10 equal parts, each representing 10% of the data.
- Percentiles divide a dataset or a distribution into 100 equal parts, each representing 1% of the data.
- IQR = Q3 – Q1, where IQR represents the spread of the data within the middle half of the distribution.
- A box-and-whisker plot is a graphical representation of a dataset that displays the distribution of values through 5 data points: min, max, Q_1, Q_2, Q_3
- A single value which can represent the whole data is called measure of central tendency, measure of location or an average.
- The middle most value of an arranged data is called median average or simply median.
- The measures which tell us how the values of a data are scattered around their average, are called measures of dispersion.
- Range is the difference of largest and smallest values of the data.
- Variance is the ratio between sum of squares of deviations from mean and the number of values.
- Standard deviation is the positive square root of variance.
- Probability is a measure of the likelihood of an event occurring.
- The percentage ratio between the standard deviation and mean of a data is called coefficient of variation i.e. C.V. = $\frac{\text{SD}}{\text{Mean}} \times 100$

MISCELLANEOUS EXERCISE-12

1. **Encircle the correct option in the following.**

i. Which of the following is not measure of dispersion?

(a) variance (b) standard deviation (c) range (d) arithematic mean

ii. What is median of the data 4, 3, 0, 2, 1?

(a) 0 (b) 2 (c) 3 (d) 4

iii. Which of the following is measure of dispersion?

(a) arithmetic mean (b) range (c) quartile (d) median

iv. Which of the following is used to compare the consistency of two data?

(a) arithmetic mean (b) standard deviation (c) C.V. (d) G.M.

v. What is sum of deviations taken from arithmetic mean?

(a) $\sum f$ (b) $\sum x$ (c) n (d) 0

vi. What is variance of five values 4, 4, 4, 4, 4?

(a) does not exist (b) 4 (c) 5 (d) 0

vii. Which of the following divides the data into four equal parts?

(a) decile (b) quartile (c) percentile (d) median

viii. Which of the following divides the data into ten equal parts?

(a) decile (b) quartile (c) percentile (d) median

ix. Which of the following divides the data into two equal parts?

(a) decile (b) quartile (c) percentile (d) median

x. Which of the following divides the data into hundred equal parts?

(a) decile (b) quartile (c) percentile (d) median

xi. line of best fit is given by the equation:

(a) $y = x^2$ (b) $y = mx^2 + c$ (c) $y = mx + c$ (d) $x = y^2$

xii. CV is given by the formula:

(a) $\frac{SD}{100}\times$ mean (b) $\frac{SD}{range}\times 100$ (c) $\frac{mean}{SD}\times 100$ (d) $\frac{100}{mean}\times SD$

xiii. Probability of drawing a king in one draw is -------- than probability of drawing 2 kings in 2 draws.

(a) greater (b) smaller (c) equal (d) no relation

xiv. Probability of drawing 2 aces in 2 draws with replacement is -------- than probability of drawing 2 jacks in 2 draws without replacement.

(a) greater (b) smaller (c) equal (d) no relation

xv. Probability of rolling a standard cubical dice for an even number in 2 attempts is -------- than probability of rolling an odd numbers in 2 attempts.

(a) greater (b) smaller (c) equal (d) no relation

2. For the following data, draw cumulative frequency polygon, then estimate 3 quartiles graphically. Also find variance and stand deviation of the data.

x	5	10	15	20	25	30	35	40
f	2	4	6	8	10	7	5	3

3. Aatika wants to examine the relationship between the time of exercise of patients and their blood pressure. Show the data by box and whisker plots .

variables	Min/Max	Q1	median	Q3	Min/Max
Time (min)	0	15	30	45	60
Blood Pressure	140/90	130/85	120/80	120/75	115/70

a. Find IQR of both data sets b. What is type of correlation?

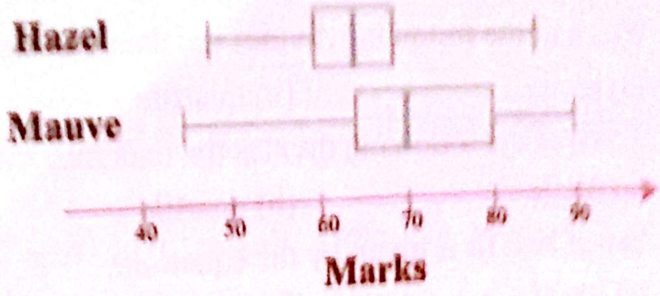
b. What conclusion can be drawn from this?

4. A sports coach wants to analyze the timed performance of athletes in an event. Draw the box and whisker plot to show a median time of 20 seconds, Q1 of 10 seconds and Q3 of 30 seconds. Find what percentage of athletes could perform in first 10 seconds? Also find IQR?

5. Given box and whisker plots show the marks of 4 boys of mauve section and 4 boys of hazel section in math test.

a) Find Median scores of both sections

b) Find IQR of scores of both sections



6. Bareera and Hadiya draw cookie packs, Hadiya after Bareera from a bag of 2 choco cookies and 3 lemon cookies. Find the probability of drawing both lemon cookies

7. A company has two machines, A & B, which produce 60% & 40% of the total products, respectively. Machine A produces 20% defective products & Machine B produces 15% defective products. What is the probability that a randomly selected product is defective?

Maths Play Ground:

Help students to make colour wheel/ number wheel and let them perform trials of moving the wheel to record the outcomes. Then ask them different related questions of probability.

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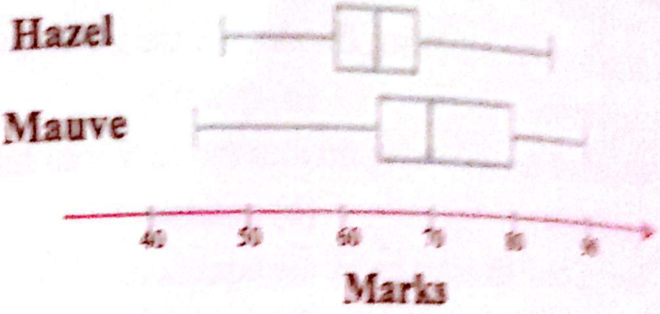

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Help students to make colour wheel/ number wheel and let them perform trials of moving the wheel to record the outcomes. Then ask them different related questions of probability.

ANSWERS

UNIT 1

Exercise 1.1

1. (i) $\sqrt{3}i$ (ii) $12i$ (iii) $\frac{2}{3}i$ (iv) $-2\sqrt{5}i$ (v) $4-2\sqrt{15}\,i$ (vi) -4

2. (i) $9+4i$ (ii) 2 (iii) $-9+5i$ (iv) -1 (v) $-6+8i$ (vi) $-10-11i$
(vii) $-16-16i$ (viii) $5+12i$ (ix) 45 (x) $16+30i$
(xi) $1+\sqrt{6}+(\sqrt{2}-\sqrt{3})i$ (xii) 3

3. (i) i (ii) i (iii) 1 (iv) $-i$ (v) $-27i$ (vii) 16

4. (i) $8+0=8$ (ii) $-17+i$ (iii) $1-13i$ (iv) $22i$ (v) $2i$ (vi) -2

5. (i) $\frac{12}{17}+\frac{3}{17}i$ (ii) $-\frac{15}{16}+\frac{18}{61}i$(iii) $\frac{2}{7}-\frac{3\sqrt{5}}{7}i$ (iv) $\frac{7}{5}-\frac{2}{5}i$ (v) $-\frac{9}{41}+\frac{40}{41}i$
(vi) $\frac{8}{5}+\frac{1}{5}i$ (vii) $\frac{7}{5}-\frac{9}{5}i$ (viii) $\frac{a^2-b^2}{a^2+b^2}+\frac{2ab}{a^2+b^2}i$ (ix) $-\frac{1}{2}+\frac{1}{2}i$ (x) 4

Exercise 1.2

1. (a) $4-5i$ (b) $3+3i$ (c) $-5+5i$ (d) $-4i$

3. (a)$\frac{1}{2}-\frac{1}{2}i$ (b) $\frac{7}{58}+\frac{3}{58}i$ (c) $\frac{5}{122}+\frac{3}{61}i$ (d) $\frac{5}{2}-\frac{1}{2}i$ (e) $3+2i$ (f) $\frac{a}{a^2+b^2}+\frac{b}{a^2+b^2}i$

4. (a) 16 (b) 2 (c) 49 (d) 40 (e) 181 (f) 137

6. Do yourself 5. b. (i) $\sqrt{10}$ (ii) $\sqrt{34}$ (iii) 1

7. (a) $\text{Re}=-1$, $\text{lm}=-2\sqrt{6}$ (b) $\text{Re}=1$, $\text{lm}=2\sqrt{2}$ (c) $\text{Re}=\frac{41}{10}$, $\text{lm}=-\frac{3}{10}$ (d) $\text{Re}=-\frac{1}{2}$, $\text{lm}=\frac{\sqrt{3}}{2}$
(e) $\text{Re}=-1$, $\text{lm}=1$ (f) $\text{Re}=\frac{1}{2}$, $\text{lm}=0$ (g) $\text{Re}=\frac{-8}{25}$, $\text{lm}=-\frac{6}{25}$

Exercise 1.3

1. $\{\pm\sqrt{7}i\}$ 2. $\{\pm 3i\}$ 3. $\{\pm 10i\}$ 4. *yes* 5. *yes* 6. No 7. Yes 8. $(x-4i)(x+4i)$

9. $(a-bi)(a+bi)$ 10. $(x-5yi)(x+5yi)$ 11. $W=1-i$, $Z=4-i$

12. $W=3-2i$, $Z=1$ 13. (a) (i) $I=9+\frac{37}{4}J$ (ii) $I=\frac{-37}{5}+\frac{2}{5}J$ (b) (i) $Z=-\frac{5}{2}-10J$
(ii) $Z=-\frac{770}{73}-\frac{380}{73}J$ (c) $\frac{225}{41}+\frac{200}{41}i$

Miscellaneous Exercise-1

1. (i) (c) (ii) (b) (iii) (a) (iv) (c) (v) (b) (vi) (c)
(vii) (d) (viii) (b) (ix) (c) (x) (d)

2. a $-10+9i$ (b) $1-3i$ 3. (a) $2x-3y+(3x+2y)i$ (b) $-45i$

4. (a) $3\sqrt{2}-\sqrt{7}i$ (b) $\sqrt{2}i$ 5. (a) $\frac{5}{4}$ (b) 245 6. (a) $\frac{-4}{5}-\frac{-3}{5}i$ (b) $\frac{3\sqrt{5}}{5}-\frac{\sqrt{5}i}{5}$

7. a. $2(x-3i)(x+3i)$ b. $-(x-5i)(x+5i)$

8. a. $3(x-\sqrt{5}i)(x+\sqrt{5}i)$ b. $6(x-\sqrt{6}i)(x+\sqrt{6}i)$

UNIT 2

Exercise 2.1

1. (i) $x^2-x-11=0$ (ii) $3x^2+26x-2=0$ (iii) $x^2-2x=0$

2. (i) $\{1,4\}$ (ii) $\{1\}$ (iii) $\{-1,8\}$ (iv) $\{3,5\}$ (v) $\{3,-10\}$

3. (i) $\{4,-8\}$ (ii) $\{0,-8\}$ (iii) $\{-3\pm 3\sqrt{2}\}$ (iv) $\left\{\frac{-2\sqrt{3}\pm 2}{\sqrt{3}}\right\}$ (v) $\left\{\frac{-1\pm\sqrt{-3}}{2}\right\}$ (vi) $\left\{\frac{5}{2},-\frac{1}{2}\right\}$

4. (i) $\{3,-3\}$ (ii) $\left\{\frac{-5\pm\sqrt{17}}{4}\right\}$ (iii) $\{-1, 24\}$ (iv) $\{0,2\}$ (v) $\left\{1,-\frac{3}{2}\right\}$ (vi) $\left\{\frac{-11\pm\sqrt{165}}{2}\right\}$

5. $\{-3,-3\}$ 6. $-1\pm\sqrt{3}i$ 8. Only two solutions, degree is 2

9. The width of the strips added is approximately 16.85 meters.
The new dimensions of the playground are approximately 76.85 meters by 46.85 meters.

10. It takes approximately 5.65 seconds for the car to travel 145 meters.

Exercise 2.2

1. (i) $ay^2+by+c=0$ (ii) $9y^2-3y+7=0$ (iii) $y^2+1=0$ (iv) $z^2-4z+3=0$
(v) $3y^2+8y+5=0$ (vi) $3z^2+7z+11=0$ (vii) $az^2+bz+c=0$
(viii) $8y^2-7y-1=0$ (ix) $y^2-16y-42=0$ (x) $4z^2+19z+13=0$

2. (i) $\{\pm2,\pm4\}$ (ii) $\{\pm3,\pm5i\}$ (iii) $\{-32,-243\}$ (iv) $\left\{\pm1,\pm\frac{2}{\sqrt{3}}\right\}$ (v) $\left\{0,-\frac{2}{5}\right\}$
(vi) $\left\{\pm\sqrt{\frac{14}{5}},\pm\sqrt{\frac{-2}{5}}\right\}$ (vii) $\left\{-1,2,\frac{1}{2}\right\}$ (viii) $\left\{3,-4,\frac{1}{3},-\frac{1}{4}\right\}$
(ix) $\left\{\frac{1\pm\sqrt{5}}{2},\frac{-3\pm\sqrt{73}}{8}\right\}$ (x) $\{1,5\}$ (xi) $\{1,2\}$ (xii) $\{2,3\}$
(xiii) $\left\{1,-8,\frac{-7\pm\sqrt{97}}{2}\right\}$ (xiv) $\{2,-4,-1\pm2i\}$ (xv) $\{-1\pm\sqrt{17},-1\pm\sqrt{41}\}$

3. (i) $\left\{\pm1,\pm\frac{1}{\sqrt{2}}\right\}$ (ii) $\left\{1,-\frac{1}{2}\right\}$ (iii) $\{\pm1\}$ (iv) $\{-1\}$

4. do yourself 5. Take $z=y-4$, The solution $y=10$, $y=5$ 6. $\left\{5, \frac{-5+5\sqrt{3}i}{2}, \frac{-5-5\sqrt{3}i}{2}\right\}$

Exercise 2.3

1. i) 144 ii) 49 iii) -28 iv) 841
2. i) Rational and unequal ii) Rational and unequal
iii) Rational and unequal iv) Imaginary
v) Irrational and unequal
3. $k=\pm24$ 4. $k=\pm6$
6. i) $m=9$ ii) $m=-1/4$ iii) $m=-1, -11/3$

Exercise 2.4

1. i) $S=5$, $P=2$ ii) $S=-3/2$, $P=1/2$ iii) $S=2/5$, $P=2/5$
iv) $S=-2$, $P=9/4$ v) $S=17/16$, $P=-3/4$ vi) $S=25.67$, $P=6.0$
2. i) $3x^2+5x-8=0$ ii) $x^2-3\sqrt{3}\,x+6=0$ iii) $x^2-4x+1=0$
iv) $x^2+25=0$ v) $x^2-14x+53=0$
3. i) -5/4 ii) -20/9 iii) 16/3 iv) 1/2
v) -5/3 vi) 8/9 vii) -80/27 viii) 25/3
4. i) $49x^2-2x+49=0$ ii) $7x^2+10x+7=0$ iii) $49x^2-2x+49=0$
iv) $49x^2+96=0$ v) $7x^2+6x+15=0$ vi) $49x^2-2x+49=0$
vii) $49x^2+143x+100=2$ viii) $49x^2+100x+100=0$ ix) $x^2+68x/49+359/343=0$
5. $x^2-60x+864=0$ 6. $2x^2+38x-7=0$ 7. $k=\pm\sqrt{5}$ 8. $k=4$

Exercise 2.5

1. S.S. = {(-1, 3), (9/5, -13/5)} 2. S.S. = {(1, 1), (-5, -8)} 3. S.S. = {(2, -9), (-3, 6)}

4. S.S. = {(1/2, 7/2)} 5. S.S. = {(9/√2, 3/√2), (-9/√2, -3/√2), (3√5/2, 3√5/2), (-3√5/2, -3√5/2)}

6. S.S. = $\left\{(\pm\sqrt{6}, \pm\sqrt{6}), \left(\sqrt{\frac{5}{2}}, -6\sqrt{\frac{5}{2}}\right), \left(-\sqrt{\frac{5}{2}}, 6\sqrt{\frac{5}{2}}\right)\right\}$ 7. S.S. = (-1/2, -4), (2, 7/2)

8. S.S. = {(-1, -1), (2, -4)} 9. S.S = {± 1/√2 , ± 3/√2} 10. S.S = {(±3, ±√2)}

Exercise 2.6

1. 10, 12 2. 20 3. 6, 5 4. (3,6) (6,3) 5. 5, 6 6. 3,5 7. Son's age = 13 , Father's age = 37 8. 83

9. (24, -4)(15, 5) 10. 4, 6 11. 625, 400 12. 8, 9 13. Appro 4.1 seconds 14. 20.82 Km/h

15. 11 cm long and 10 cm wide

Miscellaneous Exercise-2

1. (i) (c) (ii) (d) (iii) (b) (iv) (b) (v) (a) (vi) (a)
(vii) (d) (viii) (b) (ix) (c) (x) (b) (xi) (c) (xii) (d) (xiii) (a)

2. $\left\{1, -\frac{1}{2}, \frac{-1\pm\sqrt{-3}}{2}, \frac{-1\pm\sqrt{-3}}{4}\right\}$ 3. m = 3/2, x = -3 4. $\frac{c+3(b+3a)}{a}$ 5. $a = \pm 2\sqrt{2}$, b = -2 6. $L = x + 1$, then $W = x + 3$

Exercise 3.1

1. (a) 1, 2, 1-by-2 (b) 2, 2, 2-by-2 (c) 1, 1, 1-by-1 (d) 1, 1, 1-by-1

2. (a) not equal (b) not equal (c) not equal (d) equal 3. $R = \begin{bmatrix} 7 & 11 & 10 \end{bmatrix}, C = \begin{bmatrix} 7 \\ 11 \\ 10 \end{bmatrix}$

4. (a) scalar matrix (b) square matrix (c) unit matrix (d) unit matrix (e) scalar matrix

5. (a) symmetric matrix (b) skew symmetric matrix (c) skew symmetric matrix
(d) skew symmetric matrix (e) symmetric matrix

Exercise 3.2

1. (a) $x = 2, y = 9$ (b) impossible, corresponding elements are not equal . (c) impossible; orders are not equal
(d) $x = -4, y = -2, z = -8$ (e) $x = 3, y = 3, z = 6$ (f) impossible; orders are not equal .
(g) $x = 6, y = 5$ (h) $x = 2, y = 5, z = 10$ (i) $x = 4, y = 2$ (j) $x = -10, y = -6, z = -4$

2. $-R = \begin{bmatrix} -5 & 0 & -3 \\ -7 & 9 & 1 \\ 8 & -5 & -6 \end{bmatrix}, -S = \begin{bmatrix} 5 & -2 \\ -3 & 6 \\ 9 & -4 \end{bmatrix}, -T = \begin{bmatrix} -5 & 6 & -1 \end{bmatrix}$ 3. (i) $\begin{bmatrix} 20 & 12 \\ -8 & 11 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix}$

4. (i) yes (ii) no (iii) no (iv) no 8. (i) $Z = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$ (ii) $Z = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$

9. (a) (i) impossible (ii) 1-by-2 (iii) 2-by-1 (iv) impossible (v) impossible (vi) impossible
(vii) 2-by-1 (viii) impossible (ix) 1-by-2 (x) impossible (xi) 1-by-1 (xii) 1-by-1
(xiii) 2-by-2 (xiv) 2-by-2 (xv) 2-by-1

(b) (ii) $\begin{bmatrix} 8 & 0 \end{bmatrix}$ (iii) $\begin{bmatrix} 8 \\ 0 \end{bmatrix}$ (vii) $\begin{bmatrix} 5 \\ 15 \end{bmatrix}$ (ix) $\begin{bmatrix} 45 & 15 \end{bmatrix}$ (xi) $\begin{bmatrix} 50 \end{bmatrix}$ (xii) $\begin{bmatrix} 20 \end{bmatrix}$ (xiii) $\begin{bmatrix} 10 & 20 \\ 15 & 30 \end{bmatrix}$ (xiv) $\begin{bmatrix} 2 & 5 \\ 4 & 3 \end{bmatrix}$ (xv) $\begin{bmatrix} 19 \\ -2 \end{bmatrix}$

10. (i) $AB = \begin{bmatrix} 2 & 1 \\ 4 & -7 \end{bmatrix}, BA = \begin{bmatrix} 4 & 3 \\ -6 & -9 \end{bmatrix}, AB \neq BA$ (ii) $AC = CA = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

(vi) $A^2 = \begin{bmatrix} 1 & 0 \\ 8 & 9 \end{bmatrix}, B^2 = \begin{bmatrix} 4 & -1 \\ 0 & 9 \end{bmatrix}, A + B = \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix}, A - B = \begin{bmatrix} -1 & -1 \\ 2 & 6 \end{bmatrix},$

$(A+B)(A-B) = \begin{bmatrix} -1 & 3 \\ -2 & -2 \end{bmatrix}, A^2 - B^2 = \begin{bmatrix} -3 & 1 \\ 8 & 0 \end{bmatrix}$ (vii) no (viii) no (ix) no (x) yes

Exercise 3.3

1. (a) A is non singular, B is non singular, C is singular, D is non singular and E is singular. (b) $k = 1$

2. (a) $R^{-1} = \begin{bmatrix} 1 & \frac{1}{2} \\ 3 & 2 \end{bmatrix}$, but S^{-1}, T^{-1} & U^{-1} are not possible.

4. Do yourself. 5. (i) $\begin{matrix} 4x + 2y = 6 \\ 3x + y = 4 \end{matrix}$ (ii) $\begin{matrix} 5x = 10 \\ 4y = 20 \end{matrix}$ (iii) Not possible (iv) $5x + 4y =$

6. (a)
(i) $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 1 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 90 \\ 10 \end{bmatrix}$ (iii) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
(iv) $\begin{bmatrix} \frac{5}{2} & -3 \\ -4 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (b) $K = 8$ 7. (a) (i) $\{(-3, 2)\}$ (ii) $\left\{\left(0, \frac{3}{4}\right)\right\}$ (iii) impossible
(iv) impossible (b) (i) $\{(4, -1)\}$ (ii) $\left\{\left(\frac{5}{3}, \frac{-1}{2}\right)\right\}$ (iii) $\{(20, 10)\}$ (iv) impossible

Exercise 3.4

1. 50°, 40° 2. literature = 5, science=10 3. Note book =Rs.100, Book=Rs.150
4. Sundas =15 years, Zenab =7 years 5. The first truck (with a capacity of 10 tons) made 7 round trips
The second truck (with a capacity of 12 tons) made 13 round trips. 6. 8 kg of the metal with 55% and 12 kg of the
metal with 80% aluminum content. 7. 150 pounds of soybean and 200 pounds of cornmeal
8. Length=76 meters Width=19 meters

Miscellaneous Exercise-3

1.

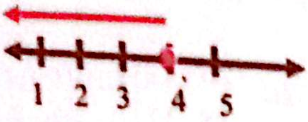
(i)	(b)	(ii)	(d)	(iii)	(b)	(iv)	(d)	(v)	(d)	(vi)	(d)
(vii)	(a)	(viii)	(a)	(ix)	(d)	(x)	(c)	(xi)	(c)	(xii)	(b)
(xiii)	(d)	(xiv)	(c)	(xv)	(b)	(xvi)	(d)				

3. impossible, matrix of coefficient is singular. 4. impossible, matrix of coefficient is singular.
5. impossible, matrix of coefficient is singular.

UNIT 4

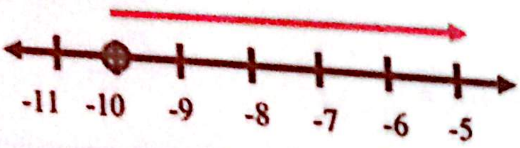
Exercise 4.1

1. $x \leq 4$

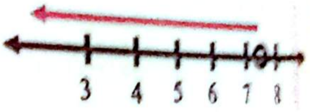


2. No solution

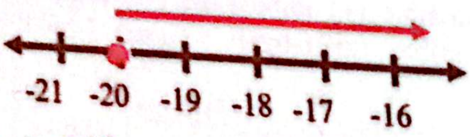
3. $x > -10$



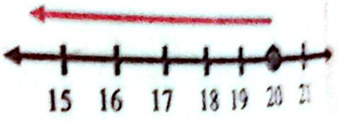
4. $x < \frac{55}{7}$



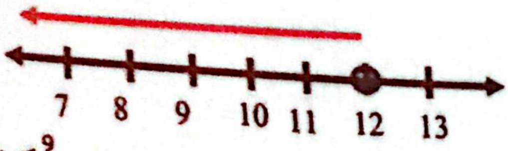
5. $x \geq -20$



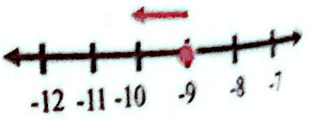
6. $x < 20$



7. $3x + 4 < 40, x < 12$



8. $2(x+8) \leq -2, x \leq -9$

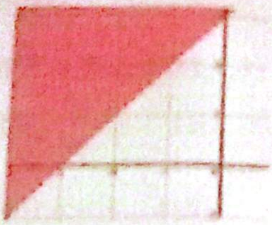


9. $x > 7$ 10. $x \leq \frac{9}{2}$ 11. $x < -1$ or $x > 3$ 12. $-1 < x < 9$
13. $x < -9$ or $x > 10$ 14. $-7 < x < 3$ 15. $x \neq 4$ 16. $4 - 15 < x < 15 + 4$
17 $10 \leq W \leq 24$ 18 There are no solutions except -1 19. $0 \leq x \leq 24.54$ 20. $L: x \leq 10m, W: 2x \leq 20$

Exercise 4.2

1. No 2. Yes 3. Yes 4. No 5. Yes 6. No

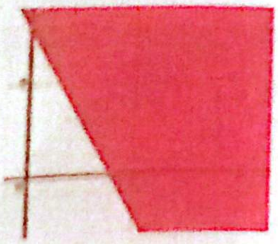
7.



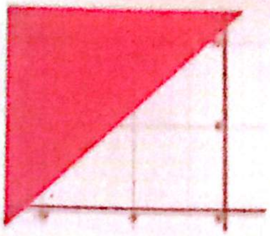
8.

9.

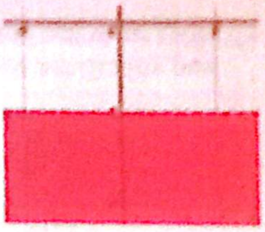
10.



11.



12.



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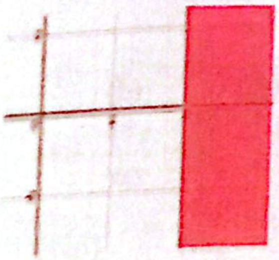
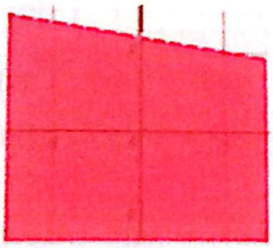
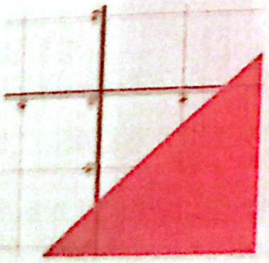


14.

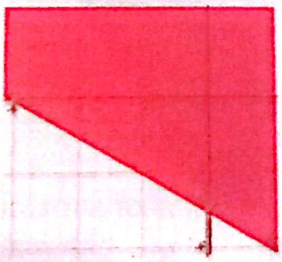


15. Do Yourself.

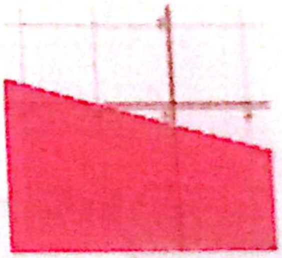
16. $x - 3 \geq y$



17. $-2y \leq x + 6$



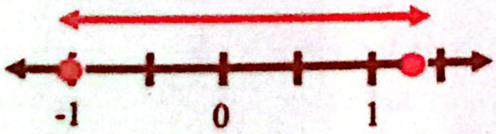
18. $x + 4y < -2$



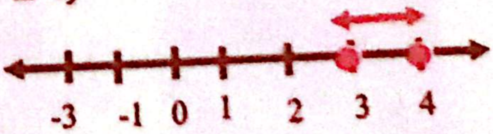
Exercise 4.3

1. $\{-8, 14\}$ 2. $\{-\frac{39}{2}, \frac{21}{2}\}$ 3. No solution 4. $\{4\}$ 5. $\{1\}$ 6. $\{-\frac{38}{5}, \frac{46}{5}\}$ 7. Absolute must be positive 8. Absolute Deviation= ±0.5 psi, Minimum Pressure=8 psi−0.5 psi=7.5 psi Maximum Pressure=8 psi+ 0.5 psi=8.5 psi

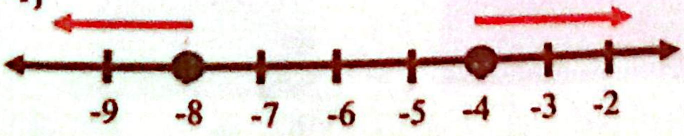
9. $\{-1 \leq x \leq \frac{11}{9}\}$



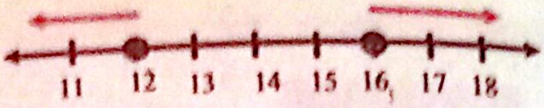
10. $\{3 \leq x \leq 4\}$



11. $\{x < -8 \text{ or } x > -4\}$

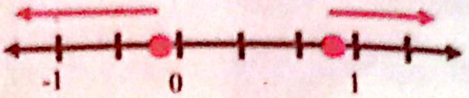


12. $\{x < 12 \text{ or } x > 16\}$,



13. $\{-\frac{8}{3} \le x \le 4\}$

14. $\{x \le -\frac{2}{15} \text{ or } x \ge \frac{14}{15}\}$

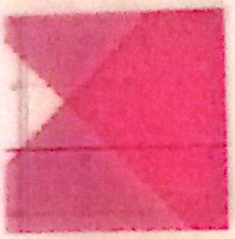


15. $\{470 \le W \le$ [illegible]$\}$

Exercise 4.4

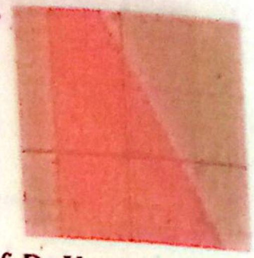
2. a. (0,0) and (−3,1) are solutions b. (3,2) is solution

3. a.



c. Both lines are parallel and superpose each other

e.



g.

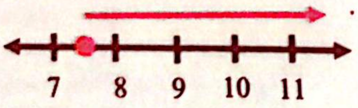
b. Do Yourself. d. Do Yourself. f. Do Yourself. h. Do Yourself.

4. No solution

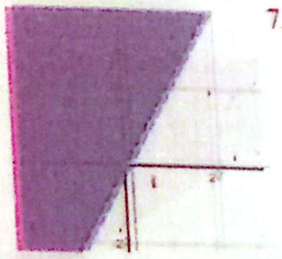
Miscellaneous Exercise-4

1. (i) (d) (ii) (c) (iii) (b) (iv) (b) (v) (c) (vi) (d)
(vii) (b) (viii) (c)

2. $x \ge \frac{15}{2}$



3. $x \ge 3$ 4. No solution 5. $x \ge -1$ 6.



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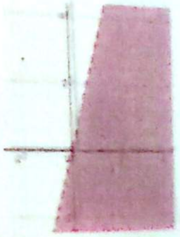

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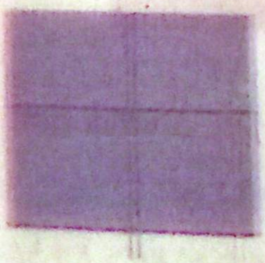

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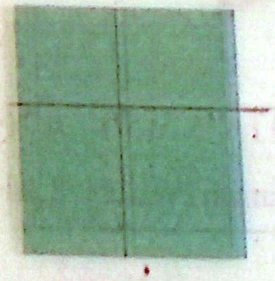

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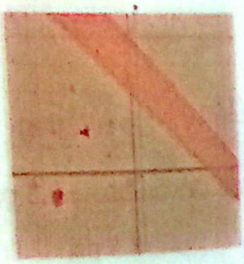

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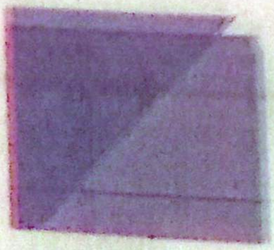

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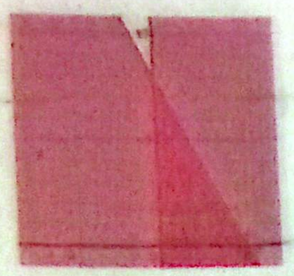
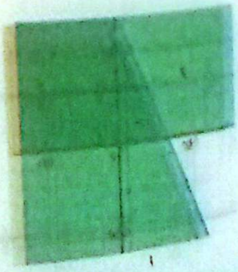


13.



14. $2 \le x \le 4$, $1 \le y \le 4$. 15. $x \ge 3$, $y \ge \frac{2}{3}x - 2$, $y \le \frac{4}{3}x - 2$

UNIT 5

Exercise 5.1

1. (i) $\frac{3x^2}{5ay^4}$ (ii) $\frac{2p^3m^3}{3k}$ (iii) $\frac{nq}{mp^3}$ (iv) $\frac{1}{5ab}$ (v) $\frac{2lmn}{3}$ (vi) -1

(vii) $x-9$ (viii) $\frac{r+3}{r-4}$

2. (i). 9 (ii) 294 (iii) 96.6 (iv) −7.5 (v) 1.002 (vi) $-3\frac{17}{56}$

3. 5050 5. $-11\frac{2}{3}$ 6. 21 cm

Exercise 5.2

1. (i) $\frac{7x}{10}$ (ii) $\frac{11x+2}{18}$ (iii) $\frac{81x+24}{28}$ (iv) $\frac{3x+2}{x+5}$ (v) $\frac{7x+3}{x-3}$

2. (i) $\frac{x+12}{5}$ (ii) $\frac{17x-101}{21}$ (iii) $\frac{-2x^3+2x^2+1}{x}$

3. (i) x^2+2x-3 (ii) $x^2-2xy+y^2$ (iii) $\frac{6(x-2)}{x}$ (iv) $\frac{x}{x-5}$

4. (i) $\frac{47x}{60}$ (ii) $\frac{29-2x}{6}$ (iii) $\frac{-4}{x^2-1}$ (iv) $\frac{3x^2-100}{15}$ (v) $\frac{9b^2cx^2}{4x^3y}$ (vi) $\frac{p^2q^2x}{z^2}$ (vii) $\frac{x+1}{x-3}$

(viii) $\frac{4}{3}$ (ix) $\frac{12a^2-4a+7}{3(4a^2-9)}$ (x) $\frac{23x}{(1+2x)(2+x)(5-9x)}$ (xi) $\frac{1}{5m}$ (xii) $\frac{1-t}{p}$ (xiii) $\frac{1}{1+x}$

Exercise 5.3

1. { 2 } 2. { 2, 3 } 3. No solution 4. only 4 (-1 is not solution) 5. { −1, −5 }
6. { −6, 8 }
7. No solution 8. approximately $2\frac{2}{9}$ or about 2 hours and 13 minutes
9. add 12 pints of yellow paint to the mixture to achieve a paint mixture that is 80% yellow.
10. Wasi takes 36 hours and Waqar takes 45

Miscellaneous Exercise-5

1(i) (b) (ii) (a) (iii) (d) (iv) (c) (v) (c)
(vi) (d) (vii) (b) (viii) (d) (ix) (a) (x) (c)
(xi) (d) (xii) (d) (xiii) (c)

2. 78.571 cm 3. (i) 24.64 square inches (ii) 1.75 feet 4. (i), (ii) and (iv)

5. 20 6, $3x-9$ 7. $x-2$ 8. (i) $\frac{7x^2+30x+16}{8-2x^2}$ (ii) $\frac{1}{3}$

9. (i) { −2, −6 } (ii) { 5, −6 }3 is not solution 10. Essential Step: Ensure that solutions do not make any denominator zero, as such values are not valid for rational equations.

UNIT 6

Exercise 6.1

1. i. Yes ii. No iii. No iv. Yes 2. i. Yes, Bijective ii. No iii. Yes, Into
3. i. 6 ii. 9/2 iii. 28/9 iv. $\frac{1}{2}(2t^2+3t+8)$
4. Injective. 5. a = t = 1 6. Yes

Exercise 6.2

1. i. x^2+2x-3 ii. $-x^2+6x-5$ iii. $4(x-1)^3$ iv. $\frac{4}{x-1}$

2. i. $4(x+1)$ ii. $4x+1$ iii. $16x$ iv. $x+2$

3. i. $f\circ g(x)=-2x+1$ ii. $f\circ g(x)=\frac{x-1}{x}$ iii. $f\circ g(x)=\frac{6}{\sqrt{x-1}}$ iv. $f\circ g(x)=x-2$

$g\circ f(x)=-2x+4$ $g\circ f(x)=\frac{4}{2-x}$ $g\circ f(x)=\frac{2}{\sqrt{3x-1}}$ $g\circ f(x)=\sqrt{x^2-2}$

4. i. $x=0,-2$ ii. $x=1\pm\sqrt{2}$ iii. $x=\pm 2$ iv. $3\pm\sqrt{5}$

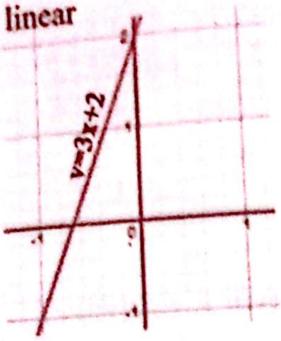
5. i. $f^{-1}(x)=\frac{1+x}{2}$ ii. $f^{-1}(x)=\frac{2+3x}{x}$ iii. $f^{-1}(x)=x^2-5$

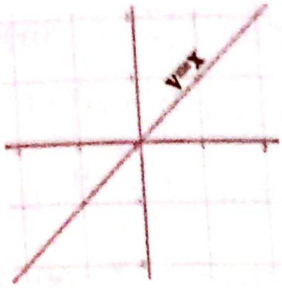
6. i. $f^{-1}(1)=8$, i. $g^{-1}\left(\frac{1}{2}\right)=-1$

Exercise 6.3

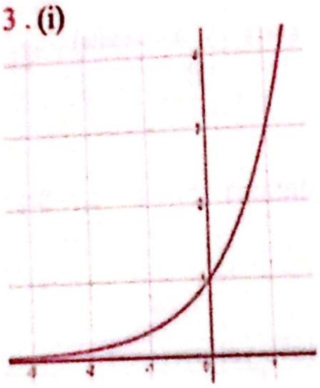
1. (i), (iii), (iv) and (iv) are graphs of functions.

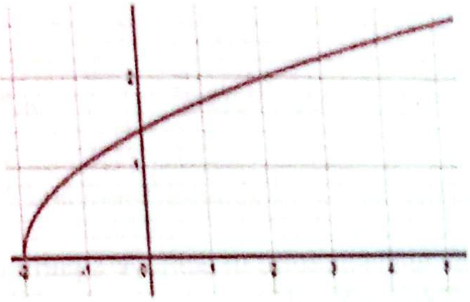
2. i. linear i. linear iii. not linear





3 . (i) (iv) (ii), (iii),(v) draw yourself





4. i. upwards, x-intercept: 1 , y-intercept: 1 ii. downwards, x-intercepts: ±√2, y-

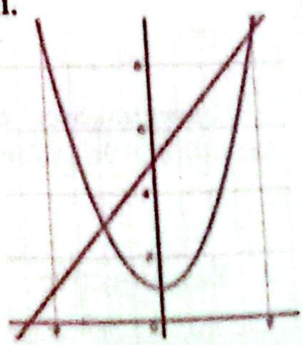
intercept: 2

iii. downwards, x-intercept: 2±√3, y-intercept: 2 iv. downwards, x-intercepts: −1,3, y-

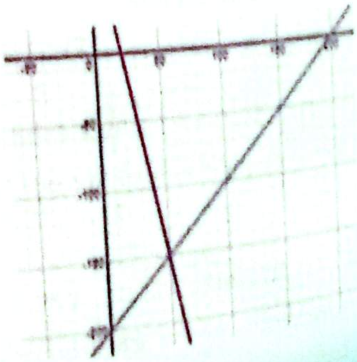
intercept: 3

5. h=4/9, k = 2

6. i. draw yourself ii.



7.



8. 63814 (approx.)

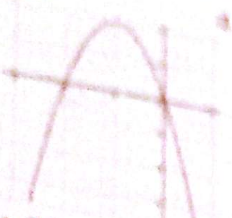
Supply will become equal to demand at x = 50. 9. draw yourself 10. 175 dollars

Miscellaneous Exercise-6

1. i. b ii. c iii. c iv. a v. c vi. c vii. b viii. c ix. d x. b

2. $f^{-1}(x) = \frac{x}{2x-3}$, $f^{-1}(x)$ is undefined at $x = \frac{3}{2}$, $f^{-1}(-1) = \frac{1}{5}$

3. downwards

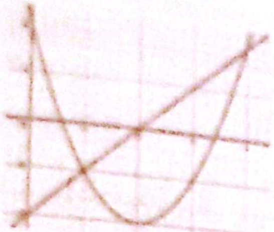


4. $y = -10x^2 + 10$

5. $x = -2, 3$

6. $F = \frac{9}{5}C + 32$, 77°F

7. (1, −1), (4, 2)



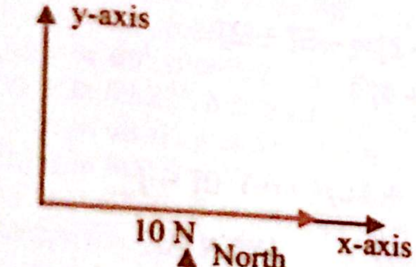
8. 17 hrs.(approx.)

9. 2 units

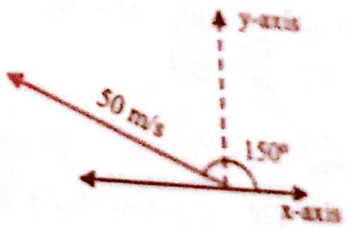
UNIT 7

Exercise 7.1

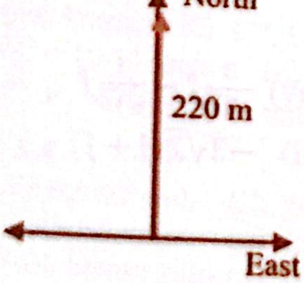
1. (i)



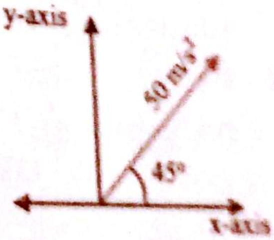
(ii)



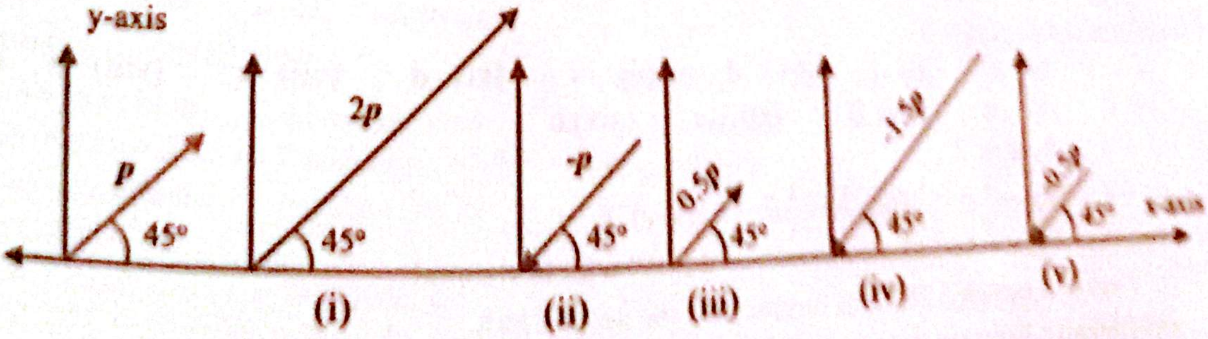
(iii)

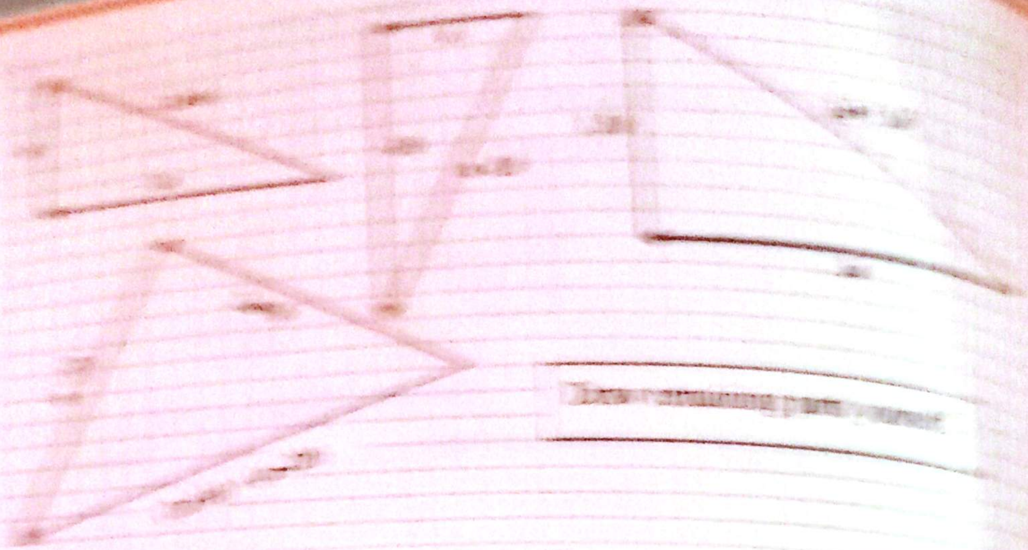


(iv)



2.





[illegible]

UNIT 8

Exercise 8.1

1. (i) 55° (ii) 42° (iii) 69° (iv) 38°

2. 144° 3. 93° 4. $\sin 120° = \sin 60° = \frac{\sqrt{3}}{2}, \cos 120° = -\cos 60° = -\frac{1}{2}, \tan 120° = -\sqrt{3}$

5. $\sin\frac{3\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}, \cos\frac{3\pi}{4} = -\cos\frac{\pi}{4} = -\frac{\sqrt{2}}{2}, \tan\frac{3\pi}{4} = -1$ 6. (i) −0.559

(ii) 0.83 (iii) −1.485 (iv) −0.674 (v) −1.789 (vi) 1.204

Exercise 8.2

1. (i) $\beta = 50°, a = 9.64m, b = 11.49m$ (ii) $\alpha = 54.5°, c = 86.1m, a = 70.1m$
(iii) $c = 19.5cm, \alpha = 50.20°, \beta = 39.8$

2. (i) $\alpha = 55°, H = 14.64cm, P = 8.4cm$ (ii) $\beta = 64.41°,\ a = 14.54cm,\ c = 33.5cm$
(iii) $c \cong 56cm, \alpha \cong 63.43°, \beta \cong 26.57°$ (vi) $b \cong 26.46cm, \alpha \cong 41.84°, \beta \cong 48.16°$
(v) $c \cong 40.5cm, a \cong 32.6cm, \beta \cong 53.75°$ (vi) $c \cong 38.7cm, b \cong 37.32cm, \beta \cong 19.5°$

3. 42.89m 4. 116.5m 5. 5.18 ft 6. 26.59 7. 48.6° 8. 24 ft 9. 61° 10. 32.47m

11. 814.3m 12. 4.7m 13. 23.2 m 14. 6.66 m 15. 9.33m

Exercise 8.3

1. (i) $\alpha = 36.1°, \beta = 85.4°, c = 10.2$ (ii) $\gamma = 48.3°, \beta = 95.7°, c = 11.8$
(iii) $\alpha = 86°, \gamma = 79°, b = 38.9$ (iv) $\alpha = 71.57°, \gamma = 63.43°, b = 30$
(v) $\alpha = 37.7°, \beta = 58.3°, \gamma = 88.0°$ (vi) $\alpha = 33.33°, \beta = 33.33°,\ \gamma = 112.89°$
(vii) $\alpha = 36.3°, \beta = 62.7°, \gamma = 81.0°$ (viii) $\alpha = 60°, \beta = 60°, \gamma = 60°$
(ix) $\alpha = 22{,}62°, \beta = 67.38°, \gamma = 90°$

3. (i) $\gamma = 80°, b \cong 13.5, c \cong 15.3$ (ii) $\gamma = 60°, a \cong 16.3, c \cong 18.4$
(iii) $\gamma = 60°, a \cong 9.8, c \cong 13.4$ (iv) $\alpha \cong 19.42°, a \cong 9.1, c \cong 23.8$
(v) $\gamma \cong 69.33°, b \cong 34.2, c \cong 32.1$ (vi) $\beta \cong 52.28°, a \cong 35.2, b \cong 41.2$

5. (i) $\alpha = 48.6°, \gamma \cong 51.4°, c \cong 10.4$ (ii) $\alpha = 71.8°, \gamma \cong 39.2°, a \cong 20.1$
(iii) $\alpha = 90°, \beta = 26°, a \cong 13.8$ (iv) $\gamma = 101.3°, \beta = 23.6°, c = 43$
(v) $\alpha \cong 57.4°, \beta \cong 35.1°, \gamma \cong 87.5°$ (vi) $\alpha \cong 83.6°, b \cong 36.2, c \cong 88$

6. Distance $= 362.7.5km$ 7. Perimeter = 1425 m, Area $\cong 84852.8m^2$

8. Largest Angle ≅ 83.41°, Smallest Angle ≅ 44.64° 9. 70.75 cm, 98.97 cm

10. (i) BC = 10.15 inch (ii) ∠BDC = 19.49° 11. (i) BC = 11.49 m (ii) ∠EDG = 56.9°

12. 284.51 ft 13. 9.51 miles 14. 25.98 cm 15. 36.71°

Exercise 8.4

1. (i) 52.27 square units (ii) 269.74 square units (iii) 63.3 square units (iv) 215.39 square units
(v) 5.11 square units (vi) Not possible (not triangle) (vii) 87.56 square units (viii) 254.73 square units
(ix) 136.04 square units (x) 263 square units (xi) 61.5 square units (xii) 187.64 square units
(xiii) 168 square units (xiv) 249.47 square units (xv) 0.0359 square units (xvi) 74.53 square units
(xvii) 0.906 square units

2. 127.28 square units 3. 387.4 square units

4. (a) $\cos\theta = \frac{1}{3}$ (b) $\sin\theta = \frac{2\sqrt{2}}{3}$ (c) $5.66\ sq.units$ (d) $6\ sq.units$

5. $A \cong 233.75$ sq units 6. $125\sqrt{3}$ sq units 7. Do yourself.

8. Length of side = 22.3 m, Third Angle = 75° 9. $27\sqrt{3}\ \text{cm}^2$

10. (a) Sine of angle B $= \frac{2}{3}$ (b) $\angle B = 41.8°$ 11. (a) Area $= (ac)\sin\theta$ (b) $\theta = 90°$

Exercise 8.5

1. (i) $r \cong 3.24, R \cong 8.53, r_1 \cong 6.48, r_2 \cong 9.26, r_3 \cong 21.6$
(ii) $r \cong 6.87, R \cong 15.18, r_1 \cong 16.31, r_2 \cong 18{,}64, r_3 \cong 32.61$
(iii) $r = 1, R = 2.5, r_1 = 2, r_2 = 3, r_3 = 3$
(iv) $r \cong 16.33, R \cong 35.75, r_1 \cong 36.74, r_2 \cong 48.99, r_3 \cong 73.48$

16. $x \approx 2.43cm$, $R \approx 7.64 cm$ (ii) $2\pi r_2 \approx 70.56 cm$ 17. $A \approx 2.72 cm^2$, $C \approx 5.85 cm$

15. (i) $2\pi r_1 \approx 20.16 cm$

Exercise 8.6

1. 18 cm 2. 78.86° 3. $6\sqrt{2}$ cm, $6\sqrt{2}$ cm 4. 13.9° 5. (i) 20cm (ii) 25 cm (iii) 26.9°
(ii) 12.44m (iii) 53.67° (iv) 64.52°
6. 46.7° 7. (i) 10.82m
(ii) 29 km (iii) 56 km
8. (i) 29 km

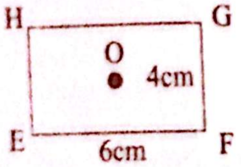
Miscellaneous Exercise-8

1. (i) a (ii) c (iii) b (iv) d (v) c (vi) d
(vii) b (viii) a (ix) c (x) d (xi) b (xii) a
2. (i) $\gamma = 145°, b = 10.6, c = 17.7$ (ii) $\beta = 32.4°, \gamma = 32.6°, c = 63.5$
(iii) $\gamma = 56.9°, \alpha = 24.9°, a = 7.37$ (iv) $b \approx 114.74, a \approx 146.99, \gamma = 20°$
(v) $b \approx 66.86, \alpha = 28.88°, \gamma = 43.12°$ (vi) $\alpha = 53.13°, \beta = 36.87°, \gamma = 90°$
3. 14.48 cm 4. a. 1.8 km b. 4.25 km 5. a. 33.73 inches b. 57.5° 6. 1717.62 m^2
7. a. 8.84 m b. 32.18 m 8. 137.3 meters and 121.6 meters
9. Yes, it is possible for Aamir to draw a parallelogram with one side of 12 cm, one diagonal of 10 cm, and one angle of 120°. The calculated length of the other diagonal is approximately 20.8 cm, which is consistent with the geometry of a parallelogram.
10. Approximately 73.8feet (higher observer) and 45.3 feet (lower observer). The distance from the pipe to the apartment complex (horizontal distance) is approximately 42.3 feet. 11. 5.22 miles 12. Both are 21.9 cm
16. 1702.3 meters 17. 24.96° , 18. 21°

UNIT 9

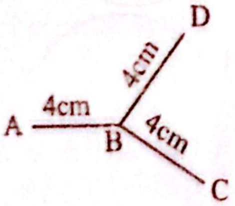
Exercise 9.1

2. (i) $x = 4cm$ (ii) $y = 6cm$ 3. $CD = 26cm, CE = 1cm$ 4. $CD = 24cm$
5. $AB = 24cm$
6.



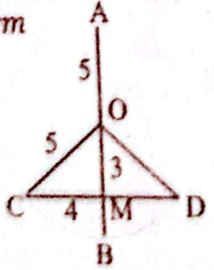
As the circle has only one centre O, therefore one and only one circle can pass through vertices of rectangle.

7.



As the circle has only one centre B, therefore one and only one circle can pass through points A, C and D.

8. (i) $OU = 16 - r$ (ii) $r^2 = 5^2 + (16 - r)^2$; $r = 8.8cm$
9. $r = 10cm$
10.



Sides of triangle OCM satisfy Pythagorean triplet (3, 4, 5). Therefore, $\angle OMC = 90°$ proving the result.

11. Prove $\triangle ACD \cong \triangle BCD$, with the help of SAS postulate, then $AC = BC$.

Exercise 9.2

1. $\angle OCF = 15°, \angle CFD = 75°$ 2. 2.24cm, Yes, they are equal. 3. 7cm
4. (i) $a = 9cm, y = 53°$ (ii) $a = 6cm, x = 45°, y = 8.49cm$
5. 16.53cm 6. 6cm, 4cm 7. 31cm 8. $\sqrt{10}cm = 3.16cm$ 9. 5.15cm

Exercise 9.3

1. (i) yes (ii) yes (iii) chord AB < chord AD
4. (i) no (ii) both are proportional 5. (i) yes (ii) CD = 4 cm (iii) 2.24 cm
6. 120° 8. 6 cm 11. (a) 130° (b) 85° (c) 75° (d) 70°
13. ∠AOB = ∠COD = 100°; Equal chords of a circle subtend equal angles at the centre.
14. (i) All the four chords make central angle of 90°. (ii) ∠PCB = 45°, ∠ACB = 90°
(iii) ∠CBD, ∠BDA, ∠DAC (iv) square (v) one and only one
15. (i) ∠PTQ = 45°, ∠RTS = 135° (ii) 1 : 3

Exercise 9.4

1. (i) Length of minor arc = 12.21 cm, length of major arc = 31.75 cm (ii) 43.96 cm; yes
2. (i) Area of minor sector = 33.51 cm^2, area of major sector = 167.55 cm^2
(ii) 201.06 cm^2 ; yes 3. 26.2 cm
4. (i) 22cm (ii) 10.5cm (iii) 11.5cm (iv) 231cm^2 (v) 52.4cm^2 (vi) 178.6cm^2
5. (i) 19.8cm (ii) 22cm (iii) 41.8cm (iv) 98cm^2 (v) 154cm^2 (vi) 56cm^2
6. 12.57cm 7. 4 cm 8. A = 12.32 ft^2 ; P = 14.4 ft 9. 35.9 m 10. 28 m
11. P = 12.48 ft ; A = 1.85 ft^2 12. 29.3 ft 13. (i) 12228.1 m^2 (ii) 65.1 m

Miscellaneous Exercise-9

1. (i) b (ii) d (iii) c (iv) c (v) b (vi) a (vii) d (viii) a (ix) d
(x) c (xi) b (xii) c (xiii) b (xiv) d (xv) b (xvi) a
(xvii) b (xviii) d

UNIT 10

Exercise 10.1

1. (i) $r = 5cm$ (ii) $d = 4\sqrt{7}cm$ (iii) $x = 8cm,\ y = 4.62cm$
(iv) $x = 25^o, y = 65^o$ (v) $a = 30^o, b = 10\sqrt{3}cm$ (vi) $a = b = 60^o$
(vii) $a = 15^o$ (viii) $x = 63^o, y = 60^o, z = 57^o$
2. (i) ∠OBA = 90° (ii) ∠OCB = 30° (iii) ∠BAC = 60° (iv) ∠ABC = 60°
4. 4 ft 5. 2cm, 6cm, 10cm 6. 5.06 cm 7. ∠PGH = 35° 8. $x = 48^o, y = 68^o$

Exercise 10.2

1. ∠Q = 30°, ∠ROS = 60° 2. ∠ABO = 30°, ∠AOB = 120°, ∠ACB = 60°
3. ∠ABC = 90°, ∠BCA = 50°, ∠CAD = 30° 4. ∠O = 60°, ∠C = 30°
5. ∠AEB = 110°, ∠B = 40°, ∠C = 40°, ∠D = 30°
6. (i) 4cm (ii) ∠ABE = 60° (iii) ∠ABC = 90° (iv) ∠EBD = 30°
7. (i) 2 (ii) 4cm (iii) ∠BQC = ∠BPO = 90° (iv) $5\sqrt{5}\ cm$ (v) 15 cm
8. (i) ∠QSR = 120° (ii) ∠PRS = 30° (iii) ∠PQS = 30° (iv) ∠QTR = 60°
(v) Angle in a segment greater than semicircle is acute angle.
9. (i) ∠BAD = 65° (ii) ∠BCD = 115° (iii) ∠ABC + ∠ADC = 180°
10. (i) ∠QRS = 85° (ii) ∠PQR = 95° (iii) ∠PSR = 85°
11. (i) ∠BAC = 30° (ii) ∠CAD = 30°
12. (i) ∠ECD = 65° (ii) ∠CDE = 95° (iii) ∠CED = 20°
13. (i) 2.2 cm (ii) 1.2 cm 14. 40 cm, 125.6 cm

Miscellaneous Exercise-10

1. (i) a (ii) c (iii) b (iv) b (v) d (vi) a (vii) d (viii) a
(x) c (xi) d (xii) c (xiii) d (xiv) b (xv) d (xvi) a
(xvii) c (xviii) c (xix) b (xx) d
2. $6\sqrt{2}\ cm$ 3. $8cm$ 4. $4.75cm$

UNIT 11

Exercise 11.1

8. (i) He will take three points on circular path and join them to obtain a triangle. Point of intersection of right bisectors of sides of the triangle is the required point.

9. 22 tiles

12. 25 cm   13. 61.68 cm   14. 5 cm

Exercise 11.3   12. 12451.51 km

11. 1.12 inches

Review Exercise-11

1. (i) d (ii) a (iii) b (iv) c (v) a (vi) c (vii) d (viii) b (ix) a
(x) d (xi) c (xii) d (xiii) b (xiv)

UNIT 12

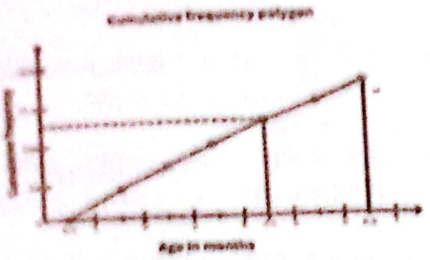
Exercise 12.1

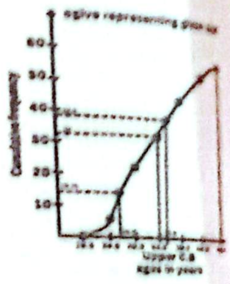
1.

Class Interval	1 – 10	11 – 20	21 – 30	31 – 40	41 – 50	51 – 60
Frequency	3	4	7	9	5	2
Cumulative freq	3	7	14	23	28	30

2. i. 30   ii. 21-30   iii. 7   iv. 31-40   v. 50.5

3. 33   a. 26.5 app   b. 37app   c. 34.5app



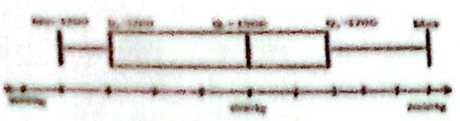


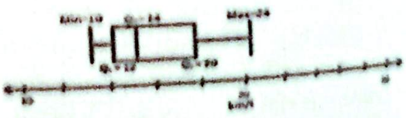
From the graph, mass of the baby at the age of $4\frac{1}{2}$ months is 12kg app.

4. Draw ogive yourself. a. 90   b. 72.5   c. 45   d. 9   e. 27   f. 1   5. do yourself
6. a. 75 app   b. 45% app   c. 3 : 1 app   d. 25   e. 55m app   f. 80m app
7. Range = Rs.125,000 & IQR = Rs. 40,000   8. She performed above 75% of of her class
9. Between Rs. 25,000,000 & Rs. 30,000,000
10. i. 38app   ii. 30app   iii. 50app   iv. 20app   v. 48app   vi. 32app

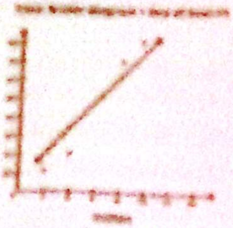
Exercise 12.2

1.





a. $IQR_1 = 500$, $IQR_2 = 8$,   b. negative   c. lighter vehicles are more fuel efficient
2. a. Rs. 500   b. 0 &700   c. Rs. 400   d. Rs. 300   e. positive



Positive correlation indicates increase in number of flowers with increased rain quantity.

4. a. Positive correlation indicates higher score with more practice time. b. draw yourself

Exercise 12.3

1 Measure of spread of data : Range , Variance , Standard deviation
2 range = 599, Variance = 28886 app 3i. 758.8 ii. 19097 iii. 138.19 iv. 18.2
4 range = Rs.60.000 , Variance = 246437000app , SD = 15698.3 CV = 22861.5
5. mean = 66.35 , Variance = 67.57 , SD = 8.22
6. range = 2.000 , Variance = 243806app. SD = 493.8
7. OT: range = 60, mean = 127.3, Variance = 204.7app , SD = 14.31 CV = 11.24
OPD: range = 60, mean = 127, Variance = 292.4app , SD = 17.1 CV = 13.46
OPD patients BP data have more variation.
8. Mean : 103, 66.9 Variance: 33.3, 147.6
Next week average temp seems to be 103F app in Jhelum and 147.6F app in Indiana.

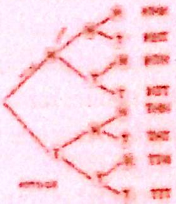
Exercise 12.4

1.

trials	First trial			
Second trial	×	1	2	3
	1	1	▨	3
	2	▨	▨	▨
	3	3	▨	9

P(Even) = 5/9

2.



3. 0.85 4. 0.7 5. 0.9 6. 5/6 7. 25/27 8. 0.28 9. 0.63 10. 0.72

Miscellaneous Exercise-12

1 i. d ii. b iii. b iv. c v. d vi. d vii. b viii. a ix. d x. c
xi. c xii. d xiii. a xiv. a xv. c
2 Draw the graph yourself , variance = 83.14 , SD = 9.12
3 a. $IQR_1 = 30$, $IQR_2 = 10/10$ b. negative c. more exercise time caused reduction in BP.
4 25%, IQR = 20 5. Hazel: M = 65 app, IQR = 84-48 = 36,
Mauve: M = 68 app, IQR = 86-45 = 41
6. 3/10 7. 0.18