بِسۡمِ اللّٰہِ الرَّحۡمٰنِ الرَّحِیۡمِ

(In the Name of Allah, the Most Merciful, the Most Compassionate.)

MATHEMATICS

10

PECTAA

Transformation, Innovation and Excellence

PUNJAB EDUCATION, CURRICULUM,
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THIS TEXTBOOK IS BASED ON UPDATED/REVISED NATIONAL CURRICULUM OF PAKISTAN 2023

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Experimental Edition

UNIT 1

Complex Numbers

Students' Learning Outcomes

After completing this unit, the students will be able to:

- Identify complex numbers, complex conjugate, absolute value or modulus of a complex number.
- Apply algebraic properties and perform basic operations on complex numbers.
- Demonstrate additive identity and multiplicative identity for the set of complex numbers.
- Find additive inverse and multiplicative inverse of a complex number z.
- Demonstrate the following properties of a complex number z.
  - $|z| = |-z| = |\bar{\bar{z}}| = |-\bar{z}|$
  - $\bar{\bar{z}} = z,\ z\bar{z} = |z|^2$
  - $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
  - $\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$; $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}$, $z_2 \neq 0$
- Apply the Geometric interpretation of a complex number, modulus of a complex number and algebraic operations of a complex number.
- Find real and imaginary parts of complex numbers of the type $(x + iy)^n$ and

  $\left(\dfrac{x_1 + iy_1}{x_2 + iy_2}\right)^n$, $x_2 + iy_2 \neq 0$, where $n = \pm 1$ and $n = \pm 2$.
- Solve the simultaneous linear equations with complex coefficients.

INTRODUCTION

This unit introduces the fascinating world of complex numbers, beginning with their identification and the exploration of related concepts such as the complex conjugate, modulus and their representation in both algebraic and geometric forms. Students will apply core algebraic operations and understand how complex numbers obey properties. The unit also extends to evaluate expressions involving powers and quotients of complex numbers, extracting real and imaginary parts and solving simultaneous linear equations with complex coefficients.

1.1 Complex Numbers

Before we explore complex numbers, let's first consider the question: *"Is there a real number whose square is negative?"* To answer this question, we can examine a few simple examples. Let's take a look at equations 1 and 2 for better understanding.

Equation 1: $x^2 - 1 = 0$

$$x^2 - 1 = 0$$
$$x^2 = 1$$
$$x = \pm\sqrt{1} \Rightarrow x = \pm 1$$

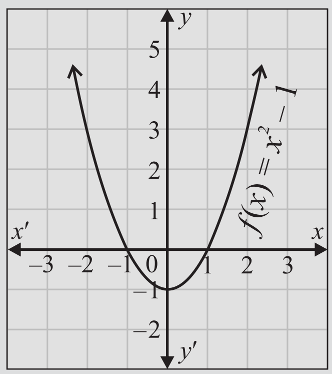


Figure 1.1

Equation 1 has two real solutions $x = -1, 1$. The graph of $f(x) = x^2 - 1$ crosses the x-axis at $(-1, 0)$ and $(1, 0)$.

Equation 2: $x^2 + 1 = 0$

$$x^2 + 1 = 0$$
$$x^2 = -1$$
$$x = \pm\sqrt{-1}$$

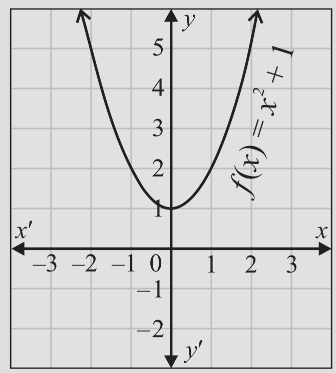


Figure 1.2

Equation 2 has no real solution, since the graph of $f(x) = x^2 + 1$ does not cross the x-axis.

This is because squaring any real number never results in a negative value. Therefore, if equation 2 has a solution, we need to introduce a new kind of number, an imaginary number, defined as the square root of −1. This imaginary unit is represented by the symbol i (iota). The imaginary number i tells us that $i^2 = -1$.

Example 1 Simplify the following:

(i) i^7 (ii) i^8 (iii) i^{17} (iv) i^{-25}

Solution

(i) $i^7 = i^6 \times i = (i^2)^3 \times i = (-1)^3 \times i = -i$

(ii) $i^8 = (i^2)^4 = (-1)^4 = 1$

Do you know?

Complex numbers are essential for many technologies like smartphone signal processing and MRI imaging.

(iii) $i^{17} = i^{16} \times i = (i^2)^8 \times i = (-1)^8 \times i = i$

(iv) $i^{-25} = \dfrac{1}{i^{25}} = \dfrac{1}{i^{24} \times i} = \dfrac{1}{(i^2)^{12} \times i} = \dfrac{1}{(-1)^{12} \times i} = \dfrac{1}{i} = \dfrac{1}{i} \times \dfrac{i}{i} = \dfrac{i}{i^2} = \dfrac{i}{-1} = -i$

We have observed that the equation $x^2 + 1 = 0$ has no solution within the real number system. To address equations like this, we expand the real number system to include new types of numbers, leading to the development of the complex number system in rectangular form.

History!

The idea of complex numbers first emerged in 16^{th} century, when Italian Mathematician Gerolamo Cardano discovered that equations could still be solved when they involved the square root of negative number. In the 18^{th} century, Carl Friedrich Gauss (German Mathematician and astronomer) expanded this early idea.

Gerolamo Cardano (1501 - 1576)

A complex number is a number expressed in the form $x + iy$, where x and y are real numbers and i is the imaginary unit, defined by $i^2 = -1$.

1.1.1 Rectangular Form of a Complex Number

A complex number is of the form $x + iy$ (or $x + yi$), where x and y are real numbers, x is called the real part and y is called the imaginary part of the complex number.

(i) If $x = 0$, the complex number is said to be pure imaginary.

(ii) If $y = 0$, the complex number is said to be real.

(iii) It is customary to denote the standard rectangular form of a complex number $x + iy$ as z and we write $x = Re(z)$ and $y = Im(z)$.

For example, $Re(5 - 7i) = 5$ and $Im(5 - 7i) = -7$

1.1.2 Equality of Complex Numbers

Two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are said to be equal if and only if $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$. i.e. $x_1 = x_2$ and $y_1 = y_2$.

For example, if $\alpha + i\beta = -2 + 5i$, then $\alpha = -2, \beta = 5$.

Example 2 If $(2x - 1) + (y + 4)i = 5 + 7i$, then find the values of x and y.

Solution Given that $(2x - 1) + (y + 4)i = 5 + 7i$

Since the two complex numbers are equal, their real and imaginary parts must be equal.

$$\therefore \quad 2x - 1 = 5 \quad , \quad y + 4 = 7$$
$$2x = 6 \quad , \quad y = 7 - 4$$
$$x = 3 \quad , \quad y = 3$$

EXERCISE 1.1

1. Simplify the following:
   (i) i^5 (ii) i^{16} (iii) $(-i)^{-19}$ (iv) $27i^{-26}$
   (v) $i^{11} + i^5$ (vi) $(i^4 + i^3 + i^2 + i)^2$ (vii) $\left(\frac{i^8}{i^5}\right)^{-5}$ (viii) $i^{13} \times i^{29}$

2. Write in terms of i.
   (i) $2 + \sqrt{-4}$ (ii) $3 - \sqrt{-7}$ (iii) $\frac{2}{5} + \frac{\sqrt{-16}}{5}$ (iv) $\sqrt{2} - \sqrt{-3}$

3. Find the values of x and y.
   (i) $(2x + 5) + (y - 3)i = 1 + 2i$ (ii) $(3x + 2) - (4 - y)i = 5 + 3i$
   (iii) $(2+i)x + (1-2i)y = 3 + 4i$ (iv) $(1-i)x + (2+i)y = 4 - i$
   (v) $(3x - 1) + (2y - 3)i = 8 + 7i$

1.2 Algebraic Operations on Complex Numbers

1.2.1 Scalar Multiplication of Complex Numbers

If $z = x + iy$ and $k \in R$, then we define $kz = (kx) + (ky)i$ or $(kx) + i(ky)$

The following diagram shows kz for $k = 2, \frac{1}{2}, -1$

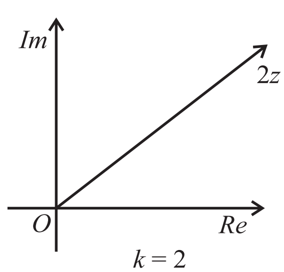


Figure 1.3

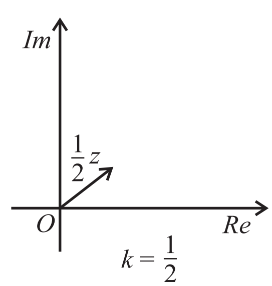


Figure 1.4

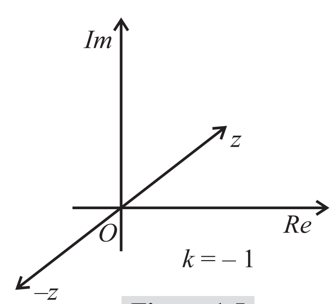


Figure 1.5

1.2.2 Addition of Two Complex Numbers

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ where x_1, x_2, y_1 and $y_2 \in R$, then

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2)$$
$$= (x_1 + x_2) + i\,(y_1 + y_2)$$

From figure 1.6, also by the parallelogram law of addition,

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

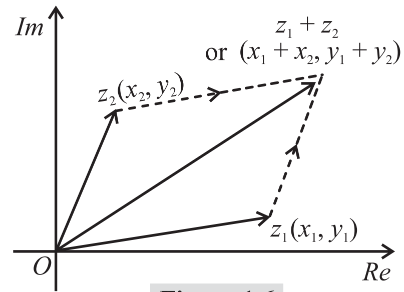


Figure 1.6

History!

The fundamental rules for addition, subtraction, multiplication and division of complex numbers were formulated by the Italian mathematician Rafael Bombelli (1526 – 1572). He is widely recognized as the first to establish a systematic algebra for complex numbers.

Example 3

Add $(3 + 4i)$ and $(5 - 2i)$.

Solution

$$(3 + 4i) + (5 - 2i)$$
$$= (3 + 5) + (4 - 2)i$$
$$= 8 + 2i$$

1.2.3 Subtraction of Two Complex Numbers

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, where $x_1, x_2, y_1, y_2 \in R$,

then $z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2) = (x_1 - x_2) + i\,(y_1 - y_2)$

$$z_1 - z_2 = z_1 + (-z_2)$$
$$= (x_1 + iy_1) + (-x_2 - iy_2)$$
$$= (x_1 - x_2) + i(y_1 - y_2)$$

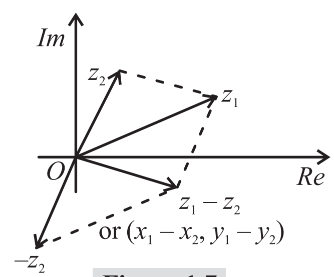


Figure 1.7

Example 4 Simplify: **(i)** $(8 + 2i) - (5 - 6i)$ **(ii)** $(4 - 3i) - (2 - 5i)$

Solution **(i)** $(8 + 2i) - (5 - 6i)$

$$= 8 + 2i - 5 + 6i$$
$$= (8 - 5) + (2i + 6i)$$
$$= 3 + 8i$$

Skilled Practice!

If $z_1 - z_2 = 4 + 6i$ and $z_2 = 3 - 2i$, then find z_1.

(ii) $(4-3i)-(2-5i)$
$= 4-3i-2+5i$
$= (4-2)+(-3+5)i = 2+2i$

1.2.4 Multiplication of Two Complex Numbers

Let $z_1 = x_1 + iy_1,\ z_2 = x_2 + iy_2$

then
$$z_1z_2 = (x_1+iy_1)(x_2+iy_2)$$
$$= x_1(x_2+iy_2)+iy_1(x_2+iy_2)$$
$$= x_1x_2 + x_1y_2i + y_1x_2i + i^2y_1y_2$$
$$= (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2) \qquad \because\ i^2 = -1$$

Example 5

(i) Simplify: $(3-4i)(5-6i)$

(ii) If $z_1 = 2+3i$ and $z_2 = 4+7i$, then find z_1z_2.

Solution

(i) $(3-4i)(5-6i)$
$= 3(5-6i)-4i(5-6i)$
$= 15-18i-20i+24i^2$
$= 15-(18+20)i-24$
$= 15-38i-24$
$= (15-24)-38i = -9-38i$

(ii) $z_1z_2 = (2+3i)(4+7i)$

Using
$(x_1+iy_1)(x_2+iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)$
$z_1z_2 = (2\times 4 - 3\times 7) + i(2\times 7 + 3\times 4)$
$= (8-21)+(14+12)i$
$= -13+26i$

1.2.5 Division of Two Complex Numbers

Let $z_1 = x_1 + iy_1,\ z_2 = x_2 + iy_2$

Now,
$$\frac{z_1}{z_2} = \frac{x_1+iy_1}{x_2+iy_2},\ z_2 \neq 0$$
$$= \frac{x_1+iy_1}{x_2+iy_2} \times \frac{x_2-iy_2}{x_2-iy_2}$$
$$= \frac{x_1x_2 - ix_1y_2 + iy_1x_2 - i^2y_1y_2}{(x_2)^2-(iy_2)^2}$$
$$= \frac{x_1x_2 - i(x_1y_2 - y_1x_2) + y_1y_2}{x_2^2+y_2^2} \qquad \because\ i^2 = -1$$
$$= \frac{x_1x_2 + y_1y_2 - i(x_1y_2 - y_1x_2)}{x_2^2+y_2^2}$$
$$= \frac{x_1x_2 + y_1y_2}{x_2^2+y_2^2} + \frac{i(y_1x_2 - x_1y_2)}{x_2^2+y_2^2}$$

Skilled Practice!

1. Simplify:
$$\left(\frac{1+2i}{3-i}\right)(2+i)$$

2. Find the complex number z:
$$\text{if } \frac{z}{2+i} = 3-i$$

Example 6 Express $\frac{3+4i}{5-7i}$ in the form of $x + iy$ or $x + yi$.

Solution

$$\frac{3+4i}{5-7i} = \frac{3+4i}{5-7i} \times \frac{5+7i}{5+7i} = \frac{15 + 21i + 20i + 28i^2}{(5)^2 - (7i)^2}$$

$$= \frac{15-28+(21+20)i}{25+49} \quad \text{as } i^2 = -1$$

$$= \frac{-13+41i}{74}$$

$$= \frac{-13}{74} + \frac{41}{74}i$$

1.2.6 Properties of Complex Numbers

The complex numbers satisfy the following properties under **addition.**	The complex numbers satisfy the following properties under **multiplication.**
(i) Closure Property For any two complex numbers z_1 and z_2, the sum $z_1 + z_2$ is also a complex number.	**(i) Closure Property** For any two complex numbers z_1 and z_2, the product $z_1 z_2$ is also a complex number.
(ii) The Commutative Property For any two complex numbers z_1 and z_2, $z_1 + z_2 = z_2 + z_1$	**(ii) The Commutative Property** For any two complex numbers z_1 and z_2, $z_1 z_2 = z_2 z_1$
(iii) The Associative Property For any three complex numbers z_1, z_2 and z_3, $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$	**(iii) The Associative Property** For any three complex numbers z_1, z_2 and z_3, $(z_1 z_2)z_3 = z_1(z_2 z_3)$
(iv) The Additive Identity There exists a complex number $0 = 0 + 0i$ such that, for every complex number z, $z + 0 = 0 + z = z$ The complex number $0 = 0 + 0i$ is known as additive identity.	**(iv) The Multiplicative Identity** There exists a complex number $1 = 1 + 0i$ such that, for every complex number z, $z \times 1 = 1 \times z = z$ The complex number $1 = 1 + 0i$ is known as multiplicative identity.

(v) The Additive Inverse	(v) The Multiplicative Inverse
For every complex number z there exists a complex number $-z$ such that, $z + (-z) = (-z) + z = 0.$ $-z$ is called the additive inverse of z.	For any non-zero complex number z, there exists a complex number w such that, $zw = wz = 1$ w is called the multiplicative inverse of z. w and it is denoted by z^{-1}.

(vi) Distributive Property of Multiplication Over Addition

For any three complex numbers z_1, z_2, and z_3

$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3 \text{ and } (z_1 + z_2) z_3 = z_1 z_3 + z_2 z_3$$

Example 7 If $z = -7 - 4i$, then show that $z + 0 = z$.

Solution

$$z + 0 = (-7 - 4i) + (0 + 0i)$$
$$= -7 - 4i + 0 + 0i$$
$$= -7 - 4i = z$$

Shows $z + 0 = z$ (0 is the additive identity)

Example 8 Verify the multiplicative identity for $z = 3 - 2i$.

Solution

$$z \times 1 = (3 - 2i)(1 + 0i)$$
$$= 3 + 0i - 2i - 0i^2$$
$$= 3 - 2i = z$$
$$1 \times z = (1 + 0i)(3 - 2i)$$
$$= 3 - 2i + 0i - 0i^2$$
$$= 3 - 2i = z$$

Hence, verified that $z \times 1 = 1 \times z = z$

Example 9 Find the additive inverse of $z = 7 - 10i$.

Solution $z = 7 - 10i$

Additive inverse of $z = -7 + 10i$

because $z + (-z) = 7 - 10i + (-7 + 10i)$
$$= 7 - 10i - 7 + 10i = 0$$

Example 10 Find multiplicative inverse of $4 - 3i$.

Solution Let $z = 4 - 3i$

Multiplicative inverse of $z = \dfrac{1}{4-3i}$

Then $z^{-1} = \dfrac{1}{4-3i}$

$$= \frac{4+3i}{(4-3i)(4+3i)} = \frac{4+3i}{(4)^2 - (3i)^2}$$

$$z^{-1} = \frac{4+3i}{16+9} = \frac{4+3i}{25} = \frac{4}{25} + \frac{3i}{25}$$

EXERCISE 1.2

1. Simplify and write in the form $a + bi$:

(i)	$(2 + 5i) + (3 - zi)$	(ii)	$(16 - 3i) + (9 + 2i)$
(iii)	$(9 - 2i) - (7 - 3i)$	(iv)	$(11 + 9i) - (9 - 7i)$
(v)	$(3 + 4i)(2 - 3i)$	(vi)	$(5 - 2i)(3 - 4i)$
(vii)	$(3 - 5i) \div (2 - 4i)$	(viii)	$(5 + 2i) \div (6 - 3i)$

2. Write additive inverse for each complex number:

(i) $3 + 2i$ (ii) $4 - 3i$ (iii) $5 - 7i$ (iv) $-\dfrac{2}{3} + \dfrac{5}{4}i$

3. Find multiplicative inverse for each complex number:

   (i) $4 + 5i$ (ii) $6 + 2i$ (iii) $7 - 3i$ (iv) $\sqrt{5} - 4i$

4. If $z_1 = 2 + 5i$, $z_2 = 1 - 3i$ and $z_3 = 2 + i$, then verify that

   (i) $z_1 + z_2 = z_2 + z_1$ (ii) $z_1 z_2 = z_2 z_1$

   (iii) $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ (iv) $(z_1 z_2)z_3 = z_1(z_2 z_3)$

   (v) $z_1 + (-z_1) = (-z_1) + z_1 = 0$

5. If $\dfrac{(1+i)^2}{2-i} = x + iy$, then find the values of x and y.

6. If $(2x + iy)(1 - i) = 4 + 2i$, then find the values of x and y.

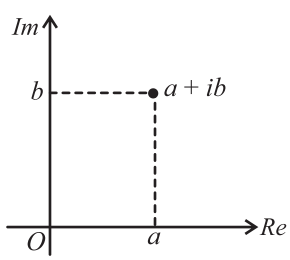
7. Find the values of a and b, if $(a + bi)(1 + 3i) = -8 + 11i$.

1.3 Complex or Argand Plane

A complex number $z = x + iy$ is exclusively determined by an ordered pair of real number (x, y). The numbers $2 - 5i$, 8 and $-7i$ are equivalent to $(2, -5)$, $(8, 0)$ and $(0, -7)$ respectively. In this manner, a complex number $z = x + iy$ can be represented by the point (x, y) in the coordinate plane.

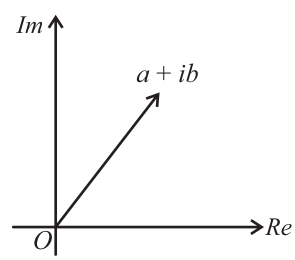
If we consider x-axis as real axis and y-axis as imaginary axis to represent a complex number, then the xy-plane is called complex plane or Argand plane. It is named in honour of the Swiss mathematician Jean Argand (1768–1822).

A complex number is represented not only by a point, but also by a position vector pointing from the origin to the point. The complex number, the corresponding point and the vector are all typically denoted by the same symbol, z. Geometrically, a complex number can be interpreted either as a point in the complex plane C or as a vector in the Argand plane.



Complex number as a point.

Figure 1.8



Complex number by a position vector pointing from origin to the point.

Figure 1.9

Here some complex numbers are plotted on the complex plane.

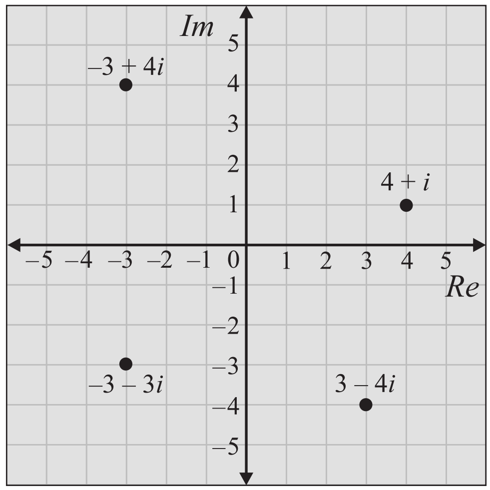


Figure 1.10

1.3.1 Conjugate of a Complex Number

The conjugate of the complex number $x + iy$ is defined as the complex number $x - iy$. The complex conjugate of z is denoted by $\bar{z}$. To get the conjugate of the complex number z, simply change i by $-i$ in z. For instance $3 - 8i$ is the conjugate of $3 + 8i$. The product of a complex number with its conjugate is a real number.

For example, $(1 + 2i)(1 - 2i) = (1)^2 - (2i)^2 = 1 + 4 = 5$

1.3.2 Geometrical Representation of Conjugate of a Complex Number

Geometrically, the conjugate of z is obtained by reflecting z on the real axis.

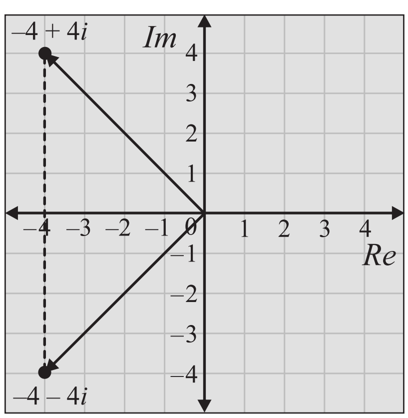


Figure 1.11

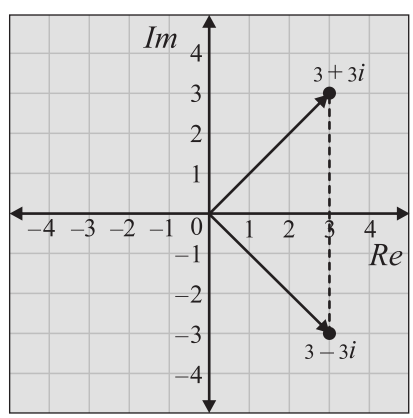


Figure 1.12

Note

Two complex numbers, $x + iy$ and $x - iy$, are known as conjugates of each other. Conjugates are particularly useful when dividing complex numbers. By multiplying both the numerator and the denominator by the conjugate of the denominator, the complex number in the denominator can be transformed into a real number.

1.3.3 Properties of Complex Conjugate

Property 1 For any two complex numbers z_1 and z_2, we have

$$\overline{z_1+z_2} = \overline{z_1} + \overline{z_2}$$

Proof: Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, where $x_1, y_1, x_2, y_2 \in R$

Now,
$$\begin{aligned}\overline{z_1+z_2} &= \overline{(x_1+iy_1)+(x_2+iy_2)}\\ &= \overline{(x_1+x_2)+i(y_1+y_2)}\\ &= (x_1+x_2)-i(y_1+y_2) = x_1+x_2-iy_1-iy_2\\ &= (x_1-iy_1)+(x_2-iy_2) = \overline{z_1}+\overline{z_2}\end{aligned}$$

Example 11 If $z_1 = 4 + 3i$, $z_2 = 5 + 2i$, then prove that $\overline{z_1+z_2} = \overline{z_1} + \overline{z_2}$.

Solution
$$\begin{aligned}z_1 + z_2 &= 4 + 3i + 5 + 2i = 9 + 5i\\ \overline{z_1+z_2} &= 9 - 5i \quad \text{... (i)}\\ \overline{z_1} &= 4 - 3i\\ \overline{z_2} &= 5 - 2i\\ \overline{z_1} + \overline{z_2} &= 4 - 3i + 5 - 2i\\ \overline{z_1} + \overline{z_2} &= 9 - 5i \quad \text{... (ii)}\end{aligned}$$

From (i) and (ii), we get

$$\overline{z_1+z_2} = \overline{z_1} + \overline{z_2}$$

Property 2 $\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$

Proof: Let $z_1 = x_1 + iy_1$

and $z_2 = x_2 + iy_2$ where $x_1, y_1, x_2, y_2 \in R$

$$\begin{aligned}z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2)\\ &= (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)\end{aligned}$$

Therefore:
$$\begin{aligned}\overline{z_1 z_2} &= \overline{(x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)}\\ &= (x_1x_2 - y_1y_2) - i(x_1y_2 + y_1x_2) \quad \text{... (i)}\end{aligned}$$

and
$$\begin{aligned}\overline{z_1}\ \overline{z_2} &= (\overline{x_1+iy_1})(\overline{x_2+iy_2})\\ &= (x_1 - iy_1)(x_2 - iy_2)\\ &= (x_1x_2 - y_1y_2) - i(x_1y_2 + y_1x_2) \quad \text{... (ii)}\end{aligned}$$

From (i) and (ii), we get

$$\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$$

Example 12 If $z_1 = 3 + 4i$, $z_2 = 2 + 3i$, then prove that $\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$.

Solution Now,

$$\begin{aligned} z_1 z_2 &= (3+4i)(2+3i) \\ &= 6 + 9i + 8i + 12i^2 \\ &= 6 + 17i - 12 \\ &= -6 + 17i \\ \overline{z_1 z_2} &= -6 - 17i \quad \text{... (i)} \end{aligned}$$

and

$$\begin{aligned} \overline{z_1}\ \overline{z_2} &= (3-4i)(2-3i) \\ &= 6 - 9i - 8i + 12i^2 \\ &= 6 - 9i - 8i - 12 \\ \overline{z_1}\ \overline{z_2} &= -6 - 17i \quad \text{... (ii)} \end{aligned}$$

From (i) and (ii), we get

$$\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$$

Property 3 $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$, $(z_2 \neq 0)$

Proof: Let $z_1 = x_1 + iy_1$
and $z_2 = x_2 + iy_2$

Now, $$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{x_1 + iy_1}{x_2 + iy_2} \times \frac{x_2 - iy_2}{x_2 - iy_2} = \frac{(x_1x_2 + y_1y_2) - i(x_1y_2 - y_1x_2)}{x_2^2 + y_2^2}$$

and $$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{(x_1x_2 + y_1y_2) + i(x_1y_2 - y_1x_2)}{x_2^2 + y_2^2} \quad \text{... (i)}$$

Now , $$\begin{aligned} \frac{\overline{z_1}}{\overline{z_2}} &= \frac{x_1 - iy_1}{x_2 - iy_2} \\ &= \frac{(x_1 - iy_1)(x_2 + iy_2)}{(x_2 - iy_2)(x_2 + iy_2)} \\ &= \frac{x_1x_2 + ix_1y_2 - iy_1x_2 - i^2 y_1y_2}{x_2^2 + y_2^2} \\ \frac{\overline{z_1}}{\overline{z_2}} &= \frac{(x_1x_2 + y_1y_2) + i(x_1y_2 - y_1x_2)}{x_2^2 + y_2^2} \quad \text{... (ii)} \end{aligned}$$

From (i) and (ii), we get

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$$

Example 13 If $z_1 = 5 + 4i$, $z_2 = 3 + 2i$, then prove that $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$.

Solution

$$\frac{z_1}{z_2} = \frac{5+4i}{3+2i}$$

$$= \frac{(5+4i)(3-2i)}{(3+2i)(3-2i)}$$

$$= \frac{15-10i+12i-8i^2}{(3)^2-(2i)^2}$$

$$= \frac{23+2i}{9+4}$$

$$\frac{z_1}{z_2} = \frac{23+2i}{13} = \frac{23}{13} + \frac{2}{13}i$$

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{23}{13} - \frac{2}{13}i \quad \text{... (i)}$$

$$\frac{\bar{z}_1}{\bar{z}_2} = \frac{5-4i}{3-2i} = \frac{(5-4i)(3+2i)}{(3-2i)(3+2i)}$$

$$= \frac{15+10i-12i-8i^2}{9+4} = \frac{15-2i+8}{9+4}$$

$$\frac{\bar{z}_1}{\bar{z}_2} = \frac{23-2i}{13} = \frac{23}{13} - \frac{2}{13}i \quad \text{... (ii)}$$

From (i) and (ii), we get

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$$

> **Skilled Practice!**
>
> Take any two complex numbers and prove that:
>
> (i) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$
>
> (ii) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$
>
> (iii) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$, $(z_2 \neq 0)$

Property 4 $\bar{\bar{z}} = z$

Proof: Let $z = x + iy$

Then $\bar{z} = x - iy$

$\bar{\bar{z}} = x + iy$

$= z$

Example 14 If $z = 7 + 3i$, then prove that $\bar{\bar{z}} = z$.

Solution $z = 7 + 3i$

Then $\bar{z} = 7 - 3i$

$\bar{\bar{z}} = 7 + 3i = z$

Hence, proved $\bar{\bar{z}} = z$

1.4 Modulus of a Complex Number

Just as the absolute value of a real number measures its distance from the origin on the real number line, the modulus of a complex number measures the distance from the origin in the complex plane. Notice that the distance from the origin to the point $P(x, y)$ lies along a radial line and forms the hypotenuse of a right triangle, where the horizontal and vertical sides have length x and y respectively.

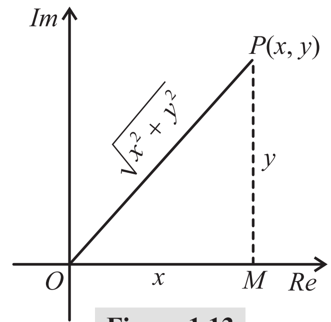


Figure 1.13

Definition

If $z = x + iy$, then the modulus of z is denoted by $|z|$ and defined as $|z| = \sqrt{x^2 + y^2}$.

For example, (i) $|i| = |0 + 1i| = \sqrt{0^2 + 1^2} = 1$

(ii) $|3 - 5i| = \sqrt{(3)^2 + (-5)^2} = \sqrt{9 + 25} = \sqrt{34}$

1.4.1 Properties of Modulus of a Complex Number

Property 1 $|z| = |\bar{z}|$

Proof: Let $z = x + iy$

Then $|z| = \sqrt{x^2 + y^2}$ ... (i)

Now $\bar{z} = x - iy$

$|\bar{z}| = \sqrt{(x)^2 + (-y)^2} = \sqrt{x^2 + y^2}$ ... (ii)

From (i) and (ii), we get

$|\bar{z}| = |z|$

Example 15 If $z = 5 + 4i$, then show that $|z| = |\bar{z}|$.

Solution $z = 5 + 4i$

$|z| = \sqrt{(5)^2 + (4)^2}$

$= \sqrt{25+16} = \sqrt{41}$ ... (i)

Now $\bar{z} = 5 - 4i$

$|\bar{z}| = \sqrt{(5)^2 + (-4)^2}$

$= \sqrt{25+16} = \sqrt{41}$ ... (ii)

From (i) and (ii), we get

$|z| = |\bar{z}|$

Property 2 $|z| = |-z| = |\bar{\bar{z}}| = |-\bar{z}|$

Proof: Let $z = x + iy$

Then $|z| = \sqrt{x^2 + y^2}$ ... (i)

$|-z| = |-x - iy| = \sqrt{(-x)^2 + (-y)^2}$

$= \sqrt{x^2 + y^2}$ ... (ii)

$\bar{z} = x - yi \Rightarrow \bar{\bar{z}} = x + yi$

$|\bar{\bar{z}}| = \sqrt{x^2 + y^2}$ ... (iii)

Here $\bar{z} = x - iy$ as $z = x + iy$

$-\bar{z} = -x + iy$

$|-\bar{z}| = \sqrt{(-x)^2 + (y)^2}$

$= \sqrt{x^2 + y^2}$ ... (iv)

From (i), (ii), (iii) and (iv), we have

$$|z| = |-z| = |\bar{\bar{z}}| = |-\bar{z}|$$

Property 3 $z\bar{z} = |z|^2$

Proof: Let $z = x + iy$

$\therefore$ $\bar{z} = x - iy$

Now $z\bar{z} = (x + iy)(x - iy)$

$= (x)^2 - (iy)^2$

$= x^2 - (i^2 y^2)$

$= x^2 - (-y^2)$ as $i^2 = -1$

$= x^2 + y^2 = |z|^2$ as $|z| = \sqrt{x^2 + y^2}$

Hence, $z\bar{z} = |z|^2$

Example 16 If $z = 5 + 3i$, then show that $z\bar{z} = |z|^2$.

Solution $z = 5 + 3i$

$\therefore$ $\bar{z} = 5 - 3i$

$z\bar{z} = (5 + 3i)(5 - 3i)$

$= (5)^2 - (3i)^2$

$= (25) - (9i^2)$

$$= 25-(-9)$$
$$= 25+9 = 34 \quad \text{... (i)}$$
$$|z| = \sqrt{(5)^2+(3)^2}$$
$$|z|^2 = \left[\sqrt{(5)^2+(3)^2}\right]^2$$
$$= 25+9 = 34 \quad \text{... (ii)}$$

From (i) and (ii), we get

$$z\bar{z} = |z|^2$$

EXERCISE 1.3

1. Find the modulus of the following complex numbers:

   (i) $4 + 3i$ (ii) $-5 - 4i$ (iii) $\frac{3}{5} - \frac{4}{5}i$ (iv) $-\sqrt{2} - \sqrt{3}i$

2. If $z_1 = 2 + 7i$ and $z_2 = 4 - 3i$, then verify that

   (i) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ (ii) $\overline{z_1 z_2} = \bar{z}_1\,\bar{z}_2$ (iii) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

3. If $z = 5 - 2i$, then verify that

   (i) $\bar{\bar{z}} = z$ (ii) $|z| = |\bar{z}|$ (iii) $|z| = |-z|$

   (iv) $z\bar{z} = |z|^2$ (v) $|z| = |-\bar{z}|$

4. If $z = 4 - 3i$, then verify that $|z| = |-z| = |\bar{\bar{z}}| = |-\bar{z}|$.

5. If $z_1 = 2 + 3i$, $z_2 = -1 + i$, then evaluate:

   (i) $Re(z_1 z_2)$ (ii) $Im(z_1 z_2)$

1.5 Finding Real and Imaginary Parts of Complex Number of $(x + iy)^n$ and $\left(\frac{x_1 + iy_1}{x_2 + iy_2}\right)^n$, where $x_2 + iy_2 \neq 0$ and $n = \pm 1$ and $n = \pm 2$

$(x + iy)^n$ when $n = \pm 1$

Let $z = (x + iy)^n$ be a complex number.

(i) For $n = 1$

We have $z = (x + iy)^1$
$$= x + iy$$
$$Re(z) = x, \; Im(z) = y$$

(ii) For $n = -1$

We have $z = (x + iy)^{-1}$

$$= \frac{1}{x+iy}$$

$$= \frac{x-iy}{(x+iy)(x-iy)}$$

$$= \frac{x-iy}{x^2+y^2}$$

$$= \frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2}$$

$$\therefore \quad Re(z) = \frac{x}{x^2+y^2}, \quad Im(z) = \frac{-y}{x^2+y^2}$$

Skilled Practice!

Find real and imaginary parts of $(x + iy)^n$ for $n = \pm 2$

Example 17 Find real and imaginary parts of $(4 + 3i)^{-1}$.

Solution Let $z = (4 + 3i)^{-1}$

$$= \frac{1}{(4+3i)}$$

$$= \frac{1(4-3i)}{(4+3i)(4-3i)}$$

$$= \frac{4-3i}{16+9} = \frac{4-3i}{25} = \frac{4}{25} - \frac{3}{25}i$$

$$\therefore \quad Re(z) = \frac{4}{25}, \quad Im(z) = -\frac{3}{25}$$

Example 18 Find real and imaginary parts of $z = (4+3i)^{-2}$.

Solution $z = (4+3i)^{-2}$

$$= \frac{1}{(4+3i)^2}$$

$$= \frac{1}{(4)^2+(3i)^2+2(4)(3i)} = \frac{1}{16-9+24i}$$

$$= \frac{1}{7+24i}$$

$$= \frac{(7-24i)}{(7+24i)(7-24i)}$$

Do you know?

$Re(z) = \dfrac{z+\bar{z}}{2}$,

$Im(z) = \dfrac{z-\bar{z}}{2i}$

$$= \frac{7-24i}{49+576}$$

$$= \frac{7-24i}{625}$$

$$= \frac{7}{625} - \frac{24}{625}i$$

$$\therefore \quad Re(z) = \frac{7}{625}, \ Im(z) = -\frac{24}{625}$$

Example 19 Find real and imaginary parts of $z = \left(\frac{4+3i}{3+2i}\right)^{-1}$.

Solution

$$z = \left(\frac{4+3i}{3+2i}\right)^{-1}$$

$$= \frac{3+2i}{4+3i}$$

$$= \frac{(3+2i)(4-3i)}{(4+3i)(4-3i)}$$

$$= \frac{12-9i+8i-6i^2}{(4)^2+(3)^2} = \frac{12-i+6}{16+9}$$

$$= \frac{18-i}{25} = \frac{18}{25} - \frac{1}{25}i$$

$$\therefore \quad Re(z) = \frac{18}{25}, \quad Im(z) = -\frac{1}{25}$$

Example 20 Find real and imaginary parts of $z = \left(\frac{1+i}{1-i}\right)^{-2}$.

Solution

$$z = \left(\frac{1+i}{1-i}\right)^{-2}$$

$$= \left(\frac{1-i}{1+i}\right)^{2} = \left(\frac{1-i}{1+i} \times \frac{1-i}{1-i}\right)^{2} = \left(\frac{1-i-i+i^2}{1^2-i^2}\right)^{2}$$

$$= \left(\frac{1-2i-1}{1+1}\right)^{2} = \left(\frac{-2i}{2}\right)^{2} = (-i)^2 = -1$$

$$\therefore \quad Re(z) = -1, \ Im(z) = 0$$

1.6 Solution of Simultaneous Linear Equations with Complex Coefficients

More than one equation which are to be satisfied by the same values of the variables involved are called simultaneous equations or a system of equations.

Example 21 Solve the given simultaneous linear equations with complex coefficients for z and w:

$$5z-(3+i)w=7-i$$
$$(2-i)z+2iw=4$$

Solution

$$5z-(3+i)w=7-i \quad \text{... (i)}$$
$$(2-i)z+2iw=4 \quad \text{... (ii)}$$

Step I From equation (i), we solve for w in terms of z.

$$5z-(3+i)w=7-i$$
$$(3+i)w=5z-7+i$$
$$w=\frac{5z-7+i}{3+i} \quad \text{... (iii)}$$

Step II Put the expression of w in equation (ii)

$$(2-i)z+2i\left(\frac{5z-7+i}{3+i}\right)=4$$
$$(2-i)(3+i)z+2i(5z-7+i)=4(3+i)$$
$$(6+2i-3i-i^2)z+10iz-14i+2i^2=12+4i$$
$$(6-i+1)z+10iz-14i-2=12+4i$$
$$(7-i)z+10iz-14i-2=12+4i$$
$$7z-iz+10iz-14i-2=12+4i$$
$$7z+9iz=12+4i+14i+2$$
$$7z+9iz=14+18i$$
$$z(7+9i)=2(7+9i)$$
$$z=\frac{2(7+9i)}{7+9i}$$
$$z=2$$

Step III

Put $z=2$ in (iii), we get

$$w=\frac{5(2)-7+i}{3+i}$$
$$=\frac{3+i}{3+i}=1$$

Hence, $z=2$, $w=1$

EXERCISE 1.4

1. Find the real and imaginary parts of the following complex numbers:

(i) $(8-3i)^2$ (ii) $(5+3i)^{-1}$ (iii) $(4-5i)^{-1}$

(iv) $(4-3i)^{-2}$ (v) $\left(\dfrac{3+2i}{4+3i}\right)^{-1}$ (vi) $\left(\dfrac{2-i}{2+i}\right)^{-2}$

(vii) $\left(\dfrac{1-2i}{1+i}\right)^{2}$

2. Solve the following simultaneous linear equations with complex coefficients for w and z:

(i) $3z+(2+i)w=11-i$
$(2-i)z-w=-1+i$

(ii) $2z+(3+i)w=9-i$
$-iz-i\,w=-1+i$

(iii) $z-4w=3i$
$2z+3w=11-5i$

(iv) $z+w=3i$
$2z+3w=2$

(v) $2z+(3+i)w=1$
$-z-(1-i)w=2$

REVIEW EXERCISE 1

1. Four possible answers are given for the following questions. Choose the correct answer:

(i) $i^2+i^4=$
(a) -1 (b) 0 (c) 1 (d) 2

(ii) Real part of $(2-3i)(2+3i)$ is:
(a) -3 (b) 1 (c) 4 (d) 13

(iii) Imaginary part of $(2-i)(2+i)$ is:
(a) 0 (b) 1 (c) 7 (d) 9

(iv) $x+iy$ will be pure imaginary number, when:
(a) $y=0$ (b) $x=0$ (c) $i=0$ (d) $x=0, y=0$

(v) What is additive inverse of $5-2i$?
(a) $5+2i$ (b) $-5-2i$ (c) $5-2i$ (d) $-5+2i$

(vi) What is multiplicative inverse of $z=1+i$?
(a) $1-i$ (b) i (c) $\frac{1}{2}-\frac{1}{2}i$ (d) $-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i$

(vii) If $z = 4 - 3i$, then $z\bar{z} =$

(a) 3 (b) 9 (c) 16 (d) 25

(viii) Conjugate of $9 - 4i$ is:

(a) $-9 - 4i$ (b) $9 + 4i$ (c) $9 + 9i$ (d) $4 - 9i$

(ix) If $z = 4 + 4i$, then $z + \bar{z} =$

(a) 8 (b) $8 + 8i$ (c) $8i$ (d) 0

(x) If $z = 5 + 4i$, then $|z| =$

(a) 9 (b) 25 (c) 41 (d) $\sqrt{41}$

2. (i) Is "0" a complex number? Explain.
   (ii) What is the result of multiplying a complex number by its conjugate?
   (iii) State the condition for two complex numbers to be equal.
3. Simplify:
   (i) i^{37} (ii) $i^{13} \times i^{11}$ (iii) $(-i)^{-9}$
   (iv) $(3 - 4i)(5 - 6i)$ (v) $(3 + 4i) \div (5 - 7i)$
4. Find additive and multiplicative inverse of $z = 8 + 9i$.
5. If $z_1 = 3 + 4i$ and $z_2 = 2 + 3i$, then verify that
   (i) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ (ii) $\overline{z_1 z_2} = \bar{z}_1\,\bar{z}_2$ (iii) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$
   (iv) $|z_1| = |-\bar{z}_1|$ (v) $\bar{\bar{z}}_2 = z_2$ (vi) $z_1\bar{z}_1 = |z_1|^2$
6. If $z_1 = 5 + 4i$, $z_2 = 3 + 2i$, then find
   (i) $z_1 z_2$ (ii) $\dfrac{z_1}{z_2}$ (iii) $\bar{z}_1\bar{z}_2$ (iv) $|z_1 z_2|$
7. Find real and imaginary parts of $z = (2 + 7i)^{-1}$.
8. Solve the given simultaneous linear equations with complex coefficients for z and w:

$$iz + (2 - i)w = 4 + i$$
$$iz + (3 + i)w = 3 + 3i$$

9. Solve $(3 - 4i)(a + bi) = 1 + 0i$ and find the values of a and b.
10. Solve the equation for x and y:

$(3 - 2i)(x + yi) = 2(x - 2yi) + 2i - 1$

UNIT 2

Quadratic Equations and Inequalities

Students' Learning Outcomes

After completing this unit, the students will be able to:

- Solve quadratic equations of the form $ax^2+bx+c=0$, $a(\neq 0)$, b, c are real numbers by factorization, quadratic formula, completing square and graphs.
- Find the intersecting point graphically when intersection occurs between
  - a linear function and coordinate axes.
  - two linear functions.
  - a linear and a quadratic function.
- Sketch and interpret the graphs of the quadratic function $y = ax^2+bx+c$, $a(\neq 0)$, b and c are real numbers.
- Establish relationship between roots and coefficients of quadratic equations.
- Form a quadratic equation when roots are given.
- Find discriminant of a given quadratic equation.
- Identify the nature of roots of a quadratic equation through discriminant.
- Solve quadratic inequalities in one unknown.
- Solve problems of "changing the subject of formula".
- Solve fractional equations that can be reduced to quadratic equations.
- Apply the concept of quadratic equations and quadratic inequalities, to real world problems (such as in physics, engineering and finance, i.e. calculating max and min heights in projectile motion, determining the maximum price on a company's budget, stability of population, growth of business, the relationship between hours worked and amount earned etc.).

INTRODUCTION

This unit explores into the study and applications of quadratic equations, a fundamental concept in algebra with wide-ranging uses in both academics and the real world. Students will learn to solve quadratic equations using factorization, the quadratic formula, completing the square and graphical methods. The unit also explores how to sketch and interpret the graphs of quadratic functions and examines the relationship between the roots and coefficients, as well as how to form equations from given roots. Key concepts like the discriminant are used to determine the nature of the roots, whether they are real, repeated or complex. Students will also solve quadratic inequalities, tackle equations involving formulas and fractions and graphically identify points of intersection between linear and quadratic functions. Finally, the unit emphasizes practical applications, demonstrating how quadratic concepts are used in different fields.

Recall

An **algebraic expression** is a combination of constants and variables joined with the signs of arithmetic operations.
For example, 5, $3x$, $2x-7y$, $3y^2-5y-2$ etc.
A **polynomial** in one variable 'x' is an expression of the form:

$$P(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$$

where $a_0, a_1, \ldots, a_n$ are real numbers and n is a non-negative integer.
If $a_n \neq 0$, then degree of polynomial is n.
For example, $5x^2 - x + 7$ is a polynomial of degree 2.

2.1 Quadratic Equation

A quadratic equation is a second degree polynomial equation in one variable.
For example, the following equations are quadratic equations:

(i) $2x^2 + 3x + 7 = 0$

(ii) $x^2 - 5x - 4 = 0$

(iii) $3y^2 - 6y + 3 = 0$

Note

If $a, b, c \in R$, $a \neq 0$ and $b \neq 0$, then equation $ax^2 + bx + c = 0$ is called the standard form of the quadratic equation in x.

If $a = 0$, the equation $ax^2 + bx + c = 0$ becomes $bx + c = 0$ which is a linear equation.

If $b = 0$, the equation $ax^2 + bx + c = 0$ becomes $ax^2 + c = 0$ which is pure quadratic equation.

2.1.1 Solution of Quadratic Equation of the Form $ax^2 + bx + c = 0$, where $a, b, c \in R$ and $a \neq 0$

a Factorization Method

The solution of a quadratic equation by factorization is based on the following result:
If $(x + l)(x + m) = 0$, then $x + l = 0$ or $x + m = 0$, where $l, m \in R$. This is called zero product property.

Example 1 Solve by factorization method: $x^2+\frac{7}{3}x=2$

Solution

$$x^2+\frac{7}{3}x = 2$$

$$3x^2+7x = 6 \quad \text{[multiplying both sides by 3]}$$

$$3x^2+7x-6 = 0$$

$$3x^2+9x-2x-6 = 0$$

$$3x(x+3)-2(x+3) = 0$$

$$(x+3)(3x-2) = 0$$

Now if either of the factor $x+3$ or $3x-2$ is zero, then their product is zero. So,

$$x+3=0 \text{ or } 3x-2=0$$

$$x=-3 \quad \text{or} \quad x=\frac{2}{3}$$

Hence, solution set is $\left\{-3, \frac{2}{3}\right\}$.

Do you know?
Every quadratic equation has at most two roots.

Note
An equation in which variable appears in the denominator of one or more terms is called a **fractional equation**.

(i) $\frac{2x}{5}-\frac{6}{x}=\frac{3}{7}, x\neq 0$

(ii) $\frac{x+3}{x-1}=\frac{2x-1}{x+5}, x\neq 1,-5$ are fractional equations.

Example 2 Solve by factorization method: $\frac{2x-3}{2}=\frac{4x-6}{x}, x\neq 0.$

Solution

$$\frac{2x-3}{2}=\frac{4x-6}{x}$$

[Multiplying both sides by LCM of the denominators. i.e., $2x$ to eliminate denominator]

$$\frac{2x(2x-3)}{2}=\frac{(2x)(4x-6)}{x}$$

$$2x^2-3x = 8x-12$$

[Transposing all terms to left hand side of '=' sign.]

$$2x^2-11x+12 = 0$$

$$2x^2-8x-3x+12 = 0$$

$$2x(x-4)-3(x-4) = 0$$

$$(2x-3)(x-4) = 0$$

$$2x-3=0 \text{ or } x-4 = 0$$

$$x=\frac{3}{2} \quad \text{or} \quad x=4$$

Hence, solution set is $\left\{\frac{3}{2}, 4\right\}$.

b Solution of a Quadratic Equation by Completing Square Method

Example 3 Solve by completing square method: $3x^2 - 32 = -10x$

Solution $3x^2 - 32 = -10x$

Step I Simplifying equation so that the terms in x^2 and x are on one side of the equation and the term without x on the other side.

$3x^2 + 10x = 32$ [Transposing]

Step II Make the coefficient of x^2 unity and positive by dividing throughout by the coefficient of x^2.

$$x^2 + \frac{10}{3}x = \frac{32}{3}$$

Step III Add to each side of the equation the square of half the coefficient of x.

$$x^2 + \frac{10}{3}x + \left(\frac{5}{3}\right)^2 = \frac{32}{3} + \left(\frac{5}{3}\right)^2$$

$$\left(x + \frac{5}{3}\right)^2 = \frac{32}{3} + \frac{25}{9}$$

$$\left(x + \frac{5}{3}\right)^2 = \frac{96+25}{9} = \frac{121}{9}$$

$$\left(x + \frac{5}{3}\right)^2 = \left(\frac{11}{3}\right)^2$$

Step IV Taking square root of both the side.

$$\therefore \quad \pm\left(x + \frac{5}{3}\right) = \frac{11}{3}$$

$$x + \frac{5}{3} = \pm\frac{11}{3}$$

$$x + \frac{5}{3} = \frac{11}{3} \quad \text{or} \quad x + \frac{5}{3} = -\frac{11}{3}$$

$$x = \frac{11}{3} - \frac{5}{3} \quad \text{or} \quad x = -\frac{11}{3} - \frac{5}{3}$$

$$x = \frac{6}{3} = 2 \quad \text{or} \quad x = \frac{-16}{3} = -5\frac{1}{3}$$

Hence, solution set is $\left\{2, -5\frac{1}{3}\right\}$.

Skilled Practice!

Solve by completing square method.

(i) $5x^2 - 18 = 2x$

(ii) $2x^2 - 7x = -20$

Challenge!

The square of every real quantity is non-negative and therefore greater than zero. Thus, $(a - b)^2$ is non-negative i.e., $a^2 - 2ab + b^2 \geq 0$. Do you agree:

$$\frac{x+y}{2} > \sqrt{xy}$$

c Solution of a Quadratic Equation by Quadratic Formula

Derivation of Quadratic Formula by Completing Square Method

After suitable reduction and transposition every quadratic equation can be written in the form: $ax^2 + bx + c = 0$, where a, b, c are real numbers and $a \neq 0$.

Now, $ax^2 + bx + c = 0$

We can rewrite the equation as:

Step I $ax^2 + bx = -c$ [Transposing]

Step II Dividing by a (the co-efficient of x^2), on both sides of the equation.

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Step III We add square of half of the co-efficient of x i.e; $\left(\frac{b}{2a}\right)^2$ on both side of the equation.

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

> **Remember!**
> $ax^2 + bx + c = 0$, can equivalently be expressed as:
> $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$ since, $a \neq 0$

Step IV Taking square root of both the sides.

$$\pm\left(x + \frac{b}{2a}\right) = \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

or $$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$ [Transposing]

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

> **Remember!**
> Moving a term from one side of an equation to the other by changing its sign is called transposition

This is the quadratic formula.

Example 4 Solve by using quadratic formula: $3x^2 + 7x - 6 = 0$.

Solution $3x^2 + 7x - 6 = 0$ (given)

$ax^2 + bx + c = 0$ (standard form)

Comparing the given equation with the standard form.

$$a = 3, b = 7, c = -6$$

Substituting the values of a, b, c in the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-7 \pm \sqrt{(7)^2 - 4(3)(-6)}}{2(3)}$$

$$= \frac{-7 \pm \sqrt{49 + 72}}{6} = \frac{-7 \pm \sqrt{121}}{6}$$

$$= \frac{-7 \pm 11}{6}$$

$$= \frac{-7 + 11}{6}, \frac{-7 - 11}{6}$$

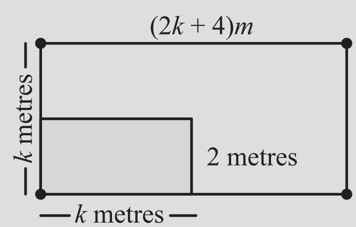
$$= \frac{4}{6}, \frac{-18}{6}$$

$$x = \frac{2}{3}, -3$$

Hence, solution set is $\left\{\frac{2}{3}, -3\right\}$

Activity

A rectangular garden is in front of Kiran's house, whose dimensions are $(2k + 4)$ metres and k metres. A smaller rectangular portion of the garden of dimensions k metres and 2 metres is levelled. Find the area of unlevelled portion of the garden.



d Solution of Quadratic Equation by Graphs

To solve the quadratic equation graphically means to find the points where the graph of quadratic equation crosses the x-axis. The x-values of these points are the solution of the quadratic equation. If the coefficient of x^2 in the quadratic equation is positive, then the graph will be $\cup$ shaped and if the coefficient of x^2 in quadratic equation is negative,then the graph will be $\cap$ shaped. To sketch the graph of quadratic function, we find its vertex by completing square method. $a(x - h)^2 + k$ is called the vertex form of quadratic function having vertex (h, k) and a is the coefficient of x^2.

Do you know?

The graph of a quadratic function is called parabola.

Example 5 **(i)** Sketch and interpret the graph of quadratic function

$$y = 2x^2 - x - 10.$$

(ii) Solve the quadratic equation $2x^2 - x - 10 = 0$ graphically.

Solution **(i)** Let us sketch the graph of $y = 2x^2 - x - 10$.

$$y = 2\left(x^2 - \frac{1}{2}x - 5\right) = 2\left[x^2 - \frac{1}{2}x + \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2 - 5\right]$$

$$= 2\left[\left(x - \frac{1}{4}\right)^2 - \frac{1}{16} - 5\right]$$

$$= 2\left[\left(x - \frac{1}{4}\right)^2 - \frac{81}{16}\right] = 2\left(x - \frac{1}{4}\right)^2 - \frac{2(81)}{16}$$

$$= 2\left(x - \frac{1}{4}\right)^2 - \frac{81}{8} = 2(x - 0.25)^2 - 10.13$$

Compare it with $y = a(x - h)^2 + k$. So, the vertex is (0.25, –10.13). To sketch the graph of quadratic function we take some more points.

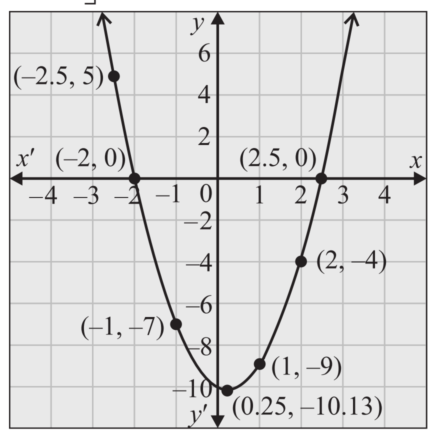


Figure 2.1

x	–2.5	–1	1	2
y	5	–7	–9	–4

By plotting the points and vertex, graph is sketched above. The graph of the given quadratic function $y = 2x^2 - x - 10$ opens upwards and it intersects the x-axis at (–2, 0) and (2.5, 0).

(ii) The graph intersects the x-axis at $x = 2.5$ and $x = -2$. Therefore, roots of the quadratic equation $2x^2 - x - 10 = 0$ are 2.5 and –2.

Hence, solution set is {2.5, –2}.

Remember!

For finding the solution of quadratic equation $ax^2 + bx + c = 0$ graphically, first of all we draw the graph of quadratic function $y = ax^2 + bx + c$.

EXERCISE 2.1

1. Write the following quadratic equations in standard form:

(i) $3x - 1 = 2x^2$ (ii) $2x(x + 1) = 4(2x + 3)$

(iii) $2x^2 - 4x = 4x + 7$ (iv) $4(3x - 2) = 9x^2$

(v) $2x + \frac{1}{x} = 5 - \frac{1}{x}, \; x \neq 0$ (vi) $\frac{6x + 6}{20 - x} = \frac{1}{x}, \; x \neq 0, 20$

2. Solve the following quadratic equations by factorization method:

(i) $x^2 - x - 6 = 0$ (ii) $x^2 + 3x - 28 = 0$

(iii) $6x^2 + 13x - 5 = 0$ (iv) $x^2 - \frac{3}{2}x = \frac{9}{2}$

(v) $\frac{3x-8}{x-2} = \frac{5x-2}{x+5}, x \neq 2, -5$ (vi) $\frac{1}{x-1} - \frac{1}{x+3} = \frac{1}{35}, x \neq 1, -3$

3. Solve the following quadratic equations by completing square method:

(i) $2x^2 + 5x + 2 = 0$ (ii) $x^2 + x = 42$

(iii) $12x^2 + 7x = 12$ (iv) $\frac{x+3}{2x-7} = \frac{2x-1}{x-3}, x \neq \frac{7}{2}, 3$

(v) $\frac{1}{1+x} - \frac{1}{3-x} = \frac{6}{35}, x \neq -1, 3$ (vi) $\frac{3x-1}{4x+7} = 1 - \frac{6}{x+7}, x \neq \frac{-7}{4}, -7$

4. Use quadratic formula to solve the following equations:

(i) $2x^2 - 5x + 3 = 0$ (ii) $2x^2 - 7x - 15 = 0$

(iii) $2x^2 + 7x = 15$ (iv) $x^2 + 11 = 7x$

(v) $\frac{x+4}{x-4} + \frac{x-2}{x-3} = 6\frac{1}{3}, x \neq 4, 3$ (vi) $\frac{3x-3}{x+1} = \frac{2x-1}{x-1}, x \neq -1, 1$

5. Solve the following quadratic equations graphically:

(i) $x^2 - 3x - 18 = 0$ (ii) $x^2 - 5x - 14 = 0$

(iii) $2x^2 + 13x + 6 = 0$ (iv) $4x^2 + 12x - 27 = 0$

2.2 Finding Intersection Point(s) Graphically

2.2.1 Finding Points of Intersection of a Linear Function and Coordinate Axes Graphically

Example 6 Find the points of intersection of $y = 2x + 4$ with coordinate axes graphically.

Solution $y = 2x+4$ is an equation of a line. Now, we draw graph of $y = 2x + 4$.

Table of values

x	1	−4
y	6	−4

Remember! To draw a straight line, it is essential to have at least two distinct points.

We plot points (1, 6), (–4, –4) and join these points.

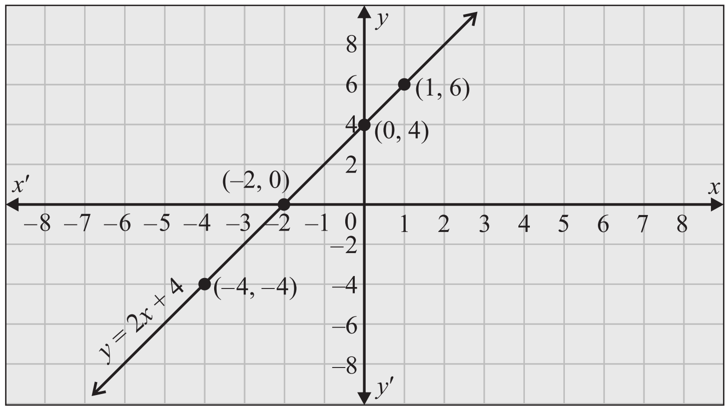


Figure 2.2

Point of intersection on x-axis is (–2,0) and the point of intersection on y-axis is (0, 4).

2.2.2 Finding Point of Intersection of Two Linear Functions Graphically

Example 7 If $x + y = 6$ and $3x + y = 12$, then find their point of intersection graphically.

Solution $x + y = 6$ and $3x + y = 12$ are two linear equations. We draw graphs of both of these equations.

Table of values for $x + y = 6$

x	0	– 4
y	6	10
Points are: (0, 6), (– 4, 10)		

Table of values for $3x + y = 12$

x	0	2
y	12	6
Points are: (0, 12), (2, 6)		

Now, we plot points for both of these equations.

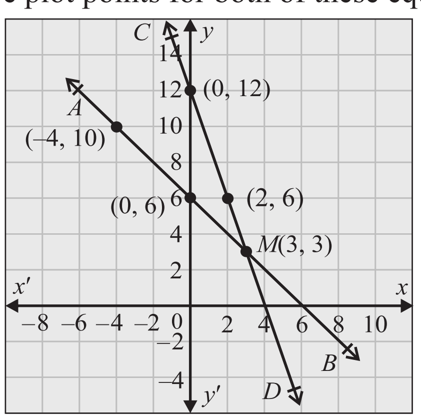


Figure 2.3

$\overleftrightarrow{AB}$ is the graph of $x + y = 6$

$\overleftrightarrow{CD}$ is the graph of $3x + y = 12$

Hence, the point of intersection is $M(3, 3)$.

2.2.3 Finding Points of Intersection of a Linear Function and a Quadratic Function Graphically

Example 8 If $y = 2 + 3x$ and $y = 2x^2 + 7x - 4$, then find their points of intersection graphically.

Solution We draw graphs of both of these equations.

Table of values for $y = 2 + 3x$

x	0	–2
y	2	–4
Points	(0, 2)	(–2, –4)

Now, $y = 2x^2 + 7x - 4$

$$= 2\left(x^2 + \frac{7}{2}x - 2\right) = 2\left(x^2 + \frac{7}{2}x + \left(\frac{7}{4}\right)^2 - \left(\frac{7}{4}\right)^2 - 2\right)$$

$$= 2\left[\left(x + \frac{7}{4}\right)^2 - \frac{49}{16} - 2\right] = 2\left[\left(x + \frac{7}{4}\right)^2 - \frac{81}{16}\right] = 2\left(x + \frac{7}{4}\right)^2 - 2\left(\frac{81}{16}\right)$$

$$= 2(x + 1.75)^2 - 10.13$$

So, the vertex is (–1.75, –10.13).

Table of values for $y = 2x^2 + 7x - 4$

x	–4	–3	0	1
y	0	–7	–4	5

We plot points of both equations and draw their graphs.

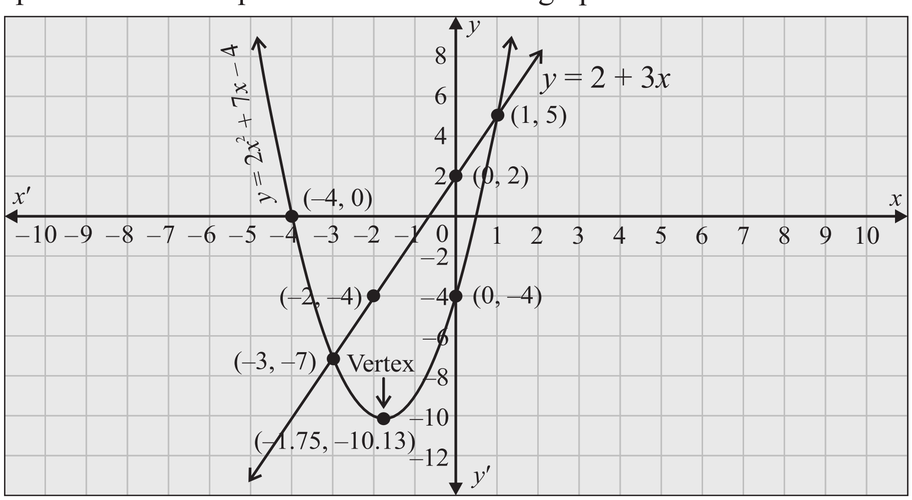


Figure 2.4

Points of intersection are : (1, 5) and (–3, –7)

EXERCISE 2.2

1. Find the points of intersection of the following linear equations with coordinate axes graphically:
   (i) $x + y = 8$ (ii) $x - y = 1$ (iii) $x - 2y = 1$
   (iv) $x - 2y + 2 = 0$ (v) $5x - 5y = 1$
2. Solve the following system of linear equations graphically:
   (i) $x + y = 8$, $3x - y = 4$ (ii) $x - y = 1$, $x + 2y = 7$ (iii) $x - 2y = 1$, $2x + y = 2$
   (iv) $y = 2x + 2$, $3x + 2y = 4$ (v) $3y = 2x + 8$, $x + y = 1$
3. Solve the following equations graphically:
   (i) $y = 8x - 32$, $y = x^2 - 6x + 8$ (ii) $y + x = 2$, $y = 2x^2 + x - 10$

2.3 Relationship Between the Roots and Coefficients of a Quadratic Equation

The standard form of a quadratic equation is $ax^2 + bx + c = 0$, $a \neq 0$ and $a, b, c \in R$. The roots of this equation are:

$$\frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \frac{-b-\sqrt{b^2-4ac}}{2a}$$

Let these roots be α, β

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a}, \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

Now S = Sum of the roots $= \alpha + \beta$

$$= \frac{-b+\sqrt{b^2-4ac}}{2a} + \frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$= \frac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a}$$

$$= \frac{-2b}{2a} = \frac{-b}{a}$$

$$S = \text{Sum of the roots} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2}$$

$$P = \text{Product of the roots} = \alpha\beta$$

$$= \left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)$$

$$= \frac{(-b)^2-\left(\sqrt{b^2-4ac}\right)^2}{(2a)(2a)} = \frac{b^2-(b^2-4ac)}{4a^2}$$

$$= \frac{b^2-b^2+4ac}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}$$

$$P = \text{Product of the roots} = \frac{c}{a} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}$$

Important Note!

(i) The standard form of a quadratic equation is $ax^2+bx+c=0$ where $a \neq 0$ and $a, b, c \in R$.

(ii) Sum of the roots $= -\frac{b}{a}$, Product of the roots $= \frac{c}{a}$

(iii) When the coefficient of x^2 in a quadratic equation is unity (one).
- Sum of roots is equal to the coefficient of x with sign changed.
- Product of the roots is equal to the constant term.

(iv) If the constant term of $ax^2+bx+c=0$ is zero, then sum of its roots $= \frac{-b}{a}$ and products of the roots = 0.

Example 9 Find the sum and the product of the roots of the equation $3x^2+5x-12=0$ without solving.

Solution Standard form of the quadratic equation is $ax^2+bx+c=0$, where $a \neq 0$ and $a, b, c \in R$. The given equation is $3x^2+5x-12=0$.

Here $a=3,\ b=5,\ c=-12$

$$S = \text{Sum of the roots} = -\frac{b}{a} = -\frac{5}{3}$$

$$P = \text{Product of the roots} = \frac{c}{a} = \frac{-12}{3} = -4$$

Example 10 Find the sum and the product of the roots of the quadratic equation

$$x^2+6x+8=0$$

Solution Coefficient of x^2 is unity.

Sum of the roots = – 6 [Coefficient of x with sign changed]

Product of the roots = 8 [Constant term]

2.3.1 Form a Quadratic Equation When Roots are Given

The standard form of a quadratic equation is $ax^2 + bx + c = 0$, where $a \neq 0$ and a, b, $c \in R$.

Let α, β be its roots. We have established a relation between the roots and coefficients.

$$S = \text{Sum of the roots} = \alpha + \beta = -\frac{b}{a} \qquad \text{... (i)}$$

$$P = \text{Product of the roots} = \alpha\beta = \frac{c}{a} \qquad \text{... (ii)}$$

The given quadratic equation is $ax^2 + bx + c = 0$.

Dividing this equation by a, we get

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Writing this equation in terms of α, β

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0 \qquad \text{[using (i) and (ii)]}$$

or $\quad x^2 - (\text{Sum of the roots})\, x + \text{Product of the roots} = 0$

or $\quad x^2 - Sx + P = 0$

Example 11 Form the equation whose roots are $\frac{2}{3}$ and $\frac{3}{4}$.

Solution **Method-I**

Here $\quad x = \frac{2}{3}$, $x = \frac{3}{4}$

$$\therefore \; x - \frac{2}{3} = 0 \quad \text{or} \quad x - \frac{3}{4} = 0$$

Both of these statements are included in

$$\left(x - \frac{2}{3}\right)\left(x - \frac{3}{4}\right) = 0$$

$$x^2 - \frac{3}{4}x - \frac{2}{3}x + \frac{1}{2} = 0$$

$$x^2 - \frac{17}{12}x + \frac{1}{2} = 0$$

$12x^2 - 17x + 6 = 0$ is the required equation.

Method-II

Roots of the required equation are $\frac{2}{3}, \frac{3}{4}$

$$S = \text{Sum of the roots} = \frac{2}{3} + \frac{3}{4}$$

$$= \frac{8+9}{12} = \frac{17}{12} \qquad \text{... (i)}$$

$$P = \text{Product of the roots} = \frac{2}{3} \times \frac{3}{4} = \frac{1}{2} \qquad \text{... (ii)}$$

Required equation is

$$x^2 - Sx + P = 0$$

Putting values of S and P in above equation

$$x^2 - \frac{17}{12}x + \frac{1}{2} = 0 \quad \text{[From (i) and (ii)]}$$

So, $\quad 12x^2 - 17x + 6 = 0$

Example 12 If α, β are roots of the equation $x^2 - 7x + 10 = 0$, then form an equation whose roots are $2\alpha + 1$ and $2\beta + 1$.

Solution As α, β are roots of $x^2 - 7x + 10 = 0$, therefore

$$\alpha + \beta = -\left(\frac{-7}{1}\right) = 7 \quad \text{... (i)} \quad , \qquad \alpha\beta = \frac{10}{1} = 10 \quad \text{... (ii)}$$

S = Sum of roots of the required equation $= 2\alpha + 1 + 2\beta + 1$

$$= 2(\alpha + \beta) + 2$$

$$= 2(7) + 2 = 16$$

P = Product of roots of the required equation $= (2\alpha + 1)(2\beta + 1)$

$$= 4\alpha\beta + 2(\alpha + \beta) + 1$$

$$= 4(10) + 2(7) + 1 = 55$$

The required equation will be

$$x^2 - Sx + P = 0$$

$$x^2 - 16x + 55 = 0 \qquad \text{[Putting values of } S \text{ and } P\text{]}$$

Example 13 Find the condition that the roots of $ax^2 + bx + c = 0$ may be equal in magnitude but opposite in sign, where $a \neq 0$.

Solution $ax^2 + bx + c = 0$ (given equation)

Let $\alpha, -\alpha$ be the roots of this equation

$$S = \text{Sum of the roots} = \alpha + (-\alpha) = -\frac{b}{a}$$

$$0 = -\frac{b}{a} \qquad \text{where } a \neq 0$$

$\therefore$ $b = 0$ is the required condition.

> **Challenge!**
> If one root of the quadratic equation is twice the other and their product is 8, find the equation and both roots.

EXERCISE 2.3

1. Form a quadratic equation whose roots are given below:
   (i) $-4, 9$ (ii) $5, -7$ (iii) $\frac{-7}{5}, \frac{-6}{5}$
   (iv) $\frac{-3}{2}, \frac{7}{2}$ (v) $3+\sqrt{5}, 3-\sqrt{5}$ (vi) $-2+\sqrt{3}, -2-\sqrt{3}$
2. Find the quadratic equation with roots exceeding by 2 than those of roots of $x^2 + 9x + 20 = 0$.
3. Find the equation whose roots are double the roots of $x^2 - px + q = 0$.
4. If α, β are the roots of the equation $x^2 + 2x + 4 = 0$, then find the equation whose roots are:
   (i) $\frac{1}{\alpha}, \frac{1}{\beta}$ (ii) $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$ (iii) $2\alpha - \frac{1}{\beta}, 2\beta - \frac{1}{\alpha}$
   (iv) α^2, β^2 (v) $2\alpha - 1, 2\beta - 1$
5. Find the condition that roots of $ax^2 + bx + c = 0$ should be reciprocals of each other.
6. Find the value of k, given that one root of $x^2 - (2k + 4)x + (7k + 1) = 0$ is 3.
7. Find the value of m in the equation $2x^2 + 3x + m = 0$ when sum of its roots is equal to double the product of its roots.
8. If α, β are the roots of $x^2 + ax + b = 0$ and α^2, β^2 are the roots of $x^2 + Ax + B = 0$, then prove that $A = 2b - a^2$, $B = b^2$.
9. If α, β are the roots of $x^2 + px + q = 0$, then find the condition that
   (i) $\alpha = \beta$ (ii) $\alpha = \frac{1}{\beta}$

2.4 Discriminant of a Quadratic Equation

The quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$ and $a, b, c \in R$ has two roots. Let these roots be α and β.

Now, $\alpha = \dfrac{-b+\sqrt{b^2-4ac}}{2a}$ and $\beta = \dfrac{-b-\sqrt{b^2-4ac}}{2a}$

Discriminant

The expression $b^2 - 4ac$, under the radical sign in quadratic formula is called the Discriminant (Disc.) of the equation, $ax^2 + bx + c = 0$. It determines the nature of character of the roots of the equation.

2.4.1 Nature of the Roots of a Quadratic Equation

Before proceeding to discuss the nature or character of the roots of a quadratic equation, it is important that the students should clearly distinguish among the terms rational, irrational, real and imaginary.

Now, $\sqrt{25}$ or 5, $4\frac{1}{2}, \frac{5}{6}$ are rational and real. $\sqrt{7}, \sqrt{5}, \sqrt{3}$ are irrational and real. $\sqrt{-3}, \sqrt{-7}, \sqrt{-4}$ are imaginary or unreal.

As we know that the two roots of $ax^2 + bx + c = 0$, where $a \neq 0$ and $a, b, c \in R$ are $\alpha = \dfrac{-b+\sqrt{b^2-4ac}}{2a}$ and $\beta = \dfrac{-b-\sqrt{b^2-4ac}}{2a}$.

Then we have the following results:

(i) If $b^2 - 4ac > 0$ and is a perfect square, then the roots are rational (real) and unequal.

(ii) If $b^2 - 4ac > 0$ and is not a perfect square, then the roots are irrational (real) and unequal.

(iii) If $b^2 - 4ac = 0$, then the roots are real and equal.

(iv) If $b^2 - 4ac < 0$, then the roots are imaginary (complex conjugates).

By applying these tests, the nature of the roots of any quadratic equation may be determined without solving the equation.

Example 14 Examine the nature of roots of the following quadratic equations:

(i) $3x^2 - 8x - 2 = 0$ (ii) $4x^2 + 4x + 1 = 0$

(iii) $2x^2 + 3x + 4 = 0$ (iv) $x^2 + 4x + 3 = 0$

Solution **(i)** Here, $a = 3,\ b = -8,\ c = -2$

Disc. $= b^2 - 4ac$

$= (-8)^2 - 4\,(3)\,(-2)$

$= 64 + 24 = 88 > 0$

As the discriminant is positive but not a perfect square. Therefore, the roots are irrational and unequal.

(ii) Here, $a = 4,\ b = 4,\ c = 1$

Disc. $= b^2 - 4ac$

$= (4)^2 - 4(4)(1)$

$= 16 - 16 = 0$

As the discriminant is zero. Therefore, the roots are real and equal.

(iii) Here, $a = 2,\ b = 3,\ c = 4$

Disc. $= b^2 - 4ac$

$= (3)^2 - 4\,(2)(4)$

$= 9 - 32$

$= -23 < 0$

Which is negative. Therefore, the roots are imaginary.

(iv) Here, $a = 1,\ b = 4,\ c = 3$

Disc. $= b^2 - 4ac$

$= (4)^2 - 4\,(1)(3)$

$= 16 - 12$

$= 4 = 2^2 > 0$

As the discriminant is positive and perfect square. Therefore, the roots are rational and unequal.

Note If the roots of $ax^2 + bx + c = 0$ are rational, then $b^2 - 4ac$ must be a perfect square.

Example 15 Find the value of k if the equation $(k+1)x^2 + (2k-1)x + (k-1) = 0$ has equal roots.

Solution Here $a = k + 1,\ b = 2k - 1,\ c = k - 1$

$$\text{Disc.} = b^2 - 4ac = 0 \qquad \text{(roots are equal)}$$

$$(2k - 1)^2 - 4(k + 1)\,(k - 1) = 0$$

$$(4k^2 - 4k + 1) - 4(k^2 - 1) = 0$$

$$4k^2 - 4k + 1 - 4k^2 + 4 = 0$$

$$-4k + 5 = 0$$

$$4k = 5$$

$$k = \frac{5}{4}$$

EXERCISE 2.4

1. Examine the nature of roots of the following quadratic equations:

(i)	$3x^2 - 9x - 2 = 0$	(ii)	$x^2 + 6x + 9 = 0$
(iii)	$2x^2 + 4x + 5 = 0$	(iv)	$7x^2 - 6x - 1 = 0$
(v)	$5x^2 - 2x + 10 = 0$	(vi)	$x^2 - 8x + 16 = 0$

2. For what values of t, the roots of $3x^2 + x + 9t = 0$ are real and unequal?
3. If the quadratic equation $16x^2 + 7px + 49 = 0$ has equal roots, then find the values of p.
4. If the quadratic equation $4u^2 + 8u + q = 0$ has unequal and real roots, find the possible values for q.
5. Find the value for m, if the quadratic equation $mx^2 - 8x + 1 = 0$ has real and equal roots.

2.5 Solution of Quadratic Inequalities in One Unknown

Inequality In Mathematics an inequality is a relation which makes a un-equal comparison between two numbers or other mathematical expressions. We have learned how to use the inequality symbols $<$ (less than), $>$ (greater than), $\leq$ (less than or equal to) and $\geq$ (greater than or equal to).

Here we learn how to solve quadratic inequalities in one unknown.

There are following four steps to solve quadratic inequality:

(i) Solve the corresponding equation. (ii) Determine intervals.
(iii) Test points in each interval. (iv) Conclusion.

Example 16 Solve the inequality $x^2 - 5x + 6 > 0$.

Solution $x^2 - 5x + 6 > 0$

Step I **Solve the corresponding or associated equation.**

Set the quadratic expression equal to zero (0) to find the roots.

$$x^2 - 5x + 6 = 0$$
$$x^2 - 3x - 2x + 6 = 0$$
$$(x - 2)(x - 3) = 0 \quad \text{[Factorize]}$$

Now, $x - 2 = 0$ or $x - 3 = 0$

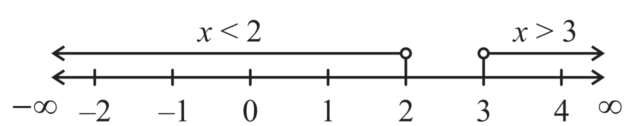
So, the solutions are $x = 2$ and $x = 3$.

Here, critical points (2, 0) and (3, 0).

Step II **Determine intervals**

The critical points divide the number line into three intervals.

Interval 1: $x < 2$
Interval 2: $2 < x < 3$
Interval 3: $x > 3$



Step III **Test points in each interval.**

Choose a test point in each interval and substitute it into the inequality.

For $x = 1: (1-2)(1-3) = 2 > 0$.

So, this interval satisfies the inequality.

For $x = 2.5: (2.5 - 2)\,(2.5 - 3) = -\,0.25 < 0$.

So, this interval does not satisfy the inequality.

For $x = 4: (4 - 2)\,(4 - 3) = 2 > 0$.

So, this interval satisfies the inequality.

Step IV **Conclusion:**

The solution of this inequality $x^2 - 5x + 6 > 0$ is $x \in (-\infty, 2) \cup (3, \infty)$

Example 17 Solve the inequality $2x^2 - 5x + 2 \leq 0$.

Solution $2x^2 - 5x + 2 \leq 0$

Step I **Solve the corresponding or associated equation.**

$$\begin{aligned} 2x^2 - 5x + 2 &= 0 \\ 2x^2 - 4x - x + 2 &= 0 \\ 2x(x-2) - 1(x-2) &= 0 \\ (2x-1)\,(x-2) &= 0 \end{aligned}$$

Now, $2x - 1 = 0$ or $x - 2 = 0$

So, the solutions are $x = \dfrac{1}{2}$ and $x = 2$

Here, critical points are $\left(\dfrac{1}{2}, 0\right)$ and $(2, 0)$

> **Note**
> For solution, we use open interval (a, b) when equation has $<$ or $>$ sign and closed interval $[a, b]$ when equation has $\leq$ or $\geq$ sign.

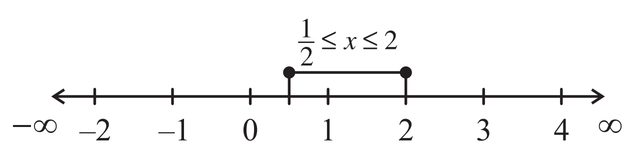
Step II **Determine Intervals**

The critical points $\left(\dfrac{1}{2}, 0\right)$, $(2, 0)$ divides the number line into three intervals.

Interval 1: $x < \dfrac{1}{2}$

Interval 2: $\dfrac{1}{2} < x < 2$

Interval 3: $x > 2$



Step III **Test points in each interval.**

For $x = 0$: $(2(0) - 1)(0 - 2) = (-1)(-2) = 2 > 0$,

So, this interval does not satisfy the inequality.

For $x = 1$: $(2(1) - 1)(1 - 2) = (1)(-1) = -1 < 0$,

So, this interval satisfies the inequality.
For $x = 3$: $(2(3) - 1)(3 - 2) = (6 - 1)(1) = 5 > 0$,
So, this interval does not satisfy the inequality.

Step IV **Conclusion:**

The solution of the inequality is: $x \in \left[\frac{1}{2}, 2\right]$.

EXERCISE 2.5

1. Solve the following inequalities:

(i) $x^2 + 3x - 4 > 0$ (ii) $2x^2 - 8x + 6 > 0$

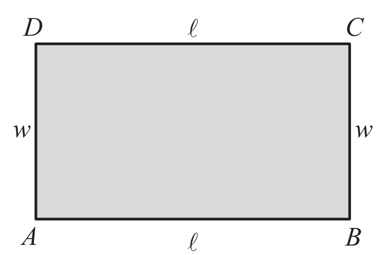
(iii) $x^2 + x - 6 < 0$ (iv) $x^2 - 6x + 9 < 0$

(v) $4x^2 - 16x + 15 \leq 0$ (vi) $-x^2 + 3x - 2 \geq 0$

2.6 Changing the Subject of Formula

A formula expresses the precise relationship between various measurable quantities in a problem or situation in an equation form.

The figure on the right is of a rectangle. Its length is ℓ units and width is w units. Let P be its perimeter.



Now, $P = \ell + w + \ell + w$
$= 2\ell + 2w$
$P = 2(\ell + w)$ units

We have constructed a formula for perimeter (P) of a rectangle. In the above formula $P = 2(\ell + w)$, the letter P on the left-hand-side is defined as the **subject** of the formula, while the letters on the right-hand-side are called terms of the formula.

Perimeter (P) of square of length (ℓ) is $P = 4\ell$.

We change the subject P and rewrite this formula as $\ell = \frac{1}{4}P$, ℓ has become subject of the formula. If value of P is given, then we can directly find the value of ℓ.

Example 18 The area of a circle is $A = \pi r^2$. Make r the subject of the formula.

Solution

$$A = \pi r^2$$

or $\quad \pi r^2 = A$

$$r^2 = \frac{A}{\pi}$$

$\therefore \quad r = \sqrt{\frac{A}{\pi}}$ [r is always positive]

EXERCISE 2.6

1. Make F the subject of the formula, $C^o = \frac{5}{9}(F^o - 32)$.
2. The formula for finding simple interest is $I = PRT$.
   (a) Make P the subject of the formula.
   (b) Make T the subject of the formula.
3. Make 'a' the subject of the formula $S = 2a + (n - 1)\,d$.
4. The volume of a cylinder is given by the formula, $V = \pi r^2 h$. Make 'h' the subject of the formula.
5. The area of a trapezoid is $A = \frac{1}{2}\,h\,(b_1 + b_2)$, make h the subject of the formula.
6. If $y = mx + c$, then make 'x' the subject of this equation.
7. Perimeter (P) of a rectangle is $P = 2\,(\ell + w)$, make ℓ as the subject of this formula.
8. The equation of a parabola is $y^2 = 4ax$, make 'x' as a subject of this equation.

9. If $P = S - C$, where S is selling price and C is cost price. Make S as subject of the equation.

10. Volume of the cone is $V = \frac{1}{3}\pi r^2 h$, make 'h' as subject of this formula.

2.7 Applications of Quadratic Equation in Real World

Example 19 A ball is thrown upward with an initial velocity of $20\,ms^{-1}$. Calculate the maximum height it reaches above ground level. Calculate time of return to the ground.

Solution **Maximum height**

At its highest point, the ball's velocity is zero. So, we have the following quantities:

Initial velocity = $v_i = 20\ ms^{-1}$

Final velocity = $v_f = 0\ ms^{-1}$

Gravitational acceleration = $g = -10\,m\ s^{-2}$

Distance (displacement) = s = ?

The relevant equation of motion is $v_f{}^2 = v_i^2 + 2gs$

Substituting values of v_i, v_f and g, we get

$$0^2 = (20)^2 + 2(-10)\,s$$

$$0 = 400 - 20s$$

so, $$s = \frac{400}{20} = 20m$$

Maximum Time of Return: When the ball returns to the point from which it was thrown, its displacement (s) is zero. We have the following information:

Displacement = s = 0

Initial velocity = $v_i = 20ms^{-1}$

Gravitational acceleration = $g = -10\ ms^{-2}$

Time = t = ?

The relevant equation of motion:

$$s = v_i\, t + \frac{1}{2}gt^2$$

Putting the values of s, v_i and g

Note

The equations of motion describe the relationship between displacement, velocity, time and acceleration for objects moving in a straight line with uniform acceleration.

First equation : $v_f = v_i + at$

Second equation : $s = v_i t + \frac{1}{2}at^2$

Third equation : $v_f^2 = v_i^2 + 2as$

$$0 = 20t + \frac{1}{2}(-10)t^2$$

$$0 = 20t - 5t^2$$

$$0 = t[20 - 5t]$$

Now $\quad t = 0$ sec $\quad$ or $\quad 20 - 5t = 0 \quad \Rightarrow \quad t = \frac{20}{5} = 4$ sec

(i) $t = 0$ sec. means, the ball had zero displacement at the time it was thrown.

(ii) $t = 4$ sec. means, the ball returned to the ground after 4 sec, it is the return time or the required time.

Example 20 A company models its profit P(rupees in thousands) by the equation $P(x) = -2x^2 + 80x - 500$, where x is the price per item (in rupees). Find the price that gives the maximum profit and calculate the maximum profit.

Solution $\quad P(x) = -2x^2 + 80x - 500$

Since $-2 < 0$, the parabola opens downward and vertex gives the maximum point.

$$x = \frac{-b}{2a} = \frac{-80}{2(-2)} = \frac{-80}{-4} = 20$$

Hence, the price per item that gives maximum profit is Rs. 20.

$$P(20) = -2(20)^2 + 80(20) - 500$$
$$= -2(400) + 1600 - 500$$
$$= 300 \quad \text{(in thousands)}$$

Hence, maximum profit is Rs. 300,000.

> **Note**
> We can also find vertex by using $x = \frac{-b}{2a}$, $y = f\left(\frac{-b}{2a}\right)$

Example 21 A ball is thrown upwards from the ground. Its height after t seconds is modeled by:

$$h(t) = -5t^2 + 20t \qquad \text{(in metres)}$$

For what time interval is the ball at least 15 metres above the ground?

Solution $\quad$ Here, $\quad -5t^2 + 20t \geq 15$

$$-5t^2 + 20t - 15 \geq 0$$
$$5t^2 - 20t + 15 \leq 0$$

By solving the quadratic equation $5t^2 - 20t + 15 = 0$.

$$t = \frac{-(-20) \pm \sqrt{(-20)^2 - 4(5)(15)}}{2(5)}$$

$$= \frac{20 \pm \sqrt{400 - 300}}{10}$$

$$= \frac{20 \pm \sqrt{100}}{10}$$

$$= \frac{20 \pm 10}{10}$$

$$= \frac{20 - 10}{10}, \frac{20 + 10}{10}$$

$$= 1, 3$$

So, the ball is at least 15 metres high when $1 \le t \le 3$ seconds.

EXERCISE 2.7

1. A town's population is modeled by $P(t) = -2t^2 + 40t + 800$, where t is years since 2020. Find the years when the population will be at least 1000.

2. A company models its profit P in thousands of rupees by the equation: $P(x) = -5x^2 + 150x - 1000$, where x is the price per item in rupees. Find the price that gives maximum profit.

3. A toy car rolls down an incline and covers a distance given by the equation $d = t^2 - 0.5t$ metres, where t is the time in seconds. Find the time when the car has travelled a distance 12.5 metres.

4. A ball's height (in metres) after t seconds is $h(t) = -4t^2 + 24t$. For what time interval is the ball at least $20m$ above the ground?

5. A ball is thrown upward with an initial velocity of $40\,\text{ms}^{-1}$. Calculate the maximum height it reaches above ground level.

6. A freelancer's earnings follow the model $E(h) = -2h^2 + 40h$, where E is earning in rupees and h is hours worked per week. What is the maximum number of hours he should work to maximize earnings?

REVIEW EXERCISE 2

1. Four possible answers are given for the following questions. Choose the correct answer:

(i) The type of the equation $2x^2 - x + 1 = 0$ is:
(a) quadratic (b) linear (c) third degree (d) Pure quadratic

(ii) What is the discriminant of $x^2 + 5x - 5 = 0$?
(a) –20 (b) 20 (c) 25 (d) 45

(iii) The solution set of $3x^2 - 9 = 0$ is:
(a) $\{3\}$ (b) $\{\pm 3\}$ (c) $\{\pm\sqrt{3}\}$ (d) $\{\sqrt{3}\}$

(iv) Sum of the roots of $3x^2 + 5x - 12 = 0$ is:
(a) $\frac{5}{3}$ (b) $-\frac{5}{3}$ (c) $\frac{12}{3}$ (d) $\frac{3}{5}$

(v) Product of the roots of $3x^2 + 5x - 12 = 0$ is:
(a) −4 (b) 3 (c) 4 (d) 5

(vi) What are the roots of $(x - 3)(x + 3) = 0$?
(a) 3, –3 (b) 3, 3 (c) –3, –3 (d) 9, 0

(vii) 3 and 2 are the roots of:
(a) $x^2 + 5x + 6 = 0$ (b) $x^2 + 6x + 5 = 0$
(c) $x^2 - 5x + 6 = 0$ (d) $x^2 + 6x - 5 = 0$

(viii) If $b^2 - 4ac > 0$ and is a perfect square, then the roots of $ax^2 + bx + c = 0$ are:
(a) equal (b) unequal (c) imaginary (d) irrational

(ix) If $b^2 - 4ac = 0$, then the roots of $ax^2 + bx + c = 0$ are:
(a) unequal (b) irrational (c) imaginary (d) equal

(x) Subject "c" of $x - 2c = b$ is:
(a) $x + b$ (b) $b - x$ (c) $\frac{x-b}{2}$ (d) $\frac{b-x}{2}$

2. Solve the following quadratic equations by factorization method, by completing square method and by quadratic formula:
(i) $8x^2 = x + 7$ (ii) $2x^2 - x - 10 = 0$

3. Form a quadratic equation whose roots are; 6, $\frac{3}{2}$.

4. Examine the nature of the roots of the following equations:
(i) $15x^2 + 11x + 2 = 0$ (ii) $x^2 - x - 1 = 0$

5. If a ball is thrown upward with a velocity v, the maximum height it reaches can be determined by using a formula $h = \frac{v^2}{2g}$. Rearrange the formula to make v the subject.

6. If the equation $x^2 + 2(1 + k)x + k^2 = 0$ has equal roots, then find the value of k.

UNIT 3

Matrices and Determinants

Students' Learning Outcomes

After completing this unit, the students will be able to:

- Display information in the form of matrix of order 2.
- Solve situations involving sum, difference and product of two matrices.
- Calculate the product of the scalar quantity and a matrix.
- Evaluate the determinant and inverse of a matrix of order 2-by-2.
- Solve the simultaneous linear equations in two variables using matrix inversion method and Cramer's rule.
- Explain, with examples, how mathematics plays a key role in the development of new scientific theories and technologies. (e.g., Mathematical models and simulations are used to design and optimize new materials and drugs, and to understand the behaviours of complex systems such as the human brain).
- Apply concepts of matrices to real world problems (such as engineering, economics, computer graphics and physics).

INTRODUCTION

The evolution of the theory of 'matrices' is the result of attempts to obtain compact and simple methods for solving system of linear equations. In 1850, it was **James Joseph Sylvester** an English Mathematician and lawyer who used the word matrix. In 1858, Arthur Cayley published **Memoir on the theory of matrices** which was remarkable for containing the first abstract definition of a matrix. The mathematicians James Joseph Sylvester (1814-1897), William Rowan Hamilton (1805-1865) and **Arthur Cayley** (1821-1895) played important role in the development of matrix theory. English Mathematician **Cullis** was the first to use modern brackets notation for matrices in 1913. The knowledge of matrices is absolutely necessary not only within the branches of Mathematics but also in other areas of science, genetics, economics, sociology, modern psychology and industrial management.

3.1 Matrices

A matrix is a rectangular array in shape, whose elements (entries) are written within square brackets in a specific order, in rows and columns. The arrays obey certain algebraic operations.

Generally, the matrices are denoted by the capital letters of the English alphabets while their elements are denoted by the small letters.

e.g $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = [5 \quad 7]$, $C = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$D = \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}$, $E = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

The time table in schools, flight schedule at air-port or arrival and departure time of trains at the railway station are written in the matrices form.

3.1.1 Rows and Columns of a Matrix

Row: Entry or entries written in the horizontal line or lines form the row or rows of a matrix.

Column: Entry or entries written in vertical line or lines form the column or columns of a matrix.

For example,

	Column 1	Column 2	Column 3	Column 4	
	↓	↓	↓	↓	

$$A = \begin{bmatrix} 2 & 3 & 7 & 1 \\ 1 & 5 & 3 & 2 \\ 9 & 1 & 8 & 4 \end{bmatrix} \begin{matrix} \leftarrow \text{Row 1} \\ \leftarrow \text{Row 2} \\ \leftarrow \text{Row 3} \end{matrix}$$

Matrix A has 3 rows and 4 columns.

	Column 1	Column 2	Column 3	
	↓	↓	↓	

$$B = \begin{bmatrix} 7 & 4 & 8 \\ 3 & 0 & 10 \\ 2 & 1 & 11 \end{bmatrix} \begin{matrix} \leftarrow \text{Row 1} \\ \leftarrow \text{Row 2} \\ \leftarrow \text{Row 3} \end{matrix}$$

Matrix B has 3 rows and 3 columns.

$$C = [\,5\,]$$

Matrix C has 1 row and 1 column.

3.1.2 Elements or Entries

If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, then 1, 2, 3, 4, 5, 6 are called elements or entries of matrix A.

3.1.3 Order or Size of a Matrix

If a matrix A has 'm' number of rows and 'n' number of columns, then the order or size of the matrix A is m-by-n. Order of matrix $B=\begin{bmatrix}5 & -1 & 0\\ 9 & 2 & 3\\ 0 & 1 & 2\end{bmatrix}$ is 3-by-3. If order of a matrix is 3-by-4, it means it has 3 rows and 4 columns. It does not mean 3 by 4 = 12, however 12 gives the number of elements in the matrix.

3.1.4 Equal Matrices

Two matrices A and B are said to be equal if and only if they are of same order and their corresponding elements are same.

For example, $A=\begin{bmatrix}2 & 3\\ 4 & -2\end{bmatrix}$, $B=\begin{bmatrix}4-2 & 3\\ 4 & 4-6\end{bmatrix}$ are equal matrices.

Similarly, $C=\begin{bmatrix}a & b\end{bmatrix}$, $D=\begin{bmatrix}3 & 4\end{bmatrix}$ will be equal matrices if and only if $a=3$ and $b=4$.

EXERCISE 3.1

1. Write the number of rows and number of columns in each matrix.

$A=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$, $B=\begin{bmatrix}3\\ 4\\ 7\end{bmatrix}$, $C=\begin{bmatrix}8 & -10 & 11\end{bmatrix}$,

$D=\begin{bmatrix}3 & 2 & 1\\ 1 & 0 & 3\\ 4 & 5 & 6\end{bmatrix}$, $E=\begin{bmatrix}5 & 9 & -2\\ -3 & 4 & 5\end{bmatrix}$

2. Write the order of each matrix.

$A=\begin{bmatrix}1\\ 0\end{bmatrix}$, $B=\begin{bmatrix}3 & 4\end{bmatrix}$, $C=\begin{bmatrix}3 & 5\\ 5 & 6\end{bmatrix}$, $D=\begin{bmatrix}0 & 3 & 4\\ 5 & 0 & 9\end{bmatrix}$, $E=\begin{bmatrix}0 & 1\\ 3 & 4\\ 5 & 6\end{bmatrix}$, $F=\begin{bmatrix}5\end{bmatrix}$

3. Which of the following matrices are equal?

$A=\begin{bmatrix}0 & 3\\ 4 & 1\end{bmatrix}$, $B=\begin{bmatrix}9\\ -7\end{bmatrix}$, $C=\begin{bmatrix}2\times 3 & 2-1\\ 2\times 2 & 4-2\\ 4+4 & 3+0\end{bmatrix}$,

$D=\begin{bmatrix}5+4\\ -8+1\end{bmatrix}$, $E=\begin{bmatrix}6 & 1\\ 4 & 2\\ 8 & 3\end{bmatrix}$, $F=\begin{bmatrix}3-3 & 3\\ 3+1 & 1\end{bmatrix}$

4. If $\begin{bmatrix}a+2 & c-3\\ b-1 & d+4\end{bmatrix}=\begin{bmatrix}5 & 8\\ 6 & 4\end{bmatrix}$, then find the values of a, b, c and d.

5. If $\begin{bmatrix} 2x+1 & 4 \\ 7 & 5 \end{bmatrix} = \begin{bmatrix} 9 & 4 \\ 7 & y \end{bmatrix}$, then find the values of x and y.

6. If $\begin{bmatrix} a+b & 2d-1 \\ 3b+2 & 4 \end{bmatrix} = \begin{bmatrix} 10 & 5 \\ 11 & c \end{bmatrix}$, then find the values of a, b, c and d.

7. If $\begin{bmatrix} p+q & 5 \\ 11 & p-2q \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 11 & 0 \end{bmatrix}$, then find the values of p and q.

3.2 Types of Matrices

(i) **Row Matrix:** A matrix having only one row is called a row matrix.

For example: $A = [0\ \ 1\ \ 4]$, $B = [a\ \ b]$, $C = [a]$ are row matrices of order 1-by-3, 1-by-2 and 1-by-1 respectively.

(ii) **Column Matrix:** A matrix having only one column is called a column matrix.

For example: $A = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$, $B = \begin{bmatrix} a \\ b \end{bmatrix}$, $C = [2]$ are column matrices of order 3-by-1, 2-by-1 and 1-by-1 respectively.

(iii) **Zero matrix or Null matrix**

If all the entries in a matrix are zero, it is called zero matrix or null matrix. It is represented by O.

For example: $[0], \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are zero matrices of order 1-by-1, 2-by-2 and 2-by-3 respectively.

(iv) **Square Matrix:** A matrix in which number of rows is equal to the number of columns is called a square matrix. A matrix of order n-by-n is often referred to as a square matrix of order n.

For example: $A = \begin{bmatrix} a & b & c \\ d & c & f \\ g & h & l \end{bmatrix}$ is a square matrix of order 3.

In a square matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

a_{11}, a_{22}, a_{33} are called entries of principal diagonal or main diagonal or leading diagonal.

(v) Rectangular Matrix: If the number of rows and number of columns in a matrix are not equal, then it is called a rectangular matrix.

For example: $A = \begin{bmatrix} a & b \end{bmatrix}$, $B = \begin{bmatrix} a \\ b \end{bmatrix}$, $C = \begin{bmatrix} 4 & 0 & 7 \\ 8 & 3 & 1 \end{bmatrix}$ are rectangular matrices.

(vi) Diagonal Matrix: A square matrix is called a diagonal matrix if at least any one of the elements of its main diagonal is non-zero and non-diagonal elements are zero.

For example: $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ are diagonal matrices of order 3-by-3 and 2-by-2 respectively.

(vii) Scalar Matrix: A diagonal matrix is called a scalar matrix if all the entries in the main diagonal are the same and non-zero.

For example: $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$, $C = \begin{bmatrix} \sqrt{3} \end{bmatrix}$ are scalar matrices of order 3-by-3, 2-by-2 and 1-by-1 respectively.

(viii) Identity Matrix or Unit Matrix: A scalar matrix is called an identity or unit matrix if all of its main diagonal entries are 1.

For example: $P = \begin{bmatrix} 1 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are identity matrices of order 1-by-1, 2-by-2 and 3-by-3 respectively. We represent these matrices as:

$$I_1 = \begin{bmatrix} 1 \end{bmatrix}, \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Remember!

A scalar matrix and identity matrix are also diagonal matrices but all diagonal matrices are not scalar or identity matrices.

(ix) Transpose of a Matrix: A matrix obtained from a given matrix by interchanging the rows and columns is called the transpose of the given matrix. If A is a matrix, then its transpose is represented as A'.

For example: $A = \begin{bmatrix} 1 & 2 & 4 \\ 8 & 0 & 5 \end{bmatrix} \Rightarrow A' = \begin{bmatrix} 1 & 8 \\ 2 & 0 \\ 4 & 5 \end{bmatrix}$

(x) **Symmetric Matrix:** A square matrix A is called symmetric if it is equal to its transpose. i.e., $A^t = A$.

For example: If $A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$, then $A^t = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} = A$

Thus, A is a symmetric matrix.

(xi) **Skew-Symmetric Matrix:** A square matrix A is called skew-symmetric if $A^t = -A$.

For example: If $B = \begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix}$, then $B^t = \begin{bmatrix} 0 & 5 \\ -5 & 0 \end{bmatrix} = -B$

Thus, B is a skew-symmetric matrix.

(xii) **Negative of a Matrix:** The negative of a matrix A is obtained by multiplying each of its elements by –1. It is denoted by $-A$.

For example: If $A = [3 \quad -6]$, then

$-A = [-3 \quad 6]$

EXERCISE 3.2

1. From the following matrices identify unit matrices, row matrices, column matrices and null matrices.

$A = [5 \quad 7 \quad 8]$, $B = [0]$, $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

$D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $E = \begin{bmatrix} 6 \\ 0 \\ 8 \end{bmatrix}$, $F = \begin{bmatrix} 7 \\ 1 \\ 9 \end{bmatrix}$

2. Identify type of the given matrices as row, column, square and rectangular matrices.

$A = \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 6 \\ 4 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 5 & 8 \\ 0 & 4 & -2 \end{bmatrix}$, $D = \begin{bmatrix} 5 & -5 \\ 2 & 7 \end{bmatrix}$,

$E = \begin{bmatrix} 3 & 2 \\ 4 & 1 \\ 5 & 0 \end{bmatrix}$, $F = [5 \quad -3 \quad 7]$, $G = \begin{bmatrix} 3 & 0 & 1 \\ 1 & 3 & 4 \\ 5 & 2 & -3 \end{bmatrix}$, $H = \begin{bmatrix} 3 & 5 \\ 4 & 4 \\ 5 & 2 \end{bmatrix}$

3. Identify diagonal, scalar and unit matrices.

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$,

$D = \begin{bmatrix} 7 & 0 \\ 0 & 6 \end{bmatrix}$, $E = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

4. Find transpose of each of the following matrices:

$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 3 \\ 7 \\ 6 \end{bmatrix}$, $C = \begin{bmatrix} 5 & -2 & 4 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 5 \\ 3 & 6 \\ 4 & 7 \end{bmatrix}$

5. Find negative of the following matrices:

$A = \begin{bmatrix} -3 & 0 \\ 5 & 6 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 3 \\ -2 & 2 \end{bmatrix}$, $C = \begin{bmatrix} -9 & 1 \\ 1 & -7 \end{bmatrix}$

6. If $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 6 \\ 5 & 8 \end{bmatrix}$, then verify that

(i) $(A^t)^t = A$ (ii) $(B^t)^t = B$

7. Show that $L = \begin{bmatrix} 2 & 3 & 4 \\ 3 & -2 & 5 \\ 4 & 5 & 0 \end{bmatrix}$ is a symmetric matrix.

3.3 Algebraic Operations on Matrices

Basic operations on matrices are:

(i) Multiplication of a matrix by a scalar.

(ii) Addition / subtraction of two matrices.

(iii) Multiplication of two matrices.

There is no concept of dividing a matrix by another matrix.

3.3.1 Multiplication of a Matrix by a Scalar

For a given matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$ and a scalar k, we define a new matrix kA

as $kA = \begin{bmatrix} ka & kb & kc \\ kd & ke & kf \end{bmatrix}$, that is, each element of A has been multiplied by a scalar

k. In particular if $k = -1$, then $-A = \begin{bmatrix} -a & -b & -c \\ -d & -e & -f \end{bmatrix}$.

3.3.2 Addition of Matrices

If A and B are two matrices of the **same order,** these can be added, then their sum denoted by $A + B$, is a matrix of the **same order**. They are said to be **conformable for addition.** $A + B$ is obtained by adding the **corresponding entries** of A and B.

Example 1 If $A = \begin{bmatrix} 3 & 8 & 2 \\ 5 & 3 & 1 \\ 6 & 0 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 1 & -2 \\ 3 & 0 & 6 \\ 2 & 3 & 4 \end{bmatrix}$, then find $A + B$ and $B + A$.

Solution Given that $A = \begin{bmatrix} 3 & 8 & 2 \\ 5 & 3 & 1 \\ 6 & 0 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 1 & -2 \\ 3 & 0 & 6 \\ 2 & 3 & 4 \end{bmatrix}$

Order of A = 3-by-3 and order of B = 3-by-3. A and B are conformable for addition

Now, $$A + B = \begin{bmatrix} 3+4 & 8+1 & 2-2 \\ 5+3 & 3+0 & 1+6 \\ 6+2 & 0+3 & 7+4 \end{bmatrix} = \begin{bmatrix} 7 & 9 & 0 \\ 8 & 3 & 7 \\ 8 & 3 & 11 \end{bmatrix}$$

and $$B + A = \begin{bmatrix} 4+3 & 1+8 & -2+2 \\ 3+5 & 0+3 & 6+1 \\ 2+6 & 3+0 & 4+7 \end{bmatrix} = \begin{bmatrix} 7 & 9 & 0 \\ 8 & 3 & 7 \\ 8 & 3 & 11 \end{bmatrix}$$

3.3.3 Subtraction of Matrices

If A and B are two matrices of the **same order,** they can be subtracted and their difference is denoted as $A - B$ or $B - A$, it is again a matrix of the **same order.** They are said to be **conformable for subtraction.** $A - B$ obtained by subtracting the corresponding entries of B from A.

Example 2 If $A = \begin{bmatrix} 5 & 9 \\ 4 & 8 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 6 \\ 2 & 1 \end{bmatrix}$, then find $A - B$ and $B - A$.

Solution $A = \begin{bmatrix} 5 & 9 \\ 4 & 8 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 6 \\ 2 & 1 \end{bmatrix}$

Now, $$A - B = \begin{bmatrix} 5-3 & 9-6 \\ 4-2 & 8-1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 2 & 7 \end{bmatrix}$$

and $$B - A = \begin{bmatrix} 3-5 & 6-9 \\ 2-4 & 1-8 \end{bmatrix} = \begin{bmatrix} -2 & -3 \\ -2 & -7 \end{bmatrix}$$

We see $A - B \neq B - A$

Example 3 If $A=\begin{bmatrix}2 & 4\\1 & 5\end{bmatrix}$, $B=\begin{bmatrix}3 & 4 & 1\\5 & 2 & 7\end{bmatrix}$, then find $A+B$ and $B-A$, if possible.

Solution $A=\begin{bmatrix}2 & 4\\1 & 5\end{bmatrix}$, $B=\begin{bmatrix}3 & 4 & 1\\5 & 2 & 7\end{bmatrix}$

Order of A is 2-by-2 and order of B is 2-by-3.Therefore, A and B are not conformable for addition and subtraction.

3.3.4 Commutative and Associative Laws of Addition of Matrices

a Commutative Law of Addition of Matrices

For any two matrices A and B of same order $A + B = B + A$. This law is called commutative law of matrices with respect to addition.

Let $A=\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix}$, $B=\begin{bmatrix}3 & 1 & 3\\4 & 2 & 2\\0 & -1 & 0\end{bmatrix}$

Then $A+B=\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix}+\begin{bmatrix}3 & 1 & 3\\4 & 2 & 2\\0 & -1 & 0\end{bmatrix}$

$$=\begin{bmatrix}1+3 & 2+1 & 3+3\\4+4 & 5+2 & 6+2\\7+0 & 8-1 & 9+0\end{bmatrix}=\begin{bmatrix}4 & 3 & 6\\8 & 7 & 8\\7 & 7 & 9\end{bmatrix} \quad ...(i)$$

and $B+A=\begin{bmatrix}3 & 1 & 3\\4 & 2 & 2\\0 & -1 & 0\end{bmatrix}+\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix}$

$$=\begin{bmatrix}3+1 & 1+2 & 3+3\\4+4 & 2+5 & 2+6\\0+7 & -1+8 & 0+9\end{bmatrix}=\begin{bmatrix}4 & 3 & 6\\8 & 7 & 8\\7 & 7 & 9\end{bmatrix} \quad ...(ii)$$

Hence, from (i) and (ii) $A + B = B + A$
Commutative law of addition of matrices holds.

b Associative Law of Addition of Matrices

For any three matrices A, B and C of same order $(A + B) + C = A + (B + C)$. This law is called associative law of matrices with respect to addition.

Let $A=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}$, $B=\begin{bmatrix}3&1&3\\4&2&2\\0&-1&0\end{bmatrix}$, $C=\begin{bmatrix}0&2&0\\3&0&3\\1&1&1\end{bmatrix}$

then $(A+B)+C=\left(\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}+\begin{bmatrix}3&1&3\\4&2&2\\0&-1&0\end{bmatrix}\right)+\begin{bmatrix}0&2&0\\3&0&3\\1&1&1\end{bmatrix}$

$$=\begin{bmatrix}1+3&2+1&3+3\\4+4&5+2&6+2\\7+0&8-1&9+0\end{bmatrix}+\begin{bmatrix}0&2&0\\3&0&3\\1&1&1\end{bmatrix}$$

$$=\begin{bmatrix}4&3&6\\8&7&8\\7&7&9\end{bmatrix}+\begin{bmatrix}0&2&0\\3&0&3\\1&1&1\end{bmatrix}=\begin{bmatrix}4&5&6\\11&7&11\\8&8&10\end{bmatrix} \quad \text{...(i)}$$

and $A+(B+C)=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}+\left(\begin{bmatrix}3&1&3\\4&2&2\\0&-1&0\end{bmatrix}+\begin{bmatrix}0&2&0\\3&0&3\\1&1&1\end{bmatrix}\right)$

$$=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}+\begin{bmatrix}3+0&1+2&3+0\\4+3&2+0&2+3\\0+1&-1+1&0+1\end{bmatrix}$$

$$=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}+\begin{bmatrix}3&3&3\\7&2&5\\1&0&1\end{bmatrix}=\begin{bmatrix}4&5&6\\11&7&11\\8&8&10\end{bmatrix} \quad \text{...(ii)}$$

Hence, from (i) and (ii) $(A+B)+C=A+(B+C)$

Associative law of addition of matrices holds.

3.3.5 Additive Identity of a Matrix

The additive identity of a matrix is a matrix that, when added to any matrix A of the same order, leaves it unchanged.

Mathematically, $A+O=A=O+A$, where O is the zero matrix (also called the **additive identity matrix**).

Let $A=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}$, $O=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$

Then $A+O=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}+\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}=A$

and $O+A=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}+\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}=A$

3.3.6 Additive Inverse of a Matrix

The additive inverse of a matrix A is another matrix $-A$ such that

$$A+(-A)=O=(-A)+A.$$

Let $A=\begin{bmatrix}-2&1&3\\4&2&0\\-3&0&5\end{bmatrix}$, then $-A=\begin{bmatrix}2&-1&-3\\-4&-2&0\\3&0&-5\end{bmatrix}$

Now $A+(-A)=\begin{bmatrix}-2&1&3\\4&2&0\\-3&0&5\end{bmatrix}+\begin{bmatrix}2&-1&-3\\-4&-2&0\\3&0&-5\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}=O$

$(-A)+A=\begin{bmatrix}2&-1&-3\\-4&-2&0\\3&0&-5\end{bmatrix}+\begin{bmatrix}-2&1&3\\4&2&0\\-3&0&5\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}=O$

Thus, A and $-A$ are additive inverse of each other.

EXERCISE 3.3

1. Which of the following matrices are conformable for addition and subtraction?

   $A=\begin{bmatrix}3&4\\-2&1\end{bmatrix}$, $B=\begin{bmatrix}3\\4\end{bmatrix}$, $C=\begin{bmatrix}5&2\end{bmatrix}$, $D=\begin{bmatrix}1&7\\2&5\end{bmatrix}$, $E=\begin{bmatrix}2\end{bmatrix}$,

   $F=\begin{bmatrix}7&11\end{bmatrix}$, $G=\begin{bmatrix}a\\b\end{bmatrix}$, $H=\begin{bmatrix}3\end{bmatrix}$, $M=\begin{bmatrix}l\\m\end{bmatrix}$

2. If $X=\begin{bmatrix}1&-1\\-2&2\end{bmatrix}$, $Y=\begin{bmatrix}1&2\\3&4\end{bmatrix}$ and $Z=\begin{bmatrix}3&0\\0&-2\end{bmatrix}$, then find the following:

   (i) $X+Y$ (ii) $Y+7Z$ (iii) $4X-Z$

(iv) $X + 2Y + 3Z$ (v) $X - 4Y + Z$ (vi) $Z - Z$

3. Find the additive inverse of the following matrices:

(i) $P = \begin{bmatrix} 5 \\ -7 \end{bmatrix}$ (ii) $Q = \begin{bmatrix} 9 & -3 \end{bmatrix}$ (iii) $R = \begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix}$ (iv) $S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

4. If $A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & -2 \\ 0 & 5 \end{bmatrix}$, then verify the following:

(i) $A + B = B + A$ (ii) $(A + B) + C = A + (B + C)$

(iii) $(2A + B) + C = 2A + (B + C)$ (iv) $3(A + B) = 3A + 3B$

5. If $A = \begin{bmatrix} 5 & 6 \\ 7 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & -6 \\ -7 & 2 \end{bmatrix}$, then show that B is additive inverse of A and A is additive inverse of B.

6. If $A = \begin{bmatrix} 6 & -2 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 8 & -1 \\ 3 & 0 \end{bmatrix}$, then verify that

(i) $(A + B)^t = A^t + B^t$ (ii) $(A - B)^t = A^t - B^t$

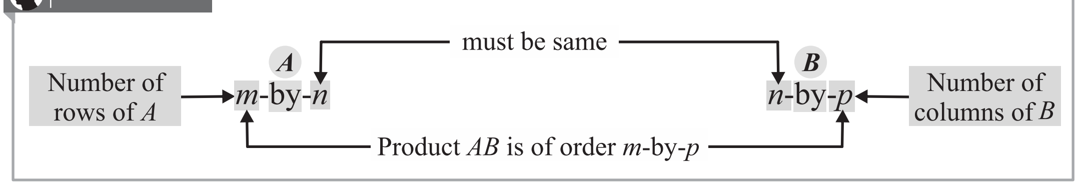
3.4 Multiplication of Matrices

A matrix A is said to be conformable for multiplication with a matrix B if the number of columns of A is equal to the number of rows of B.

Thus, if A and B are given two matrices of order m-by-n and n-by-p respectively, then the product of matrices A and B is denoted by AB and its order will be m-by-p.

The order of AB is m-by-p = number of rows of A by number of columns of B.

Remember!



Let us explain the concept of multiplication of matrices with the help of an example from our daily life.

Talmeez purchased 7 books and 6 pencils at the rate of Rs. 20 and Rs. 3 per article respectively from shop No. 1 and he purchased same number of books and pencils at the rate of Rs. 22 and Rs. 4 per article respectively from Shop No. 2. Majid purchased 5 books and 8 pencils from Shop No. 1 at the same rate Talmeez purchased. Majid purchased same number of books and pecils from Shop No. 2 at the same rate Talmeez purchased.

We show their purchase in the form of tables.

Table 1

	Books	Pencils
Talmeez	7	6
Majid	5	8

Table 2

	Shop No. 1	Shop No. 2
Rate per book	20	22
Rate per pencil	3	4

Amount paid by Talmeez at Shop No. 1.

$7 \times 20 + 6 \times 3 = 158$ rupees ...(i)

Amount paid by Talmeez at Shop No. 2

$7 \times 22 + 6 \times 4 = 178$ rupees ...(ii)

Amount paid by Majid at shop No. 1

$5 \times 20 + 8 \times 3 = 124$ rupees ...(iii)

Amount paid by Majid at shop No. 2

$5 \times 22 + 8 \times 4 = 142$ rupees ...(iv)

We show entries of the two tables in the form of matrices.

Table 1 $\begin{bmatrix} 7 & 6 \\ 5 & 8 \end{bmatrix}$ **Table 2** $\begin{bmatrix} 20 & 22 \\ 3 & 4 \end{bmatrix}$

Calculations given above is shown in the form of matrix.

$$\begin{bmatrix} 7 \times 20 + 6 \times 3 & 7 \times 22 + 6 \times 4 \\ 5 \times 20 + 8 \times 3 & 5 \times 22 + 8 \times 4 \end{bmatrix} \text{ see (i), (ii), (iii) and (iv)}$$

$$= \begin{bmatrix} 158 & 178 \\ 124 & 142 \end{bmatrix}$$

For example, $A = \begin{bmatrix} 9 & 7 \\ 4 & 3 \end{bmatrix}, B = \begin{bmatrix} 2 & 6 \\ 5 & 8 \end{bmatrix}$

Now, $AB = \begin{bmatrix} 9 & 7 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 2 & 6 \\ 5 & 8 \end{bmatrix}$

$$= \begin{bmatrix} 9 \times 2 + 7 \times 5 & 9 \times 6 + 7 \times 8 \\ 4 \times 2 + 3 \times 5 & 4 \times 6 + 3 \times 8 \end{bmatrix}$$

$$AB = \begin{bmatrix} 18+35 & 54+56 \\ 8+15 & 24+24 \end{bmatrix} = \begin{bmatrix} 53 & 110 \\ 23 & 48 \end{bmatrix}$$

Remember!

For Multiplication

If A and B are two matrices and we multiply A with B, when:

(i) Number of columns in A = Number of rows in B.

(ii) Multiplication of two matrices A and B is written as AB.

(iii) To find AB, we start with the first row of A and multiply its each element with the corresponding elements of the first column of B and add the products. This gives the first element of the first row of the product matrix AB.

Next, we multiply each element of the first row of A with the corresponding elements of second column of B and at the product. This give the second element of first row of the product matrix AB.

Similarly, we multiply all the rows of A with each column of B and add the products to get remaining elements of the matrix AB.

Example 4 If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} l & m \\ n & p \end{bmatrix}$, then find:

(i) AB, if possible. **(ii)** BA, if possible.

Solution **(i)** Number of columns in matrix $A = 2$

Number of rows in matrix $B = 2$

Thus, AB is possible

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} l & m \\ n & p \end{bmatrix}$$

$$= \begin{bmatrix} al + bn & am + bp \\ cl + dn & cm + dp \end{bmatrix}$$

(ii) Number of columns in matrix $B = 2$

Number of rows in matrix $A = 2$

Thus, BA is also possible

$$BA = \begin{bmatrix} l & m \\ n & p \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} la + mc & lb + md \\ na + pc & nb + pd \end{bmatrix}$$

We note that $AB \neq BA$

Example 5 If $A=\begin{bmatrix}2 & 0\\5 & -3\end{bmatrix}$, $B=\begin{bmatrix}-2\\1\end{bmatrix}$, then find:

(i) AB, if possible. (ii) BA, if possible.

Solution (i) Number of columns in matrix $A = 2$

Number of rows in matrix $B = 2$

Thus, AB is possible

Now $AB = \begin{bmatrix}2 & 0\\5 & -3\end{bmatrix}\begin{bmatrix}-2\\1\end{bmatrix}$

$= \begin{bmatrix}2\times(-2) + 0\times 1\\5\times(-2) + (-3)\times 1\end{bmatrix}$

$= \begin{bmatrix}-4+0\\-10-3\end{bmatrix} = \begin{bmatrix}-4\\-13\end{bmatrix}$

(ii) $BA = \begin{bmatrix}-2\\1\end{bmatrix}\begin{bmatrix}2 & 0\\5 & -3\end{bmatrix}$

Number of columns in matrix $B = 1$

Number of rows in matrix $A = 2$

Thus, BA is not possible

> **Skilled Practice!**
>
> If $A = [5 \quad 3]$ and $B = \begin{bmatrix}-2\\1\end{bmatrix}$, then find AB and its order.

3.4.1 Commutative Law of Multiplication of Matrices

Consider $A = \begin{bmatrix}5 & 2\\3 & 4\end{bmatrix}$, $B = \begin{bmatrix}1 & 5\\0 & -2\end{bmatrix}$

$AB = \begin{bmatrix}5 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}1 & 5\\0 & -2\end{bmatrix}$

$= \begin{bmatrix}5\times 1 + 2\times 0 & 5\times 5 + 2\times(-2)\\3\times 1 + 4\times 0 & 3\times 5 + 4\times(-2)\end{bmatrix}$

$= \begin{bmatrix}5+0 & 25-4\\3+0 & 15-8\end{bmatrix} = \begin{bmatrix}5 & 21\\3 & 7\end{bmatrix}$ ...(i)

Now $BA = \begin{bmatrix}1 & 5\\0 & -2\end{bmatrix}\begin{bmatrix}5 & 2\\3 & 4\end{bmatrix}$

$= \begin{bmatrix}1\times 5 + 5\times 3 & 1\times 2 + 5\times 4\\0\times 5 + (-2)\times 3 & 0\times 2 + (-2)\times 4\end{bmatrix}$

$$=\begin{bmatrix} 5+15 & 2+20 \\ 0-6 & 0-8 \end{bmatrix}=\begin{bmatrix} 20 & 22 \\ -6 & -8 \end{bmatrix} \quad ...(ii)$$

From (i) and (ii), it is verified that $AB \neq BA$

Thus, commutative law under multiplication does not hold in general in the multiplication of matrices.

3.4.2 Associative Law of Multiplication of Matrices

If A, B and C are any three matrices, then $(AB)C = A(BC)$ is called associative law of multiplication of matrices.

Let $A=\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, $B=\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$, $C=\begin{bmatrix} 1 & 2 \\ 4 & 0 \end{bmatrix}$

$$AB=\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}=\begin{bmatrix} 1(2)+0(0) & 1(1)+0(3) \\ 2(2)+1(0) & 2(1)+1(3) \end{bmatrix}=\begin{bmatrix} 2+0 & 1+0 \\ 4+0 & 2+3 \end{bmatrix}=\begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix}$$

Now, $(AB)C=\begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & 0 \end{bmatrix}$

$$=\begin{bmatrix} 2(1)+1(4) & 2(2)+1(0) \\ 4(1)+5(4) & 4(2)+5(0) \end{bmatrix}=\begin{bmatrix} 2+4 & 4+0 \\ 4+20 & 8+0 \end{bmatrix}=\begin{bmatrix} 6 & 4 \\ 24 & 8 \end{bmatrix} \quad ...(i)$$

$$BC=\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & 0 \end{bmatrix}$$

$$=\begin{bmatrix} 2(1)+1(4) & 2(2)+1(0) \\ 0(1)+3(4) & 0(2)+3(0) \end{bmatrix}=\begin{bmatrix} 2+4 & 4+0 \\ 0+12 & 0+0 \end{bmatrix}=\begin{bmatrix} 6 & 4 \\ 12 & 0 \end{bmatrix}$$

$$A(BC)=\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 6 & 4 \\ 12 & 0 \end{bmatrix}$$

$$=\begin{bmatrix} 1(6)+0(12) & 1(4)+0(0) \\ 2(6)+1(12) & 2(4)+1(0) \end{bmatrix}=\begin{bmatrix} 6+0 & 4+0 \\ 12+12 & 8+0 \end{bmatrix}=\begin{bmatrix} 6 & 4 \\ 24 & 8 \end{bmatrix} \quad ...(ii)$$

Hence, from (i) and (ii)

$(AB)C = A(BC)$

Associative law of multiplication of matrices holds.

3.4.3 Distributive Laws of Multiplication Over Addition and Subtraction

(a) Let A, B and C be any three matrices, then distributive laws of multiplication over addition are as follows:

(i) $A(B+C) = AB + AC$ (Left distributive law over addition)

(ii) $(A+B)C = AC + BC$ (Right distributive law over addition)

Let $A=\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}$, $B=\begin{bmatrix}3 & 1\\ 2 & 4\end{bmatrix}$, $C=\begin{bmatrix}1 & 0\\ 5 & 2\end{bmatrix}$

$$A(B+C)=\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}\left(\begin{bmatrix}3 & 1\\ 2 & 4\end{bmatrix}+\begin{bmatrix}1 & 0\\ 5 & 2\end{bmatrix}\right)$$

$$=\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}\begin{bmatrix}4 & 1\\ 7 & 6\end{bmatrix}$$

$$=\begin{bmatrix}1(4)+2(7) & 1(1)+2(6)\\ 0(4)+1(7) & 0(1)+1(6)\end{bmatrix}$$

$$=\begin{bmatrix}4+14 & 1+12\\ 0+7 & 0+6\end{bmatrix}=\begin{bmatrix}18 & 13\\ 7 & 6\end{bmatrix} \quad ...(i)$$

$$AB=\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}\begin{bmatrix}3 & 1\\ 2 & 4\end{bmatrix}=\begin{bmatrix}3+4 & 1+8\\ 0+2 & 0+4\end{bmatrix}=\begin{bmatrix}7 & 9\\ 2 & 4\end{bmatrix}$$

$$AC=\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 5 & 2\end{bmatrix}=\begin{bmatrix}1+10 & 0+4\\ 0+5 & 0+2\end{bmatrix}=\begin{bmatrix}11 & 4\\ 5 & 2\end{bmatrix}$$

$$AB+AC=\begin{bmatrix}7 & 9\\ 2 & 4\end{bmatrix}+\begin{bmatrix}11 & 4\\ 5 & 2\end{bmatrix}=\begin{bmatrix}7+11 & 9+4\\ 2+5 & 4+2\end{bmatrix}=\begin{bmatrix}18 & 13\\ 7 & 6\end{bmatrix} \quad ...(ii)$$

Hence, from (i) and (ii)

$$A(B+C)=AB+AC$$

Similarly, we can verify right distributive law over addition.

(b) If A, B and C be any three matrices, then distributive laws of multiplication over subtraction are as follows:

(i) $A(B-C)=AB-AC$ (Left distributive law over subtraction)

(ii) $(A-B)C=AC-BC$ (Right distributive law over subtraction)

Let $A=\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}$, $B=\begin{bmatrix}4 & 3\\ 1 & 2\end{bmatrix}$, $C=\begin{bmatrix}2 & 1\\ 0 & 1\end{bmatrix}$

$$A(B-C)=\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}\left(\begin{bmatrix}4 & 3\\ 1 & 2\end{bmatrix}-\begin{bmatrix}2 & 1\\ 0 & 1\end{bmatrix}\right)$$

$$=\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}\begin{bmatrix}4-2 & 3-1\\ 1-0 & 2-1\end{bmatrix}=\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}\begin{bmatrix}2 & 2\\ 1 & 1\end{bmatrix}$$

$$=\begin{bmatrix}1(2)+2(1) & 1(2)+2(1)\\ 0(2)+1(1) & 0(2)+1(1)\end{bmatrix}$$

$$= \begin{bmatrix} 2+2 & 2+2 \\ 0+1 & 0+1 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix} \quad ...(i)$$

$$AB = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1(4)+2(1) & 1(3)+2(2) \\ 0(4)+1(1) & 0(3)+1(2) \end{bmatrix}$$

$$= \begin{bmatrix} 4+2 & 3+4 \\ 0+1 & 0+2 \end{bmatrix} = \begin{bmatrix} 6 & 7 \\ 1 & 2 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1(2)+2(0) & 1(1)+2(1) \\ 0(2)+1(0) & 0(1)+1(1) \end{bmatrix}$$

$$= \begin{bmatrix} 2+0 & 1+2 \\ 0+0 & 0+1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$$

$$AB - AC = \begin{bmatrix} 6 & 7 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6-2 & 7-3 \\ 1-0 & 2-1 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix} \quad ...(ii)$$

Hence, from (i) and (ii)

$$A(B-C) = AB - AC$$

Similarly, we can verify right distributive law over subtraction.

3.4.4 Multiplicative Identity of a Matrix

The multiplicative identity of a matrix is a matrix that, when multiplied with any matrix A, does not change it. i.e., $AI = IA = A$

Let $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$

$$AI = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2(1)+3(0) & 2(0)+3(1) \\ 4(1)+1(0) & 4(0)+1(1) \end{bmatrix} = \begin{bmatrix} 2+0 & 0+3 \\ 4+0 & 0+1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} = A$$

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 1(2)+0(4) & 1(3)+0(1) \\ 0(2)+1(4) & 0(3)+1(1) \end{bmatrix}$$

$$= \begin{bmatrix} 2+0 & 3+0 \\ 0+4 & 0+1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} = A$$

Hence, I is multiplicative identity of a matrix A.

EXERCISE 3.4

1. Find AB and BA, if possible.

   (i) $A=\begin{bmatrix}1 & 2\\ -1 & 0\end{bmatrix}$, $B=\begin{bmatrix}3 & 2\\ 1 & -1\end{bmatrix}$ (ii) $A=[1 \quad -2]$, $B=\begin{bmatrix}3\\ -4\end{bmatrix}$

   (iii) $A=\begin{bmatrix}4\\ 4\end{bmatrix}$, $B=[2 \quad 5]$ (iv) $A=\begin{bmatrix}1 & 2\\ -1 & 1\\ 3 & 0\end{bmatrix}$, $B=\begin{bmatrix}-1 & 4 & 1\\ 2 & 3 & 1\end{bmatrix}$

   (v) $A=\begin{bmatrix}1 & 2 & 2\\ 3 & 1 & 1\end{bmatrix}$, $B=\begin{bmatrix}1 & 5\\ 2 & 4\\ 1 & 6\end{bmatrix}$

2. Verify each statement, using $A=\begin{bmatrix}5 & 1\\ -1 & 4\end{bmatrix}$, $B=\begin{bmatrix}2 & -1\\ 0 & 3\end{bmatrix}$ and $C=\begin{bmatrix}5 & 1\\ 1 & 4\end{bmatrix}$.

   (i) $AB \neq BA$ (ii) $A(B-C)=AB-AC$

   (iii) $A(BC)=(AB)C$ (iv) $(BC)^t=C^tB^t$

   (v) $(B+C)A=BA+CA$

3. If $\begin{bmatrix}4 & a\\ b & 3\end{bmatrix}\begin{bmatrix}6\\ 6\end{bmatrix}=\begin{bmatrix}6\\ 3\end{bmatrix}$, then find the values of a and b.

4. If $\begin{bmatrix}x & 1\\ y & 2\end{bmatrix}\begin{bmatrix}1 & 0\\ 3 & -1\end{bmatrix}=\begin{bmatrix}7 & -1\\ 4 & -2\end{bmatrix}$, then find the values of x and y.

3.5 Multiplicative Inverse of a Matrix

3.5.1 Determinant of a Matrix of Order 2

Let $A=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}$ be a matrix of order 2. Then the determinant of A is defined as:

$$|A|=\begin{vmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}=a_{11}a_{22}-a_{21}a_{12}$$

For example, Let $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ be a square matrix. Then $|A|=\begin{vmatrix}a & b\\ c & d\end{vmatrix}=ad-bc$ is called the determinant of matrix A and $ad-bc$ is called value of determinant. We can note that the elements of A and the elements of $|A|$ are the same. The brackets [ ] of the matrix is replaced by two vertical lines "| |" in the determinant.

Example 6 Find value of the determinant of $B = \begin{bmatrix} 8 & 2 \\ 4 & 3 \end{bmatrix}$.

Solution $|B| = \begin{vmatrix} 8 & 2 \\ 4 & 3 \end{vmatrix}$

$= 8 \times 3 - 4 \times 2$

$= 24 - 8 = 16$

Remember!

The value of the determinant is not altered by changing the rows into columns and columns into rows.

3.5.2 Singular and Non-Singular Matrices

A square matrix A is said to be **singular** if $|A| = 0$. A square matrix A is said to be **non-singular** if $|A| \neq 0$.

For example, Let $A = \begin{bmatrix} 4 & 8 \\ 2 & 4 \end{bmatrix}$

Now $|A| = \begin{vmatrix} 4 & 8 \\ 2 & 4 \end{vmatrix} = 4 \times 4 - 8 \times 2 = 16 - 16 = 0$

Thus, matrix A is a singular matrix.

For example, Let $M = \begin{bmatrix} 7 & 3 \\ 4 & 9 \end{bmatrix}$

Now $|M| = \begin{vmatrix} 7 & 3 \\ 4 & 9 \end{vmatrix} = 7 \times 9 - 4 \times 3$

$= 63 - 12 = 51 \neq 0$

Thus, matrix M is a non-singular matrix.

3.5.3 Adjoint of a Matrix

The adjoint of a square matrix is an important concept in matrix algebra, especially when finding the inverse of a matrix or solving system of linear equations.

The adjoint of a square matrix A of order 2 is obtained by interchanging the diagonal elements and changing the signs of other elements. It is denoted as Adj A.

If $A = \begin{bmatrix} 4 & 2 \\ -6 & 1 \end{bmatrix}$, then Adj $A = \begin{bmatrix} 1 & -2 \\ 6 & 4 \end{bmatrix}$

3.5.4 Multiplicative Inverse of a Non-Singular Matrix

We know $\frac{3}{4}$ and $\frac{4}{3}$ are multiplicative inverse (or reciprocal) as the product $\left(\frac{3}{4} \times \frac{4}{3}\right)$ is 1.

Similarly, if we have two square matrices of same order and their product is a unit matrix, then these matrices are called multiplicative inverse of each other.

Remember!
The inverse of a singular matrix does not exist.

Let A be any square matrix and there exists a square matrix B having the same order as A such that $AB = BA = I$ (a unit matrix). Then A and B are multiplicative inverse of each other. Hence $AA^{-1} = A^{-1}A = I.$

We denote multiplicative inverse of A by A^{-1}.

3.5.5 Adjoint Method to Find Inverse of a Square Matrix

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a non-singular matrix, then multiplicative inverse of A is:

$$A^{-1} = \frac{\text{Adj } A}{|A|}$$

i.e., $$A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example 7 If $A = \begin{bmatrix} 8 & 5 \\ 4 & 3 \end{bmatrix}$, then find multiplicative inverse of A and verify that $AA^{-1} = A^{-1}A = I$

Solution

$$A = \begin{bmatrix} 8 & 5 \\ 4 & 3 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 8 & 5 \\ 4 & 3 \end{vmatrix}$$

$$= 8 \times 3 - 5 \times 4$$

$$= 24 - 20 = 4 \neq 0$$

A is a non-singular matrix, so A^{-1} is possible.

$$A^{-1} = \frac{1}{|A|} \text{ Adj } A$$

$$= \frac{1}{4}\begin{bmatrix} 3 & -5 \\ -4 & 8 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{3}{4} & \frac{-5}{4} \\ -1 & 2 \end{bmatrix}$$

Challenge!
If $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 4 \\ 2 & 4 \end{bmatrix}$, then verify $(AB)^{-1} = B^{-1}A^{-1}$

Can we say it is also a Reversal Law of Multiplicative Inverse?

Let us verify that A and A^{-1} are multiplicative inverse of each other.

and $$AA^{-1} = \begin{bmatrix} 8 & 5 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} \frac{3}{4} & \frac{-5}{4} \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 8\times\frac{3}{4}+5\times(-1) & 8\times\left(\frac{-5}{4}\right)+5\times 2 \\ 4\times\frac{3}{4}+3\times(-1) & 4\times\left(\frac{-5}{4}\right)+3\times 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6-5 & -10+10 \\ 3-3 & -5+6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Now, $$A^{-1}A = \frac{1}{4}\begin{bmatrix} 3 & -5 \\ -4 & 8 \end{bmatrix}\begin{bmatrix} 8 & 5 \\ 4 & 3 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 3\times 8+(-5)\times 4 & 3\times 5+(-5)\times 3 \\ (-4)\times 8+8\times 4 & (-4)\times 5+8\times 3 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 24-20 & 15-15 \\ -32+32 & -20+24 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Thus, $AA^{-1} = A^{-1}A = I$

Hence, $\begin{bmatrix} 8 & 5 \\ 4 & 3 \end{bmatrix}$ and $\begin{bmatrix} \frac{3}{4} & \frac{-5}{4} \\ -1 & 2 \end{bmatrix}$ are multiplicative inverse of each other.

Example 8 Prove that $A = \begin{bmatrix} 4 & -2 \\ -6 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{3}{2} & 1 \end{bmatrix}$ are multiplicative inverse of each other.

Solution Given that $A = \begin{bmatrix} 4 & -2 \\ -6 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{3}{2} & 1 \end{bmatrix}$

Now,

$$AB=\begin{bmatrix}4 & -2\\ -6 & 4\end{bmatrix}\begin{bmatrix}1 & \frac{1}{2}\\ \frac{3}{2} & 1\end{bmatrix}=\begin{bmatrix}4\times 1+(-2)\times\frac{3}{2} & 4\times\frac{1}{2}+(-2)\times 1\\ (-6)\times 1+4\times\frac{3}{2} & (-6)\times\frac{1}{2}+4\times 1\end{bmatrix}$$

$$=\begin{bmatrix}4-3 & 2-2\\ -6+6 & -3+4\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=I$$

Remember!

If A is a singular matrix i.e; $|A|=0$, so in such a case, there is no inverse of matrix A.

and $$BA=\begin{bmatrix}1 & \frac{1}{2}\\ \frac{3}{2} & 1\end{bmatrix}\begin{bmatrix}4 & -2\\ -6 & 4\end{bmatrix}$$

$$=\begin{bmatrix}1\times 4+\frac{1}{2}\times(-6) & 1\times(-2)+\frac{1}{2}\times 4\\ \frac{3}{2}\times 4+1\times(-6) & \frac{3}{2}\times(-2)+1\times 4\end{bmatrix}$$

$$=\begin{bmatrix}4-3 & -2+2\\ 6-6 & -3+4\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=I$$

Thus, $AB=BA=I$

Therefore, $A=\begin{bmatrix}4 & -2\\ -6 & 4\end{bmatrix}$ and $B=\begin{bmatrix}1 & \frac{1}{2}\\ \frac{3}{2} & 1\end{bmatrix}$ are multiplicative inverse of each other.

Example 9 If $A=\begin{bmatrix}x & 8\\ 5 & 10\end{bmatrix}$ is a singular matrix, then find the value of x.

Solution If $A=\begin{bmatrix}x & 8\\ 5 & 10\end{bmatrix}$ is a singular matrix, then $|A|=0$

Now, $$|A|=\begin{vmatrix}x & 8\\ 5 & 10\end{vmatrix}$$

$$=10x-5\times 8=10x-40$$

Challenge!

Take any 2 by 2 matrix and check whether it is singular or non-singular. Also find its adjoint.

Since, A is singular, therefore,

$$|A|=10x-40=0$$

i.e. $10x=40$

$\Rightarrow$ $x=4$

EXERCISE 3.5

1. Find the values of each of the determinant.

 (i) $\begin{vmatrix} 10 & 5 \\ 4 & 6 \end{vmatrix}$ (ii) $\begin{vmatrix} -5 & 8 \\ -3 & -7 \end{vmatrix}$ (iii) $\begin{vmatrix} 3 & 8 \\ 0 & 2 \end{vmatrix}$

2. Find whether the following matrices are singular or non-singular.

 $A=\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$, $B=\begin{bmatrix} 7 & 21 \\ 2 & 6 \end{bmatrix}$, $C=\begin{bmatrix} 13 & 5 \\ 7 & 3 \end{bmatrix}$, $D=\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$

3. Find the value of x when $A=\begin{bmatrix} x & 6 \\ 5 & 15 \end{bmatrix}$ is a singular matrix.

4. Find the adjoint of the following matrices:

 $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B=\begin{bmatrix} 5 & -2 \\ 3 & 7 \end{bmatrix}$, $C=\begin{bmatrix} -3 & 5 \\ -3 & 2 \end{bmatrix}$

5. Find multiplicative inverse of the following matrices:

 (i) $\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$ (ii) $\begin{bmatrix} -4 & 8 \\ 7 & 2 \end{bmatrix}$ (iii) $\begin{bmatrix} 40 & 8 \\ 5 & 2 \end{bmatrix}$

 (iv) $\begin{bmatrix} 3 & 5 \\ 5 & -3 \end{bmatrix}$ (v) $\begin{bmatrix} 10 & 8 \\ 3 & 3 \end{bmatrix}$ (vi) $\begin{bmatrix} -2 & -3 \\ 4 & 5 \end{bmatrix}$

6. If $A=\begin{bmatrix} 5 & -3 \\ 2 & -1 \end{bmatrix}$, then find A^{-1} and prove that $AA^{-1} = A^{-1}A = I.$

7. Show that the following matrices are multiplicative inverse of each other.

 (i) $\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 4 & 15 \\ 2 & 8 \end{bmatrix}, \begin{bmatrix} 4 & -\frac{15}{2} \\ -1 & 2 \end{bmatrix}$

8. Prove that $(AB)^{-1} = B^{-1}A^{-1}$, if

 (i) $A=\begin{bmatrix} -3 & -2 \\ 5 & 6 \end{bmatrix}$, $B=\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$ (ii) $A=\begin{bmatrix} 1 & 2 \\ 8 & 0 \end{bmatrix}$, $B=\begin{bmatrix} 0 & -1 \\ 5 & 2 \end{bmatrix}$

3.6 Solution of Simultaneous Linear Equations

A **linear equation** is an algebraic equation in which the highest power of the variable is 1. A general form of a linear equation in two variables is:

$ax + by = c$, where a, b, c are real numbers, x and y are variables and $a, b \neq 0$.

Simultaneous linear equations are a set of two or more linear equations with two or more variables that are solved together (simultaneously) to find a common solution. The general form of system of two linear equations in two variables is given below:

$$ax+by=m$$
$$cx+dy=n$$

where *a, b, c, d, m* and *n* are real numbers.

Here, we will find the solution of two simultaneous equations in two variables by the following methods:

(i) Matrix Inversion Method

(ii) Cramer's Rule

3.6.1 Matrix Inversion Method

Let $ax + by = m$
and $cx + dy = n$

These equations can be written in the matrix form as: $\begin{bmatrix} ax+by \\ cx+dy \end{bmatrix} = \begin{bmatrix} m \\ n \end{bmatrix}$

or $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} m \\ n \end{bmatrix}$ (i)

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} m \\ n \end{bmatrix}$

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Now if $|A| \neq 0$, then (i) can be written as:

$AX = B$

and $A^{-1}AX = A^{-1}B$ [pre-multiply by A^{-1}]

$IX = A^{-1}B$ [as $A^{-1}A = I$]

Thus $X = A^{-1}B$ [as $IX = X$]

$$X = \frac{\text{Adj } A}{|A|} \cdot B$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} m \\ n \end{bmatrix}}{ad-bc} = \frac{\begin{bmatrix} dm-bn \\ an-cm \end{bmatrix}}{ad-bc} = \begin{bmatrix} \dfrac{dm-bn}{ad-bc} \\ \dfrac{an-cm}{ad-bc} \end{bmatrix}$$

$$x = \frac{dm-bn}{ad-bc}, y = \frac{an-cm}{ad-bc}$$

Example 10 Solve the following by matrix inversion method:

$$2x + 3y = 13$$
$$4x - 5y = -7$$

Solution Given that $2x + 3y = 13$

$$4x - 5y = -7$$

Writing the equations in matrix form:

$$\begin{bmatrix} 2x+3y \\ 4x-5y \end{bmatrix} = \begin{bmatrix} 13 \\ -7 \end{bmatrix}$$

or $$\begin{bmatrix} 2 & 3 \\ 4 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 13 \\ -7 \end{bmatrix} \qquad \text{...(i)}$$

Let $A = \begin{bmatrix} 2 & 3 \\ 4 & -5 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 13 \\ -7 \end{bmatrix}$

Then (i) can be written as

$$AX = B$$

or $$X = A^{-1}B$$

Now $$A = \begin{bmatrix} 2 & 3 \\ 4 & -5 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 3 \\ 4 & -5 \end{vmatrix}$$
$$= (2)(-5) - (3)(4)$$
$$= -10 - 12$$
$$= -22 \neq 0$$

$$\text{Adj } A = \begin{bmatrix} -5 & -3 \\ -4 & 2 \end{bmatrix}$$

Therefore $A^{-1} = \frac{1}{|A|}\text{Adj } A$

$$= \frac{1}{-22}\begin{bmatrix} -5 & -3 \\ -4 & 2 \end{bmatrix}$$

Now $X = A^{-1}B$ can be written as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-22}\begin{bmatrix} -5 & -3 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 13 \\ -7 \end{bmatrix}$$

$$= \frac{1}{-22}\begin{bmatrix} (-5)\times 13 + (-3)\times(-7) \\ (-4)\times 13 + \quad 2\times(-7) \end{bmatrix}$$

$$= \frac{1}{-22}\begin{bmatrix} -65+21 \\ -52-14 \end{bmatrix}$$

$$= \frac{1}{-22}\begin{bmatrix} -44 \\ -66 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Thus $x = 2, y = 3$

Solution set is $\{(2, 3)\}$

Challenge!

Take any two linear equations in two variables and solve them by matrix inversion method.

History!

Gabriel Cramer was a Genevan mathematician. Cramer showed promised in Mathematics from an early age. At 18 he received his doctorate in Mathematics.

In 1750 he published Cramer's rule, giving a general formula for the solution for any unknown in linear equation system having a unique solution, in terms of determinants applied by the system. This rule is still standard.

Gabriel Cramer
(July 31, 1704 - January 4, 1752)

3.6.2 Cramer's Rule

Consider the following system of linear equations.

$$ax + by = m$$
$$cx + dy = n$$

These equations can be written in the matrix form as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} m \\ n \end{bmatrix}$$

$$AX = B \qquad \text{...(i)}$$

Where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} m \\ n \end{bmatrix}$

If $|A| \neq 0$, then from equation (i)

$$A^{-1}AX = A^{-1}B \qquad \text{[pre multiplication by } A^{-1}\text{]}$$

$$IX = A^{-1}B$$

or $$X = A^{-1}B \qquad \text{[as } IX = X\text{]}$$

$$X = \frac{\text{Adj } A}{|A|} \times B \qquad \left[A^{-1} = \frac{\text{Adj} A}{|A|} \right]$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} m \\ n \end{bmatrix}}{|A|}$$

$$= \frac{\begin{bmatrix} dm - bn \\ -cm + an \end{bmatrix}}{|A|}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \dfrac{dm - bn}{|A|} \\ \dfrac{-cm + an}{|A|} \end{bmatrix}$$

or $$x = \frac{dm - bn}{|A|}, \quad y = \frac{an - cm}{|A|}$$

or $$x = \frac{\begin{vmatrix} m & b \\ n & d \end{vmatrix}}{|A|} = \frac{|A_x|}{|A|}, \quad y = \frac{\begin{vmatrix} a & m \\ c & n \end{vmatrix}}{|A|} = \frac{|A_y|}{|A|}$$

Example 11 Use Cramer's rule to solve the system of equations:

$2x + 3y = 13$

$4x - 5y = -7$

Solution $2x + 3y = 13$

$4x - 5y = -7$

Matrix form of the equations is:

$$\begin{bmatrix} 2 & 3 \\ 4 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 13 \\ -7 \end{bmatrix}$$

Challenge!

Take any two linear equations in two variables and solve them by using Cramer's rule.

Here, we have

$$A = \begin{bmatrix} 2 & 3 \\ 4 & -5 \end{bmatrix}, A_x = \begin{bmatrix} 13 & 3 \\ -7 & -5 \end{bmatrix}, A_y = \begin{bmatrix} 2 & 13 \\ 4 & -7 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 3 \\ 4 & -5 \end{vmatrix}$$

$$= (2)(-5) - (3)(4)$$

$$= -10 - 12$$

$$= -22 \neq 0$$

$$|A_x| = \begin{vmatrix} 13 & 3 \\ -7 & -5 \end{vmatrix}$$

$$= (13)(-5) - (3)(-7)$$

$$= -65 + 21 = -44$$

$$x = \frac{|A_x|}{|A|} = \frac{-44}{-22} = 2$$

$$|A_y| = \begin{vmatrix} 2 & 13 \\ 4 & -7 \end{vmatrix}$$

$$= (2)(-7) - (13)(4)$$

$$= -14 - 52 = -66$$

$$y = \frac{|A_y|}{|A|} = \frac{-66}{-22} = 3$$

Solution set = $\{(2,3)\}$

3.7 Applications of Matrices in Real World Problems

Example 12 Three forces act on a particle which must be in equilibrium i.e. $F_1 + F_2 + F_3 = 0$, where $F_1 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$, $F_2 = \begin{bmatrix} -3 \\ x \end{bmatrix}$, $F_3 = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$. Find the value of x.

Solution Since $F_1 + F_2 + F_3 = 0$

$$\begin{bmatrix} 5 \\ 2 \end{bmatrix} + \begin{bmatrix} -3 \\ x \end{bmatrix} + \begin{bmatrix} -2 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5-3-2 \\ 2+x-4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ x-2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x-2=0$$

$$x=2$$

Example 13 Eight years ago Huria's was $\frac{3}{4}$ of Jannat's age. After four years Huria's age will be $\frac{6}{7}$ of Jannat's age. Find their present ages by using matrices.

Solution Suppose Huria's present age is x years and Jannat's present age is y years. Eight years ago their ages were $x – 8$ and $y – 8$. According to first condition:

$$x-8 = \frac{3}{4}(y-8)$$

or $\quad 4x – 32 = 3y – 24$

or $\quad 4x – 3y = – 24 + 32$

$$4x-3y = 8 \qquad ...(i)$$

After 4 years their ages will be $x + 4$ and $y + 4$.

Applying the second condition:

$$x+4 = \frac{6}{7}(y+4)$$

or $\quad 7x + 28 = 6y + 24$

$$7x-6y = -4 \qquad ...(ii)$$

Matrix form of (i) and (ii) is

$$\begin{bmatrix} 4 & -3 \\ 7 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \end{bmatrix}$$

Here, $A = \begin{bmatrix} 4 & -3 \\ 7 & -6 \end{bmatrix}$, $A_x = \begin{bmatrix} 8 & -3 \\ -4 & -6 \end{bmatrix}$, $A_y = \begin{bmatrix} 4 & 8 \\ 7 & -4 \end{bmatrix}$

$$|A| = \begin{vmatrix} 4 & -3 \\ 7 & -6 \end{vmatrix}$$

$$= (4)(-6) - (7)(-3)$$

$$= -24 + 21$$

$$= -3 \neq 0$$

Applying Cramer's rule

$$x = \frac{|A_x|}{|A|} = \frac{\begin{vmatrix} 8 & -3 \\ -4 & -6 \end{vmatrix}}{-3}$$

$$= \frac{-48-12}{-3} = \frac{-60}{-3} = 20$$

$$y = \frac{|A_y|}{|A|} = \frac{\begin{vmatrix} 4 & 8 \\ 7 & -4 \end{vmatrix}}{-3}$$

$$= \frac{-16-56}{-3} = \frac{-72}{-3} = 24$$

∴ Present ages of Huria and Jannat are 20 years and 24 years respectively.

3.8 The Role of Mathematics in Scientific Theories and Technological Advancement

Mathematics is often described as the language of science. It plays a fundamental role in developing scientific theories and advancing modern technologies. From understanding the laws of nature to designing life-saving drugs, Mathematics provides the tools needed to describe, predict and optimize real-world phenomena.

One major area where Mathematics contributes is in the use of mathematical models and simulations. Scientists and engineers use these models to represent complex systems such as weather patterns, population growth or the functioning of the human brain. These models allow researchers to test hypotheses, predict future outcomes, and visualize scenarios that would be too costly, dangerous or impossible to replicate in the real world.

In medicine and materials science, for example, mathematical simulations are used to design and optimize new materials or drug molecules. This saves time and resources by reducing the need for trial-and-error experiments. Similarly, in fields like Physics and astronomy, equations and models help us to understand the formation of galaxies, black holes and subatomic particles.

Drug Discovery

Mathematical algorithms help simulate how a drug will interact with cells or proteins in the body, speeding up the discovery of effective treatments.

Neuroscience

Mathematical models are used to simulate brain activity, helping researchers understand learning, memory and diseases like epilepsy.

Climate Science

Complex equations describe the interaction of air, water and energy, helping scientists to predict weather and long-term climate change.

EXERCISE 3.6

1. Solve by matrix inversion method, if possible.

(i) $2x + 5y = 19$, $4x - 3y = -1$

(ii) $3x + 2y = 7$, $5x - y = 16$

(iii) $x - 2y = 9$, $2x + 7y = -4$

(iv) $3x + 2y = 2$, $x - 2y = -2$

2. Use Cramer's rule to solve the following pair of linear equations, if possible.

(i) $x + 4y = 4$, $2x - y = 5$

(ii) $x + 2y = 7$, $3x - 2y = -3$

(iii) $2x - 5y = -6$, $4x - 3y = -12$

(iv) $3x + 2y = -1$, $5x + 6y = 5$

3. An electrical engineer wants to determine the current in two branches A and B of a simple electrical circuit. The system of the equations is:

$$x + y = 7$$
$$2x - y = 2$$

where x is the current in branch A and y is the current in branch B. Find x and y by using matrices.

4. Three forces act on a particle and must be in equilibrium i.e. $F_1 + F_2 + F_3 = 0$, where $F_1 = \begin{bmatrix} 8 \\ x \end{bmatrix}, F_2 = \begin{bmatrix} -2 \\ -7 \end{bmatrix}, F_3 = \begin{bmatrix} y \\ -1 \end{bmatrix}$. Find the value of x and y.

5. Two support beams, A and B are holding up a combined load of 100 kN. Twice the load on beam A and three times the load on beam B equals 240 kN. Find the load of beam A and beam B by using matrices.

6. In a 2D game world, two characters are moving along straight paths. One character moves along a line where the total of twice their horizontal position and vertical position is 5, while the other moves along a line where their horizontal position is one more than their vertical position. Find their point of intersection by using matrices.

7. Two years ago a man was 5 times as old as his son was. After 6 years he will be 3 times as old as his son. Find their present ages by using matrices.

8. Two cyclists are 44 km apart and start out at the same time. If they go towards one another they meet in 2 hours, but if they go in the same direction the faster overtakes the slower in $7\frac{1}{2}$ hours. Find their speeds by using matrices.

9. The numerator of a fraction is 7 less than the denominator. If the numerator is increased by 3, the new fraction can be cancelled down to $\frac{3}{4}$. Find the original fraction by using matrices.

REVIEW EXERCISE 3

1. Four possible answers are given for the following questions. Choose the correct answer.

(i) If $\begin{bmatrix} a+2 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$, then $a =$

(a) 3 (b) 5 (c) 6 (d) 7

(ii) $A = [3 \quad 5 \quad 0]$ is a _______ matrix.

(a) row (b) square (c) column (d) null

(iii) $B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is a _______ matrix.

(a) identity (b) square (c) row (d) column

(iv) $M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a/an _______ matrix.

(a) rectangular (b) identity (c) column (d) row

(v) If $A^t = -A$, then A is _______ matrix.

(a) symmetric (b) row
(c) rectangular (d) skew-symmetric

(vi) If $A = [3 \quad 4]$, $B = [7 \quad 8]$, then $A + B =$

(a) [21 32] (b) [24 28]
(c) [10 12] (d) [11 11]

(vii) If $A = [3 \quad 4]$, $B = [7 \quad 8]$, then $B - A =$

(a) [10 12] (b) [11 11]
(c) [4 4] (d) [–4 –4]

(viii) What is the additive inverse of $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$?

(a) $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$ (b) $\begin{bmatrix} -3 \\ -4 \end{bmatrix}$ (c) $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ (d) $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

(ix) If $A = \begin{bmatrix} 3 & 1 & 3 \\ 2 & 0 & 2 \end{bmatrix}$, then order of A^t is:

(a) 3-by-2 (b) 2-by-3 (c) 3-by-3 (d) 2-by-2

(x) $\begin{vmatrix} 3 & 1 \\ 0 & 4 \end{vmatrix} =$

(a) 11 (b) 12 (c) 13 (d) –11

2. If $A = \begin{bmatrix} 4 & 2 \\ 7 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 5 \\ 8 & 8 \end{bmatrix}$, then find

(i) $(A - B)^t$ (ii) $B^t - A^t$ (iii) $2A + 3B$

3. If $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 6 \\ 2 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 2 \\ 1 & 7 \end{bmatrix}$, then verify that

(i) $2(A + B) = 2A + 2B$ (ii) $(A + B) + C = A + (B + C)$
(iii) $(A + B)C = AC + BC$ (iv) $C(A - B) = CA - CB$
(v) $(AB)^{-1} = B^{-1}A^{-1}$ (vi) $AA^{-1} = A^{-1}A = I$
(vii) $(AB)^t = B^t A^t$ (viii) $(AB)C = A(BC)$

4. If $A = \begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 3 \\ 2 & 4 \end{bmatrix}$, then find

(i) $|B|$ (ii) $\text{Adj}\,B$ (iii) A^{-1}
(iv) $A^{-1}A$ (v) $(AB)^t$ (vi) $(B^t)^t$

5. Use matrix inversion method and Cramer's rule to solve the following pair of linear equations, if possible:

(i) $3x + 4y = 7$, $5x - y = 2$ (ii) $x - 6y = -15$, $2x + 6y = -3$ (iii) $2x + y = 5$, $x + 3y = 3$

6. Find two numbers by using matrices such that twice the first added to the second makes 21 and twice the second added to the first makes 27.

7. 4 knives and 6 forks cost Rs. 136, whereas 6 knives and 5 forks cost Rs. 164. Find the cost of a knife and a fork by using matrices.

8. A shop employs, 5 men and 3 women, pays total daily wages Rs. 3500. If the number of men is reduced to 2 and 3 extra women are taken on, the daily wages amount to Rs. 5000. Find daily wages of a man and a woman by using matrices.

UNIT 4

Functions and Graphs

Students' Learning Outcomes

After completing this unit, the students will be able to:

- Explain operations on functions and compositions of functions.
- Define inverse functions and demonstrate their domain and range with examples.
- Formulate composite functions as defined by $(f \circ g)(x) = f(g(x))$.
- Apply concepts from functions to real-world problems (such as finance, transportation and sales).
- Plot graphs of absolute valued functions.
- Solve absolute value linear equations and inequalities in one variable and express the solution as a range of values on a number line.
- Apply concepts of absolute valued functions to real-world problems (such as to calculate energy wave, magnitude and distance).

INTRODUCTION

This unit explores the foundational and applied concepts of functions, beginning with operations on functions and compositions of functions, enabling students to combine multiple functions and understand their interactions. Students will define inverse functions, determine their domains and ranges and illustrate these with clear examples. The unit also emphasizes the formulation and interpretation of composite functions. Students will apply function concepts to real-life contexts such as finance, transportation and sales, making mathematics practical and relevant. Additionally, the unit covers absolute value functions graphing them, solving linear equations and inequalities involving absolute values and expressing solutions on a number line.

4.1 Function

If X and Y are two non-empty sets, then a function from a set X to a set Y is a rule that assigns each element of set X exactly one element in set Y. It is written as $f: X \rightarrow Y$, where the set X is called the domain and the set Y is called the co-domain of function f. While the range is the

History!

Gottfried Wilhelm Leibniz is credited with first introducing the word "function" in a mathematical context. He used the term in 1673 to describe a quantity related to a curve, such as its coordinates or slope.

subset of the co-domain that contains only the actual output values produced by the function f.

If $f(a) = 1$, then 1 is called image of a under f and a is called a pre-image of 1.

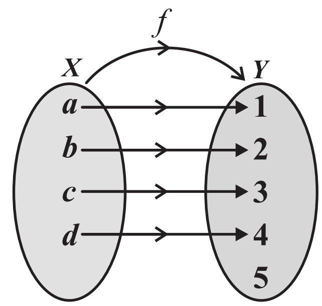


Figure 4.1

For example, $f(a) = 1, f(b) = 2, f(c) = 3, f(d) = 4$

We can see that all elements in X have only one image in Y. Therefore, f is a function.

Domain $X = \{a, b, c, d\}$

Co-domain $Y = \{1, 2, 3, 4, 5\}$

Range of $f = \{1, 2, 3, 4\}$

The symbol $f(x)$ is often used to briefly denote a function of x. If $y = f(x)$, then by substituting a succession numerical value for 'x', we can obtain a corresponding succession values for y which stands for the values of the function. Hence in this connection it is sometimes conventional to call 'x' the independent variable and 'y' the dependent variable.

Remember!
Range is always a subset of the co-domain.

Example 1 If $f(x) = x^2 + 3x + 1$, then evaluate

(i) $f(0)$ (ii) $f(-2)$ (iii) $f(a)$ (iv) $f(x+1)$

Solution Given that $f(x) = x^2 + 3x + 1$

(i) $f(0) = (0)^2 + 3(0) + 1 = 1$

(ii) $f(-2) = (-2)^2 + 3(-2) + 1$
$= 4 - 6 + 1 = -1$

(iii) $f(a) = a^2 + 3a + 1$

(iv) $f(x+1) = (x+1)^2 + 3(x+1) + 1$
$= x^2 + 2x + 1 + 3x + 3 + 1$
$= x^2 + 5x + 5$

Skilled Practice!
If $f(x) = 2x^2 + 5x$, then find $f(x+2)$.

Example 2 Find the domain and range of $f(x) = (x+1)^2$.

Solution As $f(0) = (0+1)^2 = 1$; $f(4) = (4+1)^2 = 25$; $f(-10) = (-10+1)^2 = 81$;

$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}+1\right)^2 = \frac{9}{4}$; $f\left(\sqrt{3}\right) = \left(\sqrt{3}+1\right)^2 = 7.46$; $f(-1) = 0$; so on

For every real number x, $f(x) = (x+1)^2$ is a non-negative real number. So,

Domain f = set of all real numbers ; Range f = set of all non-negative real numbers.

4.1.1 Operations on Functions

Operations on functions involves performing algebraic operations (addition, subtraction, multiplication and division)

a Addition of Functions

If $f(x)$ and $g(x)$ are two functions, then their sum is written as:

$$(f+g)(x) = f(x) + g(x)$$

Example 3 Let $f(x) = 2x + 3$ and $g(x) = x^2 + x + 1$. Add $f(x)$ and $g(x)$.

Solution

$f(x) = 2x + 3$

$g(x) = x^2 + x + 1$

$(f+g)(x) = f(x) + g(x)$

$= (2x + 3) + (x^2 + x + 1) = x^2 + 3x + 4$

b Subtraction of Functions

If $f(x)$ and $g(x)$ are two functions, then their difference is written as:

$$(f-g)(x) = f(x) - g(x)$$

Example 4 Let $f(x) = 5x + 2$, $g(x) = x^2 - 3$. Find $f(x) - g(x)$

Solution

$f(x) = 5x + 2$

$g(x) = x^2 - 3$

$(f-g)(x) = f(x) - g(x)$

$= (5x + 2) - (x^2 - 3)$

$= 5x + 2 - x^2 + 3 = -x^2 + 5x + 5$

c Multiplication of Functions

If $f(x)$ and $g(x)$ are two functions, then their product is written as:

$(f\cdot g)(x) = f(x)\cdot g(x)$

Example 5 Let $f(x) = x + 1$, $g(x) = 2x - 3$, find $f(x)\cdot g(x)$.

Solution

$f(x) = x + 1$

$g(x) = 2x - 3$

$(f\cdot g)(x) = f(x)\cdot g(x)$

$= (x + 1)(2x - 3) = 2x^2 - 3x + 2x - 3$

$= 2x^2 - x - 3$

d Division of Functions

If $f(x)$ and $g(x)$ are two functions, then their quotient is written as:

$$\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)} \quad \text{for } g(x)\neq 0$$

Example 6 Let $f(x)=x^2-4$, $g(x)=x+2$. Find $\left(\frac{f}{g}\right)(x)$, where $x\neq -2$

Solution

$f(x) = x^2 - 4$

$g(x) = x + 2$

$$\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}$$

$$=\frac{x^2-4}{x+2}$$

$$=\frac{(x+2)(x-2)}{(x+2)} = x-2$$

4.1.2 Composition of Functions

Composition of function is a process of combining two or more functions to produce a new single function.

The composition operator "o" takes two functions f and g and results a new function. Let us have a look at its definition.

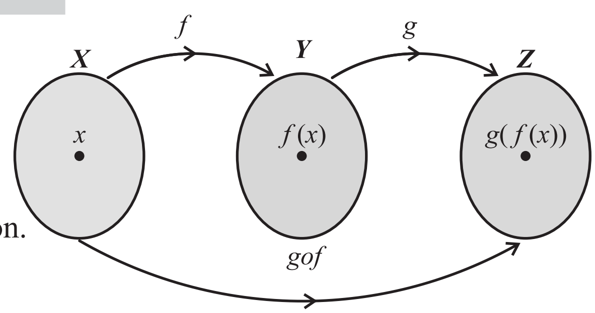
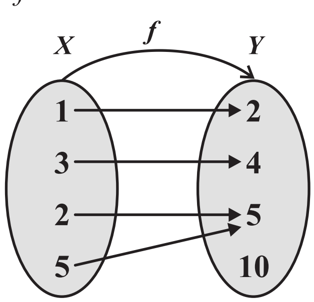


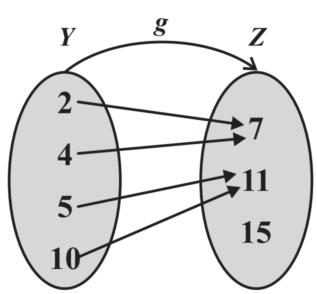
Figure 4.2

Let $f: X\to Y$ and $g: Y\to Z$ be two functions. Then the composition of f and g denoted by gof is defined as the function $gof: X\to Z$ given by $gof(x)=g(f(x))$



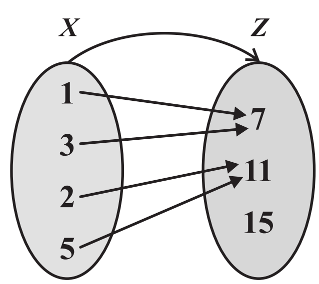
$f: X\to Y$

Figure 4.3



$g: Y\to Z$

Figure 4.4



$gof: X\to Z$

Figure 4.5

Example 7 If $f(x) = 2x + 3$, $g(x) = x^2$, then find

(i) $(fog)(x)$ **(ii)** $(gof)(x)$ **(iii)** $(fof)(x)$ **(iv)** $(gog)(x)$

Solution $f(x) = 2x + 3, g(x) = x^2$

(i)
$$\begin{aligned}(fog)(x) &= f(g(x))\\ &= f(x^2)\\ &= 2x^2 + 3\end{aligned}$$

(ii)
$$\begin{aligned}(gof)(x) &= g(f(x))\\ &= g(2x+3)\\ (gof)(x) &= (2x+3)^2\\ &= 4x^2 + 12x + 9\end{aligned}$$

(iii)
$$\begin{aligned}(fof)(x) &= f(f(x))\\ &= f(2x+3)\\ &= 2(2x+3)+3\\ &= 4x+9\end{aligned}$$

(iv)
$$\begin{aligned}(gog)(x) &= g(g(x))\\ &= g(x^2)\\ (gog)(x) &= (x^2)^2 = x^4\end{aligned}$$

Remember!
In general,
$(fog)(x) \neq (gof)(x)$

One-to-One Function
A function f is said to be one-to-one (or injective) if each input maps to a unique output.
Onto Function
A function is called onto (or surjective) if every element in the codomain has at least one pre-image in the domain.
Bijective Function
A function is called bijective if it is both one-to-one and onto.

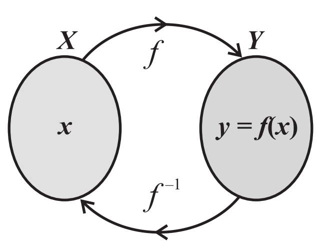
4.1.3 Inverse Function

A function f takes an input x from a set X and maps it to an output y in a set Y. i.e.,

$$f : X \rightarrow Y,\ f(x) = y$$

Then the inverse function of f denoted by f^{-1}, reverses this process. It takes y as input and gives x as output:

$$f^{-1} : Y \rightarrow X \quad , \quad f^{-1}(y) = x$$



Domain of f = Range of f^{-1}
Range of f = Domain of f^{-1}

Note
If $y = f(x)$ then $f^{-1}(y) = x$ is called inverse of $f(x)$

Let us understand it with the help of an example:

Consider $X = \{1, 2, 3\}$, $Y = \{a, b, c\}$

and $f : \{(1, a), (2, b), (3, c)\}$, then $f^{-1} : \{(a, 1), (b, 2), (c, 3)\}$

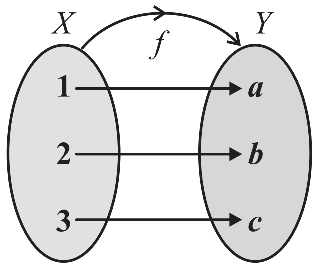


Figure 4.6

$f(1) = a$

$f(2) = b$

$f(3) = c$

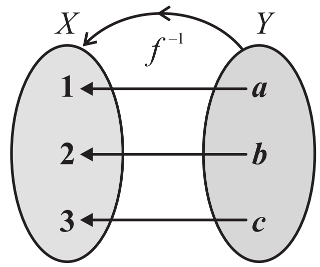


Figure 4.7

$f^{-1}(a) = 1$

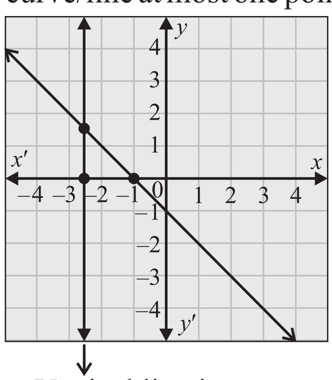
$f^{-1}(b) = 2$

$f^{-1}(c) = 3$

It is important to note that a function has an inverse if and only if it is one-to-one and onto (bijective).

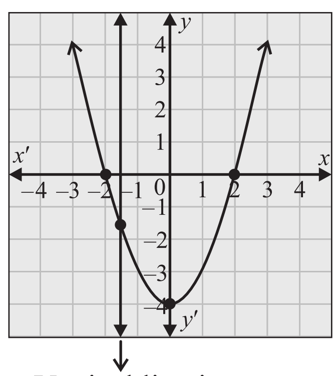
4.1.4 Vertical Line Test

A curve/line drawn in a graph represents a function, if every vertical line intersects the curve/line at most one point.



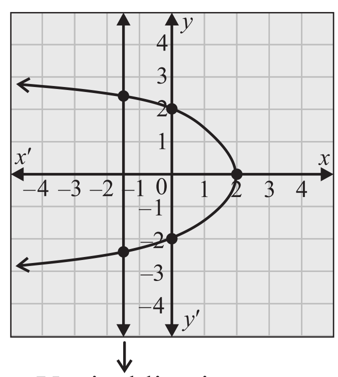
Vertical line intersects at one point

Figure 4.8



Vertical line intersects at one point

Figure 4.9



Vertical line intersects at two points

Figure 4.10

Graphs of Figure 4.8 and Figure 4.9 represent graphs of functions and graph of Figure 4.10 does not represent the graph of a function.

4.1.5 Horizontal Line Test

If every horizontal line intersects the curve at most one point, then the graph is a one-to-one function.

The vertical line intersects the graph of $f(x) = x^2$ at only one point, so it is the graph of a function. It fails in horizontal test, so it is not a one to one function.

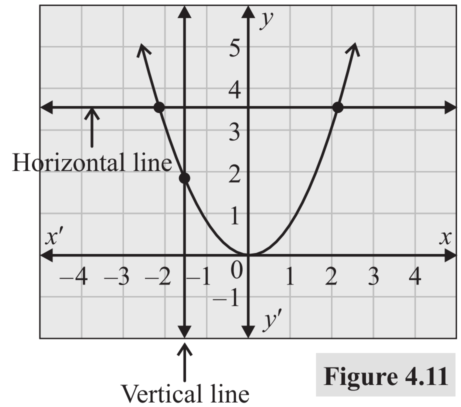


Figure 4.11

Example 8 Find the inverse of $f(x) = 2x + 3$. Also find domain and range of $f^{-1}(x)$.

Solution

$$f(x) = 2x + 3$$

$$y = 2x + 3 \quad \text{[Replace } f(x) \text{ with } y\text{]}$$

$$y - 3 = 2x$$

$$\therefore \quad x = \frac{y-3}{2} \quad \text{[Solve for } x\text{]}$$

$$f^{-1}(y) = \frac{y-3}{2} \quad [\because \; y = f(x) \Rightarrow f^{-1}(y) = x]$$

$$f^{-1}(x) = \frac{x-3}{2} \quad \text{[Replacing } y \text{ by } x\text{]}$$

Domain $f(x)$: All real numbers
Range $f(x)$: All real numbers

Domain $f^{-1}(x)$: All real numbers
Range $f^{-1}(x)$: All real numbers

Example 9 If $f(x) = x^3$, find $f^{-1}(x)$. Also verify that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

Solution

$$f(x) = x^3$$

$$y = x^3 \quad \text{[Replace } f(x) \text{ with } y\text{]}$$

$$x^3 = y$$

$$x = y^{\frac{1}{3}} \quad \text{[Solve for } x\text{]}$$

$$f^{-1}(y) = y^{\frac{1}{3}} \quad [y = f(x) \Rightarrow f^{-1}(y) = x]$$

$$f^{-1}(x) = x^{\frac{1}{3}} \quad \text{[Replacing } y \text{ by } x\text{]}$$

$$f\left(f^{-1}(x)\right) = f(x^{\frac{1}{3}}) = (x^{\frac{1}{3}})^3 = x$$

> **Challenge!**
> If $f(x) = -\frac{1}{4}x + 2$, then verify that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

$$f^{-1}\left(f(x)\right) = f^{-1}(x^3) = (x^3)^{\frac{1}{3}} = x$$

Hence $\quad f\left(f^{-1}(x)\right) = f^{-1}(f(x)) = x$

EXERCISE 4.1

1. If $f(x) = 2x + 5$, $g(x) = 3x - 2$, then find
   (i) $f(x) + g(x)$ (ii) $f(x) - g(x)$
2. If $f(x) = x + 2$, $g(x) = 2x + 4$, then find
   (i) $f(x) \cdot g(x)$ (ii) $g(x) \cdot f(x)$ (iii) $\dfrac{f(x)}{g(x)}$ (iv) $\dfrac{g(x)}{f(x)}$
3. For the functions f and g, find
   (a) $(fog)(x)$ (b) $(gof)(x)$ (c) $(fof)(x)$ (d) $(gog)(x)$

   Where,

   (i) $f(x) = 2x + 3$; $g(x) = x^3$ (ii) $f(x) = \dfrac{2}{x}, x \neq 0$; $g(x) = 2x^2 - 1$
   (iii) $f(x) = 2x - 1$; $g(x) = \dfrac{x+1}{2}$
4. Find the value of k, such that $(fog)(x) = (gof)(x)$, where $f(x) = 3x + 2$, $g(x) = 6x - k$.
5. Given that $f(x) = 3x + 2$ and $g(x) = 2x + 3$. Find
   (i) $g(f(4))$ (ii) $f\left(f(3)\right)$ (iii) $f\left(g(-2)\right)$
6. Find $f^{-1}(x)$ in each of the following:
   (i) $f(x) = 2x - 3$ (ii) $f(x) = 4x^3 - 1$
   (iii) $f(x) = \sqrt{x-1}, x \geq 1$ (iv) $f(x) = \dfrac{x+1}{3x-2}, x \neq \dfrac{2}{3}$
7. The functions f and g are defined such that $f(x) = 4x + 2$ and $g(x) = 6x - 18$.
   (i) Find $f^{-1}(x)$ and $g^{-1}(x)$. (ii) Find x if $f^{-1}(x) = g^{-1}(x)$.
8. Verify that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$
   (i) $f(x) = x - 6$ (ii) $f(x) = 7x - 4$
   (iii) $f(x) = \dfrac{x-3}{4}$ (iv) $f(x) = \dfrac{x-4}{x+2}, x \neq -2$
9. Without finding $f^{-1}(x)$, find domain and range of $f^{-1}(x)$:
   (i) $f(x) = 12x - 3$ (ii) $f(x) = \dfrac{1}{2}x + 8$

(iii) $f(x) = \dfrac{x}{1+x}$, $x \neq -1$ (iv) $f(x) = \sqrt{x-2}$, $x \geq 2$

10. Given that $f(x) = x^2 + 9$ and $g(x) = x + 21$. Find the values of a such that: $f(a) = g(a)$

4.2 Graphs of Absolute Valued Functions

An absolute valued function is a type of function that includes an algebraic expression enclosed within absolute value bars. Its general form is $f(x) = |x|$. The absolute value of a number represents its distance from zero on the number line and is always non-negative.

The graph of $f(x) = |x|$ has V-shape and is symmetric with respected to the y-axis. The point where the graph changes direction is called the vertex (0, 0).

The vertex form of an absolute value function is $g(x) = a|x - h| + k$, where $a \neq 0$ and (h, k) is the vertex. Its graph is symmetric about the line $x = h$.

Remember!

The graphs of all other absolute value functions are variations of the parent function $f(x) = |x|$ obtained through transformations.

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

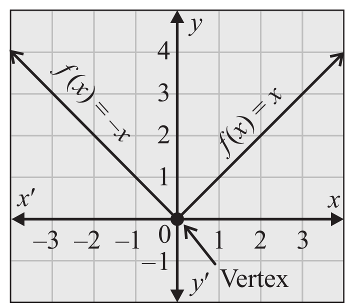


Figure 4.12

Example 10 Plot the graph of the following functions:

(i) $f(x) = |x| + 2$ (ii) $g(x) = |x - 3|$

Solution (i) $f(x) = |x| + 2$ (ii) $g(x) = |x - 3|$

(i)

Step I Make a table of values.

x	–4	–2	0	2	4...
$f(x)$	6	4	2	4	6...

Step II Plot the ordered pairs.

Step III Join the ordered pairs and draw the V-shaped graph.

(ii)

Step I Make a table of values.

x	–4	–2	0	3	5...
$g(x)$	7	5	3	0	2...

Step II Plot the ordered pairs.

Step III Join the ordered pairs and draw the V-shaped graph.

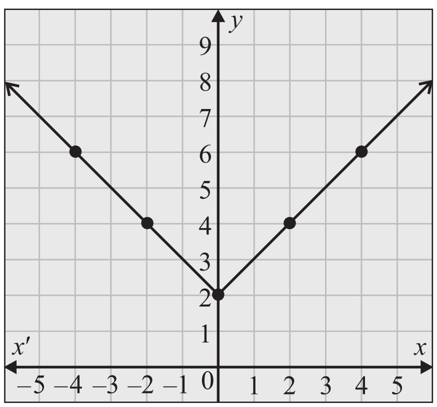


Figure 4.13

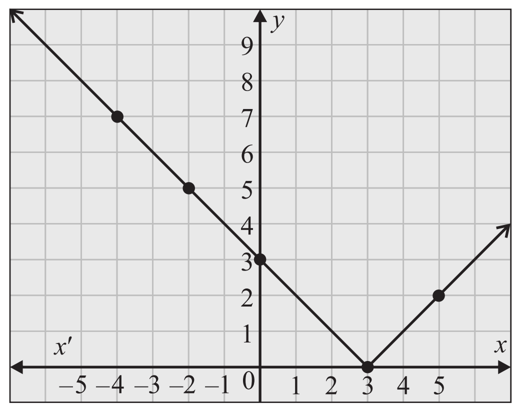


Figure 4.14

Example 11 Plot the graph of $f(x) = 2|x+1| - 3$.

Solution $f(x) = 2|x+1| - 3$

Step I Make a table of values.

x	−2	−1	0	0.5...
$f(x)$	−1	−3	−1	0 ...

Step II Plot the ordered pairs.

Step III Join the ordered pairs and draw the V-shaped graph of the function. Notice that the vertex of the graph of f is $(-1, -3)$ and the graph is symmetric about $x = -1$.

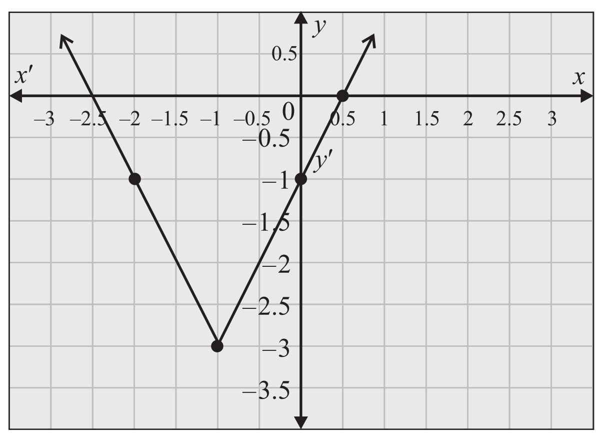


Figure 4.15

4.3 Solution of Absolute Valued Linear Equation in One Variable

An absolute value linear equation is an equation in which the absolute value is applied to a linear expression.

Example 12 Solve $2|x-3| = 16$ and express the solution on number line.

Solution

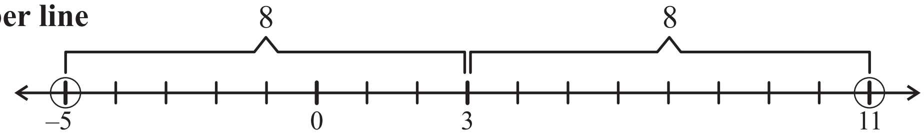
$$2|x-3| = 16$$
$$|x-3| = 8$$
$$\pm(x-3) = 8 \quad \text{or} \quad x-3 = \pm 8$$

Now $x - 3 = 8$ or $x - 3 = -8$

$x = 11$ or $x = -5$

Solution set = $\{11, -5\}$

Number line



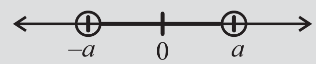
4.4 Solution of Absolute Value Linear Inequality in One Variable

An absolute value linear inequality in one variable is an inequality that involves the absolute value of a linear expression with only one variable. It has forms such as:

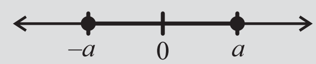
$$|ax+b| < 0,\ |ax+b| > 0\ ,\ |ax+b| \geq 0\ ,\ |ax+b| \leq 0$$

Note

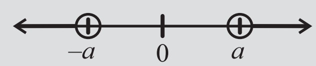
(i) $-a < x < a$ is equivalent to $(-a, a)$



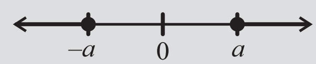
(ii) $-a \leq x \leq a$ is equivalent to $[-a, a]$



(iii) $x < -a$ or $x > a$ is equivalent to $(-\infty, -a) \cup (a, \infty)$



(iv) $x \leq -a$ or $x \geq a$ is equivalent to $(-\infty, -a] \cup [a, \infty)$



Example 13 Solve the inequality $|x - 1| \geq 5$ and express the solution on number line.

Solution

$$|x - 1| \geq 5$$

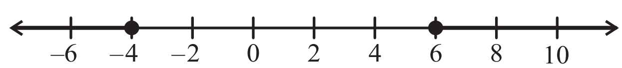
$$\pm(x-1) \geq 5$$

$-(x-1) \geq 5$ or $x - 1 \geq 5$

$x - 1 \leq -5$ or $x - 1 \geq 5$

$x \leq -4$ or $x \geq 6$

The solution is $(-\infty, -4] \cup [6, \infty)$



Example 14 Express the solution of $40|2x - 5| + 1 > 201$ on number line.

Solution First isolate the absolute value expression on one side of the inequality.

$$40|2x - 5| + 1 > 201$$

$$40|2x - 5| > 200$$

$$\frac{40|2x-5|}{40} > \frac{200}{40} \quad \text{(Divide each side by 40)}$$

$$|2x - 5| > 5$$

$$\pm(2x-5) > 5$$

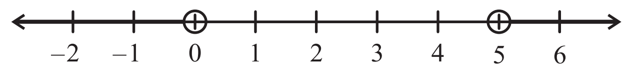
$-(2x-5)>5$ or $2x-5>5$

$2x-5<-5$ or $2x-5>5$

$2x<0$ or $2x>10$

$x<0$ or $x>5$

The solution is $(-\infty,0)\cup(5,\infty)$



EXERCISE 4.2

1. Plot graph for the following absolute valued functions:

   (i) $f(x)=|x-2|$ (ii) $f(x)=3|x+3|-4$

   (iii) $f(x)=5|x|$ (iv) $f(x)=|x+2|+3$

   (v) $f(x)=2|x+4|-3$ (vi) $f(x)=2|x+1|-6$

2. Solve and express the solution on number line:

   (i) $|x-2|=6$ (ii) $|2x+1|=7$ (iii) $|4x-9|=3$ (iv) $|7-2x|=1$

3. Solve and express the solution on number line:

   (i) $|5x+8|\le 3$ (ii) $|4x-12|\le 0$

   (iii) $|3-4x|>0$ (iv) $1-2\left|\frac{3}{2}x-5\right|>-3$

   (v) $|3x-2|\le 12$ (vi) $|1-2x|>5$

4.5 Solving Real-Life Problems Involving Function

Functions are widely used to model and solve real-life problems in areas such as finance, transportation and sales. In finance, functions can represent income, expenses, interest or profit over time. In transportation, functions can describe the relationship between distance, speed and time, helping to calculate travel times or fuel consumption. In sales, functions are used to model revenue, cost and profit based on the number of items sold. Solving these problems involves identifying the relevant variables, expressing their relationship as a function or equation and using algebraic methods to find unknown values, make decisions or predict outcomes.

Example 15 Let $E(n)=1{,}500n$ is the total earning (in Rs.) function, where n is the number of days. If labourer works 10 days, find his earning.

Solution Given that: $E(n) = 1{,}500\,n$

If the labourer works for 10 days, his earning will be $E(10) = 1{,}500 \times 10$

$= \text{Rs. } 15{,}000$

Example 16 Let $f(x) = 200 + 15x$ is a function of charging taxi fare (in Rs.), where x represents the number of kilometres. Find proposed fare for a distance of 50 km.

Solution

$$f(x) = 200 + 15x$$

$$\text{Required fare} = f(50) = 200 + 15 \times 50$$

$$= 200 + 750 = \text{Rs. } 950$$

4.6 Solving Real-Life Problems Involving Absolute Valued Functions

Absolute value functions are essential for solving real-life problems where distance, magnitude or fluctuations are involved, regardless of direction. In energy wave analysis, such as sound or seismic waves, absolute value functions model the amplitude or strength of a wave. In physics and engineering, magnitude is often expressed using absolute values to describe the size of a force, charge or vector, independent of direction. In distance-related problems, such as the location of an object from a reference point, absolute value functions measure how far one quantity is from another, ensuring the result is always non-negative. These functions help describe situations with symmetry or two-sided variation.

Example 17 A sensitive device operates best at $220V$ but can tolerate a variation of up to $5V$. The condition is modeled as: $|V - 220| \leq 5$. What is the range of acceptable voltages?

Solution

$$|V - 220| \leq 5$$

or $$-5 \leq V - 220 \leq 5$$

or $$-5 + 220 \leq V \leq 5 + 220 \quad \text{or} \quad 215 \leq V \leq 225$$

So, acceptable voltage is between $215V$ and $225V$.

Example 18 An earthquake sensor issues a warning if the recorded magnitude differs from a safe level of 4.5 by more than 1.2 units. The condition is modeled by $|M - 4.5| > 1.2$.

For which values of magnitude M will the sensor issue a warning?

Solution

$$|M - 4.5| > 1.2$$

$$\pm(M - 4.5) > 1.2$$

$$-(M - 4.5) > 1.2 \quad \text{or} \quad M - 4.5 > 1.2$$

$$M - 4.5 < -1.2 \quad \text{or} \quad M - 4.5 > 1.2$$

$$M < 3.3 \quad \text{or} \quad M > 5.7$$

Hence, the sensor issues a warning when $M < 3.3$ or $M > 5.7$.

EXERCISE 4.3

1. A function $B(t) = 5{,}000 + 200t$ represents the total balance (in rupees) after t months. What will be the balance after 6 months?
2. A function $f(k) = 150 + 20k$ represents the total fare (in rupees) for k kilometres. How much will the fare be for a 12 kilometres ride?
3. The cost of manufacturing fancy sofa set would be fixed charges Rs. 5500 which is modeled as $f(n) = 5500\,n$, where n is the number of sofa sets. Find the cost of 50 sofa sets.
4. A function $T(d) = \dfrac{d}{60}$ represents the time T in hours to travel a distance d kilometres. How long will it take to travel 180 km?
5. A company charges Rs. 100 for an encoding work. In addition, the company charges Rs. 5 per page of printed output. The model of function $f(x) = 100 + 5x$, where x represents the number of pages printed out. How much will company charge for 55 page encoding and printing work?
6. A chemical reaction is stable at $37°C$. The process must be stopped if the temperature deviates by more than $2.5°C$. The condition is modeled as: $|T - 37°| > 2.5°$, T be the temperature. For what temperature values must the process be stopped?
7. A factory produces metal rods that must be 2.5 metres long, with a tolerance of ± 0.04 metres. An absolute value inequality models this: $|x - 2.5| \le 0.04$. What is the range of acceptable lengths?
8. A machine part must be aligned so that its centre is exactly at 0. If it shifts more than 0.1 mm, the part is rejected. The model is given by $|x| > 0.1$. What positions of the centre cause rejection?

REVIEW EXERCISE 4

1. Four possible answers are given for the following questions. Choose the correct answer.

   (i) If $f(x) = \dfrac{5x-6}{3}$, then what is the value of $f(3)$?

   (a) -1 (b) 3 (c) 9 (d) 15

(ii) A function f from X to Y is represented by:

(a) $f: XY$ (b) $f: Y \to X$ (c) $f: X \to Y$ (d) $f: \frac{X}{Y}$

(iii) $(fog)(x) =$

(a) $(f+g)(x)$ (b) $(f-g)(x)$ (c) $f(g(x))$ (d) $f(x) \div g(x)$

(iv) If $f(x) = 2x + 3$, $g(x) = x + 1$, then $f(x) + g(x) =$

(a) $3x$ (b) $3x + 4$ (c) 4 (d) $2x^2 + 3$

(v) If $f(x) = 5x + 2$, $h(x) = 2x - 2$, then $f(x) - h(x) =$

(a) $3x$ (b) $5x^2 - 4$ (c) $3x + 4$ (d) $-3x - 4$

(vi) If $f(x) = 3x + 1$, $g(x) = 2x$, then $g(x) \times f(x) =$

(a) $6x + 2x$ (b) $5x^2 + 1$ (c) $x + 1$ (d) $6x^2 + 2x$

(vii) If $f(x) = x^2 - 4$, $g(x) = x + 2, x \neq -2$, then $\frac{f(x)}{g(x)} =$

(a) $\frac{1}{x-2}$ (b) $\frac{1}{x+2}$ (c) $x + 2$ (d) $x - 2$

(viii) What is the shape of the graph of an absolute value function?

(a) U-shaped (b) V-shaped

(c) L-shaped (d) M-shaped

(ix) If a graph represents a function, then every vertical line must intersect it at:

(a) 4 points (b) 3 points (c) 2 points (d) 1 point

(x) If $f(x) = x^3$, then $f(-2) =$

(a) -8 (v) 8 (c) 4 (d) -6

2. If $f(x) = 25 - x^2$ and $g(x) = 5 + x$, then find

(i) $f(x) + g(x)$ (ii) $f(x) - g(x)$ (iii) $f(x) \cdot g(x)$

(iv) $\frac{f(x)}{g(x)}$ (v) $f(7)$ (vi) $g(-8)$

3. If $f(x) = x^3$ and $g(x) = 14 + 2x$, then find

(i) $(fog)(x)$ (ii) $(gof)(x)$

(iii) $(fof)(x)$ (iv) $(gog)(x)$

4. Find $f^{-1}(x)$, if

(i) $f(x) = 9x - 1$ (ii) $f(x) = \dfrac{5}{x-1}$, $x \neq 1$

(iii) $f(x) = \sqrt{x-5}$, $x \geq 5$ (iv) $f(x) = \dfrac{3-x}{2}$

5. Plot graph for the following absolute valued functions:

(i) $f(x) = 7|x|$ (ii) $f(x) = |x+6| - 2$

6. Solve and express the solution on number line:

(i) $|3x - 2| = 1$ (ii) $|6x + 1| = 9$

7. Solve and express the solution on number line:

(i) $|7 - 2x| \leq 1$ (ii) $|6x + 18| \leq 24$

8. Given that $f(x) = 3x + 7$, find

(i) $f^{-1}(x)$ (ii) the value of x for which $f(x) = f^{-1}(x)$

9. A company earns a profit of Rs. 100 for each item sold after a fixed monthly expense of Rs. 10,000. A function $P(x) = 100x - 10{,}000$ represents the profit after selling x items. How many items must be sold to break even (profit = 0)?

10. A store offers a 15% discount on a product. A function $D(p) = 0.85p$ gives the selling price after the discount on an original price p. What is the selling price of an item that originally costs Rs. 2,000?

11. A GPS system is considered accurate if the actual position differs from the reported position by no more than 6 metres. If the actual location is at point $x = 100$, the allowed range is defined as: $|x - r| \leq 6$, where r is the reported location. What is the range of acceptable reported locations?

UNIT 5

Algebraic Fractions

Students' Learning Outcomes

After completing this unit, the students will be able to:

- Describe rational expressions.
- Factorize and simplify rational expressions.
- Demonstrate manipulation of algebraic fractions.
- Perform operations on rational expressions (limited to numerators and denominators that are monomials, binomials or trinomials).
- Solve different types of equations reducible to quadratic form of the type:
  - Exponential Equation
  - Reciprocal Equation
  - $ax^{2n} + bx^n + c = 0;\ a \neq 0, n = 2$
  - $(x+a)(x+b)(x+c)(x+d) = k$, where $a+b = c+d$
- Apply the concept of rational equations (limited to numerators and denominators that are monomials, binomials or trinomials) to real-world problems (such as the amount of work a person can do in certain amount of time, rates and work).

INTRODUCTION

This unit introduces the concept of rational expressions and equations, focusing on how to describe, factorize and simplify them effectively. Students will explore the manipulation of algebraic fractions and perform various operations on rational expressions involving monomials, binomials and trinomials. The unit also covers solving complex equations that are reducible to quadratic form, including exponential and reciprocal equations. Finally, students will apply these mathematical concepts to solve real-world problems involving work rates, shared tasks and time-related scenarios.

5.1 Algebraic Fraction

In Mathematics, an algebraic fraction is a fraction where the numerator, denominator or both contains algebraic expressions.

For example, $\frac{3}{x}, \frac{2x}{5y}, \frac{2x+9}{10x}, \frac{x^2+8}{x^2+x+3}$ are all algebraic fractions. The denominator is never zero.

5.1.1 Rational Expression

A rational expression is the qoutient $\frac{P(x)}{Q(x)}$, where $Q(x) \neq 0$, of two polynomials $P(x)$ and $Q(x)$. For example, $\frac{5}{x-2}$, $\frac{x+5}{5}$, x^2+5x+4, $\frac{x+1}{x+2}$ are all rational algebraic expressions.

5.1.2 Simplification of Rational Expressions

Simplification of a rational expression means reducing the fraction to its lowest form by cancelling out the common factors of the numerator and the denominators.

Example 1 Reduce to lowest form $\frac{24a^3c^2x^2}{18a^3x^2-12a^2x^3}$

Solution

$$\frac{24a^3c^2x^2}{18a^3x^2-12a^2x^3}$$

$$= \frac{24a^3c^2x^2}{6a^2x^2(3a-2x)} = \frac{4a^{3-2}c^2\,x^{2-2}}{3a-2x}$$

$$= \frac{4ac^2}{3a-2x}$$

> **Note**
> Factorise where possible because there might be a factor that will be cancelled.

Example 2 Reduce to lowest form $\frac{6x^2-8xy}{9xy-12y^2}$

Solution

$$\frac{6x^2-8xy}{9xy-12y^2}$$

$$= \frac{2x(3x-4y)}{3y(3x-4y)} = \frac{2x}{3y}$$

EXERCISE 5.1

1. Reduce the following rational expressions to lowest forms:

(i) $\frac{3a^2-6ab}{2a^2b-4ab^2}$ (ii) $\frac{abx+bx^2}{acx+cx^2}$ (iii) $\frac{ac}{a^2x^2-ax}$

(iv) $\frac{15a^2b^2c}{100(a^2-a^2b)}$ (v) $\frac{4x^2-9y^2}{4x^2+6xy}$ (vi) $\frac{20(x^3-y^3)}{5x^2+5xy+5y^2}$

(vii) $\frac{x(2a^2-3ax)}{a(4a^2x-9x^3)}$ (viii) $\frac{x^2-5x}{x^2-4x-5}$ (ix) $\frac{3x^2+6x}{x^2+4x+4}$

(x) $\dfrac{x^2+xy-2y^2}{x^3-y^3}$ (xi) $\dfrac{2x^2+17x+21}{3x^2+26x+35}$

5.2 Manipulation of Rational Expressions

Definition:

Manipulation is to apply operations like addition, subtraction, multiplication and division while maintaining the equality or value of the expression.

5.2.1 Addition and Subtraction of Rational Expressions with Unlike Denominators

Steps:

(i) Determine the Least Common Multiple (LCM) of the denominators.

(ii) Rewrite each fraction as an equivalent fraction with the LCM obtained in step (i)

(iii) Follow the same step for doing addition or subtraction of rational expression with like denominators.

Skilled Practice!

Simplify:

(i) $\dfrac{4a}{a^2-1}-\dfrac{a+1}{a-1}$

(ii) $\dfrac{a^3}{a-b}-\dfrac{b^3}{b-a}$

(iii) $\dfrac{4-x^2}{x}-\dfrac{x+2}{x+2}+5$

Example 3 Simplify: $\dfrac{x-2y}{4}+\dfrac{x+y}{6}-\dfrac{2x-y}{15}$

Solution LCM of 4, 6 and 15 = 60

$$\frac{x-2y}{4}+\frac{x+y}{6}-\frac{2x-y}{15}$$

$$=\frac{15(x-2y)+10(x+y)-4(2x-y)}{60}$$

$$=\frac{15x-30y+10x+10y-8x+4y}{60}=\frac{17x-16y}{60}$$

Example 4 Simplify: $\dfrac{4}{x^2-2xy-3y^2}+\dfrac{1}{x^2+3xy+2y^2}$

Solution

$$\frac{4}{x^2-2xy-3y^2}+\frac{1}{x^2+3xy+2y^2}$$

$$=\frac{4}{x^2+xy-3xy-3y^2}+\frac{1}{x^2+2xy+xy+2y^2}$$

$$=\frac{4}{x(x+y)-3y(x+y)}+\frac{1}{x(x+2y)+y(x+2y)}$$

$$=\frac{4}{(x-3y)(x+y)}+\frac{1}{(x+y)(x+2y)}$$

$$=\frac{4(x+2y)+1(x-3y)}{(x-3y)(x+y)(x+2y)}$$ (Taking LCM)

$$= \frac{4x+8y+x-3y}{(x-3y)(x+y)(x+2y)} = \frac{5x+5y}{(x-3y)(x+y)(x+2y)}$$

$$= \frac{5(x+y)}{(x-3y)(x+y)(x+2y)} = \frac{5}{(x-3y)(x+2y)}$$

Example 5 Simplify: $\frac{1}{x^2-5x+6} + \frac{1}{x^2-3x+2} - \frac{1}{x^2-8x+15}$

Solution

$$\frac{1}{x^2-5x+6} + \frac{1}{x^2-3x+2} - \frac{1}{x^2-8x+15}$$

$$= \frac{1}{(x-2)(x-3)} + \frac{1}{(x-2)(x-1)} - \frac{1}{(x-5)(x-3)}$$

$$= \frac{(x-1)(x-5)+(x-3)(x-5)-(x-1)(x-2)}{(x-1)(x-2)(x-3)(x-5)} \quad \text{(Taking LCM)}$$

$$= \frac{(x^2-6x+5)+(x^2-8x+15)-(x^2-3x+2)}{(x-1)(x-2)(x-3)(x-5)}$$

$$= \frac{x^2-6x+5+x^2-8x+15-x^2+3x-2}{(x-1)(x-2)(x-3)(x-5)}$$

$$= \frac{x^2-11x+18}{(x-1)(x-2)(x-3)(x-5)}$$

$$= \frac{(x-9)(x-2)}{(x-1)(x-2)(x-3)(x-5)} = \frac{x-9}{(x-1)(x-3)(x-5)}$$

EXERCISE 5.2

1. Simplify:

(i) $\frac{3}{x-y} + \frac{1}{y-x}$

(ii) $\frac{4x}{x^2-3x+2} - \frac{4}{1-x} - \frac{5}{x-2}$

(iii) $\frac{x+y}{12x-6y} + \frac{x-y}{18x-9y}$

(iv) $\frac{2}{x+2y} - \frac{x-6y}{x^2-4y^2}$

(v) $\frac{1}{x+1} - \frac{2}{x+2} - \frac{2x+3}{x^2+3x+2}$

(vi) $\frac{5}{x^2+x-6} - \frac{1}{2x^2-7x+6}$

(vii) $\frac{2x}{x+y} - \frac{y}{x-y} - \frac{2y^2}{x^2-y^2}$

(viii) $\frac{7}{2x^2-x-6} - \frac{8}{3x^2-4x-4}$

(ix) $\dfrac{x+2}{x^2-x-12}-\dfrac{x}{x^2+6x+9}$

(x) $\dfrac{1}{x^2-4x+3}+\dfrac{1}{x^2-3x+2}-\dfrac{1}{x^2-5x+6}$ (xi) $\dfrac{x^3}{x-y}+\dfrac{y^3}{y-x}$

2. Subtract $\dfrac{1}{x^2+2}$ from $\dfrac{2x^3+x^2+3}{(x^2+2)^2}$

5.2.2 Multiplication and Division of Rational Expressions

a Multiplication

Example 6 Simplify: $\dfrac{x^2-5x+6}{x^2-25}\times\dfrac{x^2+5x}{2x-6}$

Solution

$$\frac{x^2-5x+6}{x^2-25}\times\frac{x^2+5x}{2x-6}=\frac{x^2-2x-3x+6}{x^2-5^2}\times\frac{x(x+5)}{2(x-3)}$$

$$=\frac{(x-2)(x-3)}{(x-5)(x+5)}\times\frac{x(x+5)}{2(x-3)}=\frac{x(x-2)}{2(x-5)}$$

Example 7 Simplify: $\dfrac{3a^2-6ab}{6ab}\times\dfrac{10b^2}{4ab-8b^2}$

Solution

$$\frac{3a^2-6ab}{6ab}\times\frac{10b^2}{4ab-8b^2}$$

$$=\frac{3a(a-2b)}{6ab}\times\frac{10b^2}{4b(a-2b)}$$

$$=\frac{30}{24}=\frac{5}{4}$$

Skilled Practice!

Simplify:

(i) $\dfrac{32x^2y^2z^5}{3x^3y^3}\cdot\dfrac{243x^6}{x^4y^2z}$

(ii) $\dfrac{5a^2}{4a-8}\cdot\dfrac{6a-12}{10a}$

b Division

To divide by a rational expression, multiply by its reciprocal as in multiplication.

Example 8 $\dfrac{2a^2-2c^2}{15a^2c^2}\div\dfrac{3a+3c}{5ac}$

Solution

$$\frac{2a^2-2c^2}{15a^2c^2}\div\frac{3a+3c}{5ac}$$

$$=\frac{2a^2-2c^2}{15a^2c^2}\times\frac{5ac}{3a+3c}$$

$$=\frac{2(a^2-c^2)}{15a^2c^2}\times\frac{5ac}{3(a+c)}$$

$$= \frac{2(a+c)(a-c)}{15a^2c^2} \times \frac{5ac}{3(a+c)} = \frac{2(a-c)}{9ac}$$

Example 9 Simplify: $\frac{x^3-a^3}{x^2-2bx+b^2} \times \frac{x^2-bx-cx+bc}{ax-x^2} \div \frac{x^2-cx}{x-b}$

Solution $\frac{x^3-a^3}{x^2-2bx+b^2} \times \frac{x^2-bx-cx+bc}{ax-x^2} \div \frac{x^2-cx}{x-b}$

$$= \frac{x^3-a^3}{x^2-2bx+b^2} \times \frac{x^2-bx-cx+bc}{ax-x^2} \times \frac{x-b}{x^2-cx}$$

$$= \frac{(x-a)(x^2+ax+a^2)}{(x-b)(x-b)} \times \frac{(x-b)(x-c)}{x(a-x)} \times \frac{x-b}{x(x-c)}$$

$$= -(a-x)(x^2+ax+a^2) \times \frac{1}{x(a-x)} \times \frac{1}{x}$$

$$= \frac{-(x^2+ax+a^2)}{x^2}$$

EXERCISE 5.3

1. Simplify:

(i) $\frac{24lm}{5l+10m} \times \frac{5l^2-20m^2}{16mn}$ (ii) $\frac{x^2-3x+2}{x^2+3x-4} \times \frac{2x^2+8x}{3x+6}$

(iii) $\frac{a^2-4b^2}{a^2+2ba} \times \frac{2a^2+10ab}{a^2+3ab-10b^2}$ (iv) $\frac{(a+b)^2-c^2}{(a+c)^2-b^2} \times \frac{a^2-(b-c)^2}{(a+c)^2-c^2}$

(v) $\frac{x^3-8}{x^2-4} \div \frac{x^2+2x+4}{x^2+4x+4}$ (vi) $\frac{\left(a^2+ab\right)^2}{\left(a^2-ab\right)^2} \div \left(\frac{a+b}{a-b}\right)^2$

(vii) $\frac{x^2-x-6}{x^2-x-20} \div \left\{\frac{x^3-3x^2}{x^2+4x} \times \frac{x^3-5x^2-14x}{x^2-12x+35}\right\}$

(viii) $\frac{3x^2-3xy}{10x^2+10xy-20y^2} \times \frac{5xy+10y^2}{6x^2+6xy+6y^2} \div \frac{2x^2-2y^2}{4x^3-4y^3}$

(ix) $\frac{2x^2-98}{x^3-125} \times \frac{x^2-3x-10}{3x-21} \div \frac{x^2+5x-14}{x^2+5x+25}$

(x) $\frac{x^2-2x}{2x+6} \times \frac{x^2+x-6}{x^2-5x} \times \frac{6x-30}{x^2-2x}$

5.3 Solution of Equations Reducible to Quadratic Equations

Some algebraic equations are not in standard quadratic form, but they can be reduced or transformed into a quadratic equation by suitable substitution. Such type of equations are called equations reducible to quadratic equations.

Type (i) $ax^{2n} + bx^n + c = 0; \quad a \neq 0, n = 2$

Example 10 Solve: $2x^4 - 11x^2 + 12 = 0$

Solution

$$2x^4 - 11x^2 + 12 = 0$$

Let $x^2 = y$, then $x^4 = y^2$, so that

$$2y^2 - 11y + 12 = 0$$
$$2y^2 - 3y - 8y + 12 = 0$$
$$y(2y - 3) - 4(2y - 3) = 0$$
$$(2y - 3)(y - 4) = 0$$
$$2y - 3 = 0, \quad y - 4 = 0$$
$$y = \frac{3}{2}, \quad y = 4$$

Put $x^2 = y$

$$\therefore \quad x^2 = \frac{3}{2}, \quad x^2 = 4$$

$$x = \pm\sqrt{\frac{3}{2}} = \pm\frac{\sqrt{6}}{2}, \quad x = \pm 2 \Rightarrow S.S = \left\{\pm 2, \pm\frac{\sqrt{6}}{2}\right\}$$

Type (ii) **Exponential Equation**

Equations in which variable occurs in exponents are called exponential equations.

For example, $2^x - 3 \cdot 2^{2x} = -56$

Example 11 Solve: $4^x - 3 \cdot 2^{x+3} + 128 = 0$

Solution

$$4^x - 3 \cdot 2^{x+3} + 128 = 0$$

The equation can be written as

$$\left(2^2\right)^x - 3 \cdot 2^x \cdot 2^3 + 128 = 0$$

or $\quad 2^{2x} - 24 \cdot 2^x + 128 = 0 \quad$ ...(i)

Let $\quad 2^x = y$, then $2^{2x} = y^2$

$\therefore$ (i) becomes

$$y^2 - 24y + 128 = 0$$
$$\Rightarrow \quad y^2 - 16y - 8y + 128 = 0$$
$$y(y - 16) - 8(y - 16) = 0$$
$$(y - 16)(y - 8) = 0$$
$$y - 16 = 0, y - 8 = 0$$
$$y = 16, y = 8$$

But $y = 2^x$

$$2^x = 16 \quad , \quad 2^x = 8$$
$$2^x = 2^4 \quad , \quad 2^x = 2^3$$
$$x = 4 \quad , \quad x = 3$$

$\therefore$ Solution set $= \{4, 3\}$

Type (iii) Reciprocal Equation

An equation which remains unchanged when variable is replaced by its reciprocal.

For example, $\left(x^2 + \frac{1}{x^2}\right) - 3\left(x + \frac{1}{x}\right) + 4 = 0$

Example 12 Solve: $3\left(x^2 + \frac{1}{x^2}\right) + 4\left(x + \frac{1}{x}\right) - 14 = 0$

Solution

$$3\left(x^2 + \frac{1}{x^2}\right) + 4\left(x + \frac{1}{x}\right) - 14 = 0$$

Let $x + \frac{1}{x} = y$ ...(i)

So $\left(x + \frac{1}{x}\right)^2 = y^2$

$\Rightarrow \quad x^2 + \frac{1}{x^2} + 2 = y^2$

or $\quad x^2 + \frac{1}{x^2} = y^2 - 2$

The given equation becomes

$$3(y^2 - 2) + 4y - 14 = 0$$
$$3y^2 - 6 + 4y - 14 = 0$$
$$3y^2 + 4y - 20 = 0$$
$$3y^2 + 10y - 6y - 20 = 0$$
$$y(3y + 10) - 2(3y + 10) = 0$$
$$(3y + 10)(y - 2) = 0$$
$$3y + 10 = 0 \quad , \quad y - 2 = 0$$

$$y = -\frac{10}{3}, \; y = 2$$

Put $y = -\frac{10}{3}$ in (i), we get

$$x + \frac{1}{x} = -\frac{10}{3}$$

$$3x^2 + 3 = -10x \qquad \text{[Multiplying by } 3x\text{]}$$

$$3x^2 + 10x + 3 = 0$$

$$3x^2 + 9x + x + 3 = 0$$

$$3x(x + 3) + 1(x + 3) = 0$$

$$(x + 3)(3x + 1) = 0$$

$$x + 3 = 0, \; 3x + 1 = 0$$

$$x = -3, \; -\frac{1}{3}$$

Now, put $y = 2$ in (i), we get

$$x + \frac{1}{x} = 2 \qquad \text{[Multiplying by } x\text{]}$$

$$x^2 + 1 = 2x$$

$$x^2 - 2x + 1 = 0 \qquad \text{[Transposing]}$$

$$(x - 1)^2 = 0$$

$$x - 1 = 0$$

$$\Rightarrow \qquad x = 1$$

$$\text{Solution set} = \left\{1, -3, -\frac{1}{3}\right\}$$

Type (iv) **$(x + a)(x + b)(x + c)(x + d) = k$, where $a + b = c + d$ and k is any constant.**

Example 13 Solve: $(x + 2)(x + 3)(x - 5)(x - 6) = -12$

Solution Here $2 - 5 = 3 - 6$

Re-arrange the factors, we get

$$[(x + 2)(x - 5)][(x + 3)(x - 6)] = -12$$

$$(x^2 - 3x - 10)(x^2 - 3x - 18) = -12 \qquad \text{...(i)}$$

$$\text{Let} \qquad x^2 - 3x = y \qquad \text{...(ii)}$$

(i) becomes: $(y-10)(y-18) = -12$

$$y^2 - 28y + 180 = -12$$

$$y^2 - 28y + 192 = 0 \qquad \text{[Transposing]}$$

$$y^2 - 12y - 16y + 192 = 0$$

$$y(y-12) - 16(y-12) = 0$$

$$(y-12)(y-16) = 0$$

$$y - 12 = 0,\ y - 16 = 0$$

$$y = 12, 16$$

Put $y = 12$ in (ii), we get

$$x^2 - 3x = 12 \quad \text{or}$$

$$x^2 - 3x - 12 = 0$$

$$x = \frac{3 \pm \sqrt{(-3)^2 - 4 \times 1 \times (-12)}}{2(1)}$$

$$= \frac{3 \pm \sqrt{57}}{2}$$

Now put $y = 16$ in (ii), we get

$$x^2 - 3x = 16$$

$$x^2 - 3x - 16 = 0$$

$$x = \frac{3 \pm \sqrt{(-3)^2 - 4 \times 1 \times (-16)}}{2(1)} = \frac{3 \pm \sqrt{73}}{2}$$

$\therefore$ Solution set $= \left\{ \dfrac{3 \pm \sqrt{57}}{2}, \dfrac{3 \pm \sqrt{73}}{2} \right\}$

> **Skilled Practice!**
> Solve:
> $(x+1)(x+2)(x-5)(x-6) = 144$

EXERCISE 5.4

1. Solve the following equations:

(i) $5x^4 - 19x^2 + 12 = 0$ (ii) $4x^4 - 27x^2 + 18 = 0$

(iii) $5x^4 - 22x^2 + 8 = 0$ (iv) $4 \cdot 2^{2x+1} - 9 \cdot 2^x + 1 = 0$

(v) $4^{1+x} + 4^{1-x} - 10 = 0$ (vi) $3^x + 3^{3-x} - 12 = 0$

(vii) $5^{1+3x} + 5^{2-3x} - 126 = 0$ (viii) $\left(x^2 + \dfrac{1}{x^2}\right) + 2\left(x + \dfrac{1}{x}\right) - 33 = 0$

(ix) $2\left(x^2+\frac{1}{x^2}\right)-25\left(x-\frac{1}{x}\right)-17=0$ (x) $2x^4-5x^3-14x^2-5x+2=0$

(xi) $(x+1)(x+2)(x-4)(x-5)+8=0$ (xii) $(x+3)(x+4)(x+5)(x+6)=-1$

5.4 Real World Problems

Example 14 A tradesman buys a number of articles for Rs. 300. Four of them are broken in transit but by selling the remaining at a profit of Rs. 1.25 each, he gains Rs.350 altogether. How many articles did he buy?

Solution Suppose he buys x articles for Rs. 300

$$\therefore \quad \text{Each cost} = \text{Rs. } \frac{300}{x}$$

He sold $(x-4)$ articles for Rs. 350

$$\text{Each sold for Rs. } \frac{350}{x-4}$$

Now, $$\frac{350}{x-4}-\frac{300}{x}=1.25$$

$$\frac{350}{x-4}-\frac{300}{x}=\frac{125}{100}=\frac{5}{4}$$

Multiplying both sides by $4x(x-4)$, we get

$$350(4x)-300(x-4)(4)=5x(x-4)$$

$$1400x-1200x+4800=5x^2-20x$$

$$5x^2-20x+1200x-1400x-4800=0$$

$$5x^2-220x-4800=0 \quad \text{(Dividing by 5)}$$

$$x^2-44x-960=0$$

$$x^2-60x+16x-960=0$$

$$x(x-60)+16(x-60)=0$$

$$(x-60)(x+16)=0$$

$$x-60=0,\ x+16=0$$

$$\therefore \quad x=60,\ -16$$

Negative value is rejected.

Number of articles purchased = 60

Challenge!

Amna bought 50 kg of fruits consisting of mangoes and grapes. She paid twice as much per kg for the mango as she did for the grapes. If Amna bought Rs. 18000 worth of mangoes and Rs. 6000 worth grapes, then how many kg's of each fruit did she buy?

Example 15 A train is scheduled to cover a distance of 120 km at a certain average speed, owing to a service checkup this average speed is reduced by 5 km/h, and in consequence the journey takes 20 minutes more than the scheduled time. What was the scheduled speed?

Solution Let x km/h be scheduled speed;

then scheduled time for 120 km $= \dfrac{120}{x}$ hrs

$$20 \text{ minute} = \frac{20}{60} \text{ hrs} = \frac{1}{3} \text{ hrs}$$

But $(x - 5)$ km/h is actual speed: therefore, actual time for 120 km $= \dfrac{120}{x-5}$

$$\therefore \quad \frac{120}{x-5} - \frac{120}{x} = \frac{1}{3}$$

Multiply each term by $3x(x - 5)$

$$360x - 360(x - 5) = x(x - 5)$$

$$x^2 - 5x - 1800 = 0$$

$$(x - 45)(x + 40) = 0$$

$$x - 45 = 0, x + 40 = 0$$

$$\therefore \quad x = 45, \ x = -40$$

Negative value is rejected.

The scheduled speed is 45 km/h.

EXERCISE 5.5

1. A train travels a distance of 240 km at a uniform rate, if it had finished 4 km an hour slower, it would have taken 2 hours more over the journey. Find its rate of travelling.
2. Arshia and Ibraheem complete a job together in 4 hours. If Arshia takes 6 hours, then find how much time will Ibraheem take?
3. One pipe fills water in 5 hours, another in 8 hours. Second pipe closed after 2 hours. Find the total time.
4. Huria can complete a project in 12 hours by working alone. If Abdul Hadi join her and they finish it together in 5 hours, how long would it take Abdul Hadi to do the project alone?
5. Two cars start from opposite towns and head towards each other. The distance between them is 240 km. One travels at x km/h and the other at $(x + 10)$ km/h. They meet after 2 hours. Find the speed of each car.
6. Rashid can paint a house in 6 days, but if he gets a helper he can do it in 4 days. How long would it take the helper to paint the house alone?

7. Ishmal runs 10 km in $\frac{2x}{x+4}$ hours. If her average speed is 8 km/h, find the value of x.

8. Fahad runs 600 m at a certain pace, and then doubling his pace, does another 600 m. If he took $2\frac{1}{2}$ to cover the distance 1200 m, find the pace he started at, in metres per seconds.

9. A cyclist travels 30 km at a certain speed. If the speed had been 5 km/h faster, the journey would have taken 1 hour less. Find the speed.

REVIEW EXERCISE 5

1. Four possible answers are given for the following questions. Choose the correct answer:

(i) The expression $\frac{2x-1}{x^2+4}$ is:

(a) a polynomial (b) an algebraic fraction
(c) a numerical fraction (d) an equation

(ii) Lowest form of $\frac{18a^2bc^2}{12ac}$ is:

(a) $\frac{9abc}{6}$ (b) $\frac{3abc}{2}$ (c) $\frac{18b}{12}$ (d) $\frac{3}{2}$

(iii) Lowest form of $\frac{15xyz}{10}$ is:

(a) $\frac{3xyz}{2}$ (b) $\frac{3}{2}$ (c) xyz (d) $\frac{xyz}{2}$

(iv) Simplified form of $\frac{3x+1}{5}+\frac{2x+3}{5}$ is:

(a) $\frac{5x+4}{10}$ (b) $\frac{5x+4}{5}$ (c) $\frac{5x+4}{25}$ (d) $5x+4$

(v) Simplified form of $\left(\frac{x^2-y^2}{x+y}\right)(x-y)$ is:

(a) $(x+y)^2$ (b) x^2+y^2 (c) $x-y$ (d) $(x-y)^2$

(vi) $(x^3-y^3)\div(x-y)$ in simplified form is:

(a) x^2-xy+y^2 (b) x^2+xy+y^2 (c) x^2-y^2 (d) x^2+y^2

(vii) An equation of the form $\frac{5}{x}+\frac{1-x}{3}=\frac{1}{6x}$ is called:

(a) radical equation (b) reciprocal equation

(c) fractional equation (d) exponential equation

(viii) An equation of the form $5^x+64\cdot 5^{-x}-20=0$ is called:

(a) radical equation (b) exponential equation

(c) reciprocal equation (d) fractional equation

(ix) Roots of $y^2 - 24y + 128 = 0$ are:

(a) 8, –16 (b) 8, 16 (c) –8, 16 (d) –8, –16

(x) Linear factors of $3x^2 + 10x + 3 = 0$ are:

(a) $(x + 3)$, $(3x + 1)$ (b) $(x + 3)$, $(3x - 1)$

(c) $(x - 3)$, $(3x + 1)$ (d) $(x - 3)$, $(3x - 1)$

2. Reduce the following to the lowest form:

(i) $\frac{x^2-7x+12}{x^2-6x+9}$ (ii) $\frac{x^2+3x-18}{7x-21}$

3. Simplify:

(i) $\frac{5}{x^2+x-6} - \frac{1}{2x^2-7x+6}$ (ii) $\frac{x^3-27}{x^2-9}\div\frac{x^2+3x+9}{x^2+6x+9}$

(iii) $\frac{a^2-(b-c)^2}{(a+b)^2-c^2}\times\frac{a^2-(b+c)^2}{(a-b)^2-c^2}$

4. Solve the following equations:

(i) $x^4-16x^2+63=0$ (ii) $4x^4-16x^2+15=0$

(iii) $3^{2x} - 12\cdot 3^x + 27 = 0$ (iv) $\left(x^2+\frac{1}{x^2}\right)-3\left(x+\frac{1}{x}\right)-2=0$

(v) $\left(x^2+\frac{1}{x^2}\right)+\left(x+\frac{1}{x}\right)=4$ (vi) $(x + 9)(x - 3)(x - 7)(x + 5) = 385$

(vii) $(x - 1)(x - 2)(x + 5)(x - 8) = -360$

5. Ahmad takes 2 hours to paint 50 glasses. Faiza takes 2 hours to paint 45 glasses. Working together, how long should it take them to paint 150 glasses?

6. A tap can fill a tank in 6 hours. Another tap can fill it in 9 hours. If both taps are opened together, how long will it take to fill the empty tank?

7. Maham bought a certain number of toys for Rs. 300. If each toy had cost Rs. 5 less, she could have bought two more for the same amount. How many toys did she buy?

UNIT 6

Vectors in Plane

Students' Learning Outcomes

After completing this unit, the students will be able to:

- Introduce a rectangular coordinate system in the plane.
- Represent vectors as directed line segment.
- Express a vector in terms of two non-zero and non-parallel coplanar vectors.
- Express a vector in terms of a position vector.
- Express translation by a vector.
- Find the magnitude of a vector.
- Add and subtract vectors.
- Multiply a vector by a scalar.
- Solve geometrical problems involving the use of vectors.
- Apply concepts from geometrical problems involving the use of vectors (such as parallel and perpendicular lines in geometrical shapes, vector projectile motion, crosswinds aviation, military usage and designing roller coasters).

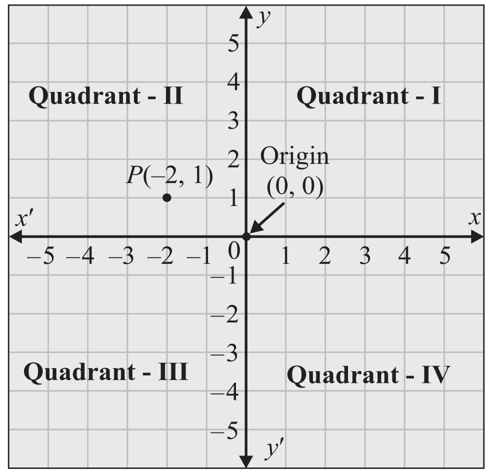
INTRODUCTION

This unit introduces the foundational concepts of vectors and the rectangular coordinate system in the plane, providing a powerful mathematical tool to describe and analyze movement, direction and position. Students will learn how to represent vectors as directed line segments and express them using position vectors or combinations of two non-zero, non-parallel coplanar vectors. The unit covers essential operations such as finding the magnitude of a vector, vector addition and subtraction and scalar multiplication, enabling students to solve both abstract and practical problems. Emphasis is placed on applying vector concepts to real-world scenarios, including geometrical analysis, projectile motion, navigation in crosswinds, military applications and engineering designs like roller coasters, equipping students with the tools to interpret and model physical phenomena using vectors.

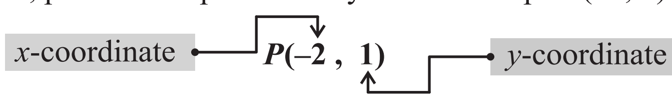
6.1 Rectangular Coordinate Plane

A rectangular coordinate plane is formed by the intersection of a horizontal number line called the **axis of *x*** or shortly ***x*-axis** and a vertical number line called the **axis of *y*** or shortly ***y*-axis**.

The axes meet at a point called the **origin** and divide the coordinate plane into four **quadrants.**



Points in a coordinate plane are represented by **ordered pairs.** The first element is the ***x*-coordinate** and the second element is the ***y*-coordinate.** In the given figure, point *P* is represented by the ordered pair (–2, 1).

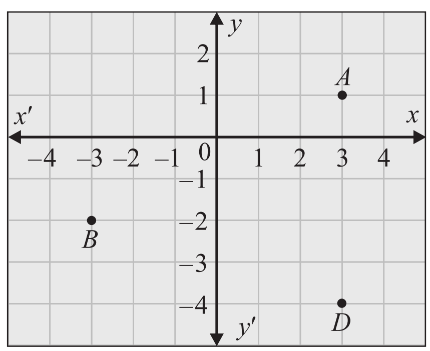


Example 1 Write the coordinates of the following points from the given figure:

(i) *A*

(ii) *B*

(iii) *D*



Solution

(i) Point *A* is 3 units to the right of the origin and 1 unit up. So, the *x*-coordinate is 3 and the *y*-coordinate is 1. The coordinates of *A* are (3, 1).

(ii) Point *B* is 3 units to the left of the origin and 2 units down. So, the *x*-coordinate is –3 and the *y*-coordinate is –2. The coordinates of *B* are (–3, –2).

(iii) Point *D* is 3 units to the right of the origin and 4 units down. So, the *x*-coordinate is 3 and the *y*-coordinate is – 4. The coordinates of *D* are (3, – 4).

6.2 Scalars and Vectors

There are two types of physical quantities that are used in Physics and Mathematics.

Scalar: A scalar is a quantity having magnitude but no direction. Mass, time, volume and power are the examples of scalar quantities.

Vector: A vector is a quantity having both magnitude and direction. Displacement, velocity, acceleration, force and momentum are examples of vector quantities.

6.2.1 Representation of a Vector as Directed Line Segment

A vector has a tail and a tip. Consider the diagram in Figure 6.1.

The tail point A is called the **initial point** and the tip point B is called the **terminal point** of the vector $\vec{AB}$. The initial point of a vector is also taken as **origin** of the vector.

We can also represent a vector with $\underline{a}$, $\vec{a}$ or $\boldsymbol{a}$.

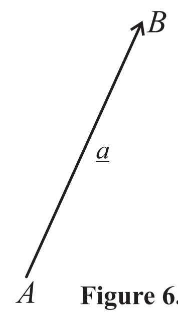


Figure 6.1

(i) A vector that has the same magnitude as the given vector but opposite direction is called **negative of a vector** as shown in Figure 6.2.

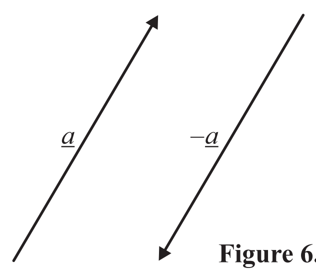


Figure 6.2

(ii) Two vectors $\underline{a}$ and $\underline{b}$ are **equal** if they have the same magnitude and direction regardless of the position of their initial points. Thus $\underline{a} = \underline{b}$ in Figure 6.3.

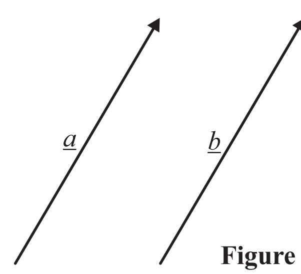


Figure 6.3

(iii) A vector whose initial and terminal points are coincident is called a **zero or null vector.** Zero vector has zero magnitude and it is parallel to every vector.

6.2.2 Vectors and Coordinates

When a Cartesian coordinate system is added to a vector diagram, it becomes possible to assign coordinates to both the initial and terminal points (tail and head) of each vector. This allows vectors to be expressed in terms of their coordinates, which can then be used effectively in calculations. To simplify this

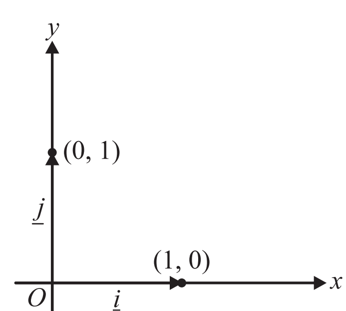


Figure 6.4

process, we introduce two special vectors along the coordinate axes: $\underset{\sim}{i}$ along the x-axis and $\underset{\sim}{j}$ along the y-axis. Both vectors originate at the origin and point in the positive direction of their respective axes.

6.2.3 Position Vectors

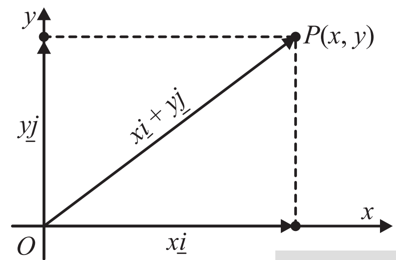


Figure 6.5

Let O (0, 0) be the origin and $P(x, y)$ be any point in the plane. Then the vector $\overrightarrow{OP} = (x-0)\underset{\sim}{i} + (y-0)\underset{\sim}{j} = x\underset{\sim}{i} + y\underset{\sim}{j}$ is called the **position vector** of P with respect to O. x and y are horizontal and vertical components respectively.

6.2.4 Magnitude of a Vector

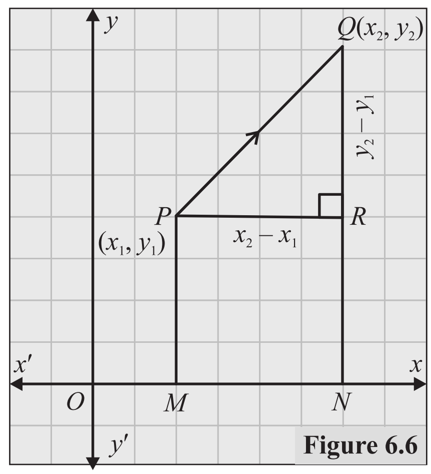


Figure 6.6

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the given points as shown in figure 6.6.

Draw $\overline{PM}$ and $\overline{QN}$ perpendiculars on the x–axis.

Draw $\overline{PR} \perp \overline{QN}$

Now, $|\overline{PR}| = |x_2 - x_1|$

$|\overline{QR}| = |y_2 - y_1|$

> **Note**
> Magnitude of a vector is also known as norm or length of a vector.

PRQ is a right angled triangle.

$$\therefore \quad |\overrightarrow{PQ}| = \sqrt{(\overline{PR})^2 + (\overline{QR})^2}$$

$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

or $$|\overrightarrow{PQ}| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

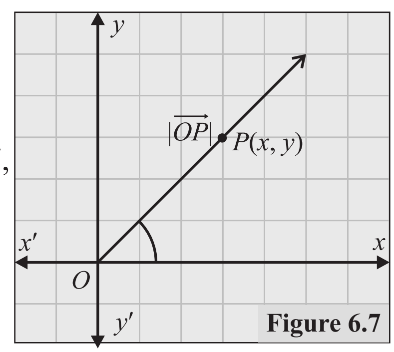


Figure 6.7

If $P(x, y)$ be any point in the plane as shown in Figure 6.7, then the magnitude of position vector $\overrightarrow{OP} = x\underset{\sim}{i} + y\underset{\sim}{j}$ from the origin is $|\overrightarrow{OP}| = \sqrt{x^2 + y^2}$.

Example 2 If $P(2, 4)$ and $Q(-5, 3)$, then find

(i) $\overrightarrow{PQ}$ in the form $x\underset{\sim}{i} + y\underset{\sim}{j}$ (ii) $|\overrightarrow{PQ}|$

Solution (i) $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (-5-2)\underset{\sim}{i} + (3-4)\underset{\sim}{j} = -7\underset{\sim}{i} - \underset{\sim}{j}$

(ii) $|\overrightarrow{PQ}| = \sqrt{(-7)^2 + (-1)^2} = \sqrt{49 + 1} = \sqrt{50} = 5\sqrt{2}$

> **Skilled Practice!**
> If $P(4, 7)$ and $Q(5, 10)$, then find the magnitude of the vector $\overrightarrow{PQ}$.

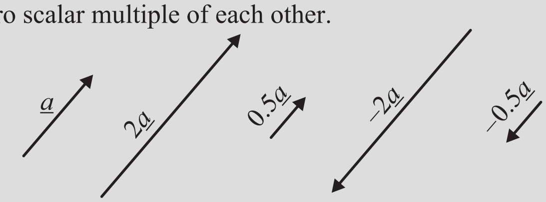
6.2.5 Scalar Multiplication of a Vector

If $\underline{a}$ is a vector and λ is a scalar (a number), then multiplication of a vector $\underline{a}$ with scalar λ is scalar multiplication and it is written as $\lambda\underline{a}$.

$\lambda\underline{a}$ is a new vector that has its magnitude scaled by $|\lambda|$.

Note

Two vectors are said to be parallel, if they are non zero scalar multiple of each other.



(i) If $\lambda > 0$, then both $\underline{a}$ and $\lambda\underline{a}$ are in the same direction.

(ii) If $\lambda < 0$, then both $\underline{a}$ and $\lambda\underline{a}$ are in opposite directions.

Example 3 If $\underline{a} = 3\underline{b}$, where $\underline{a} = 9\underline{i} - k\underline{j}$ and $\underline{b} = 3\underline{i} - 12\underline{j}$, then find the value of k.

Solution Given that $\underline{a} = 3\underline{b}$

$$9\underline{i} - k\underline{j} = 3(3\underline{i} - 12\underline{j})$$
$$9\underline{i} - k\underline{j} = 9\underline{i} - 36\underline{j}$$

Two vectors are equal, if their horizontal and vertical components are equal.

$$\therefore \quad k = 36$$

Example 4 Show that $\underline{a} = 5\underline{i} - 2\underline{j}$ and $\underline{b} = -10\underline{i} + 4\underline{j}$ are parallel.

Solution Given that:

$$\underline{a} = 5\underline{i} - 2\underline{j}$$
$$\underline{b} = -10\underline{i} + 4\underline{j}$$
$$\underline{b} = -2(5\underline{i} - 2\underline{j})$$
$$\underline{b} = -2\underline{a}$$

Note

(i) If $\underline{a} = \underline{b}$, then $\underline{a}$ and $\underline{b}$ have the same direction and $|\underline{a}| = |\underline{b}|$.

(ii) If $\underline{a}$ is the zero vector, then $|\underline{a}| = 0$

This shows that $\underline{a}$ and $\underline{b}$ are parallel with $\lambda = -2$

6.2.6 Unit Vector

A unit vector is a vector whose magnitude is 1. If $\underline{a}$ is any vector, then its unit vector is represented as $\underline{\hat{a}}$ (read as $\underline{a}$ hat). It is defined as $\underline{\hat{a}} = \dfrac{\underline{a}}{|\underline{a}|}$.

Example 5 Find a unit vector in the direction of $\underline{a} = -5\underline{i} + 2\underline{j}$.

Solution Given that $\underline{a} = -5\underline{i} + 2\underline{j}$

$$|\underline{a}| = \sqrt{(-5)^2 + (2)^2}$$

$$= \sqrt{25+4} = \sqrt{29}$$

$$\hat{\underline{a}} = \frac{\underline{a}}{|\underline{a}|} = \frac{-5\underline{i}+2\underline{j}}{\sqrt{29}}$$

$$= \frac{-5}{\sqrt{29}}\underline{i} + \frac{2}{\sqrt{29}}\underline{j}$$

Challenge!

(i) Can a zero vector be a unit vector? Explain.

(ii) If $\underline{a}$ is a unit vector, then what is the value of $|\underline{a}|$?

EXERCISE 6.1

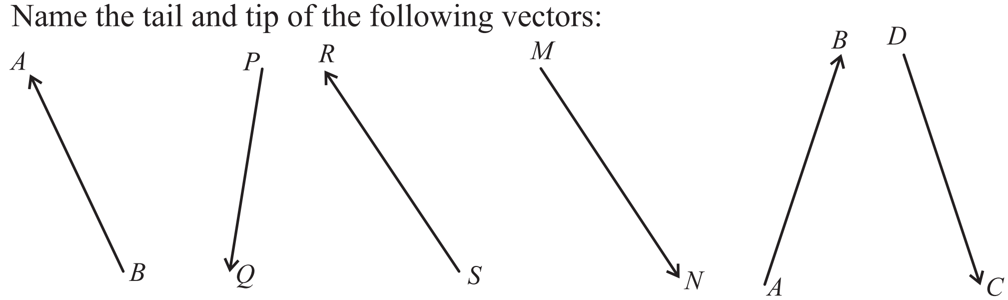
1. Name the quadrant in which each point lies.

   (i) (4, 3) (ii) (5, −4) (iii) (−6,2) (iv) (−4, −4)

2. Plot the following points on the coordinate plane:

   (i) $A(3, -3)$ (ii) $B(-3, 3)$ (iii) $C(5, 7)$ (iv) $D(-2, -4)$

3. Name the tail and tip of the following vectors:



(i) (ii) (iii) (iv) (v) (vi)

4. Write the vector $\overrightarrow{AB}$ in the form of $x\underline{i} + y\underline{j}$:

   (i) $A(1, -7), B(-2, 4)$ (ii) $A(8, 9), B(12, 3)$

5. Find the magnitude of the $\underline{a}$:

   (i) $\underline{a} = -3\underline{i} + 2\underline{j}$ (ii) $\underline{a} = 4\underline{i} - 3\underline{j}$ (iii) $\underline{a} = \frac{1}{2}\underline{i} + \frac{3}{2}\underline{j}$

6. Find a unit vector in the direction of the vector given below:

   (i) $\underline{a} = -4\underline{i} + 5\underline{j}$ (ii) $\underline{a} = 6\underline{i} + 8\underline{j}$

   (iii) $\underline{a} = \frac{1}{\sqrt{6}}\underline{i} + \frac{1}{\sqrt{6}}\underline{j}$ (iv) $\underline{a} = \frac{1}{2}\underline{i} - \frac{3}{4}\underline{j}$

7. If $\underline{a} = 5\underline{i} - 7\underline{j}$, $\underline{b} = -\underline{i} - \underline{j}$ and $\underline{c} = 2\underline{i} + 3\underline{j}$, then find unit vector parallel to $\underline{a} + \underline{b} - 3\underline{c}$.

8. If $\underline{a} = 3\underline{i} - \underline{j}$, $\underline{b} = -2\underline{i} + 4\underline{j}$ and $\underline{c} = \underline{i} + 2\underline{j}$, then find unit vector parallel to $3\underline{a} - 2\underline{c} + 4\underline{b}$.

9. Which of the following vectors are parallel?

(i) $\underline{a} = 6\underline{i} + \underline{j},\ \underline{b} = 12\underline{i} + 2\underline{j}$ (ii) $\underline{a} = -2\underline{i} + 3\underline{j},\ \underline{b} = 6\underline{i} - 9\underline{j}$

(iii) $\underline{a} = 5\underline{i} - 4\underline{j},\ \underline{b} = 6\underline{i} - 3\underline{j}$ (iv) $\underline{a} = 3\underline{i} - 7\underline{j},\ \underline{b} = 6\underline{i} - 14\underline{j}$

10. Find a vector thrice in length of $3\underline{i} - 2\underline{j}$, but opposite in direction.

11. Find two vectors that are double in magnitude of $3\underline{i} - 5\underline{j}$, one in the same direction of it and other in its opposite direction.

6.3 Addition of Vectors

(i) Triangle Law of Addition:

If two vectors $\underline{u}$ and $\underline{v}$ are represented by the two adjacent sides $\overrightarrow{AB}$ and $\overrightarrow{BC}$ of a triangle such that the terminal point of $\underline{u}$ coincides with the initial point of $\underline{v}$, then the third side $\overrightarrow{AC}$ of the triangle gives the vector sum $\underline{u} + \underline{v}$ that is:

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC} \quad \Rightarrow \quad \underline{u} + \underline{v} = \overrightarrow{AC}$$

By head to tail rule

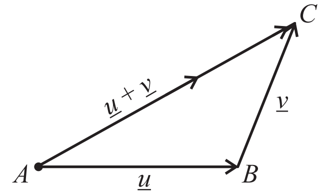


Figure 6.9

(ii) Parallelogram Law of Addition:

If two vectors $\underline{u}$ and $\underline{v}$ are represented by two adjacent sides $\overrightarrow{AB}$ and $\overrightarrow{AC}$ of a parallelogram as shown in the Figure 6.10, then diagonal $\overrightarrow{AD}$ gives the sum or resultant of $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{AC}$

$\overrightarrow{AD} = \underline{u} + \underline{v}$

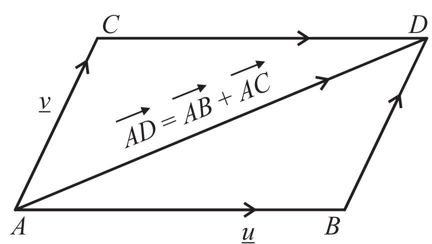


Figure 6.10

Example 6 If $\underline{a} = 2\underline{i} + 3\underline{j},\ \underline{b} = 4\underline{i} - 6\underline{j}$, then find $\underline{a} + \underline{b}$.

Solution

$$\begin{aligned}\underline{a} + \underline{b} &= (2\underline{i} + 3\underline{j}) + (4\underline{i} - 6\underline{j})\\ &= (2+4)\underline{i} + (3-6)\underline{j}\\ &= 6\underline{i} - 3\underline{j}\end{aligned}$$

Skilled Practice!

If $\underline{a} = 10\underline{i} - \frac{1}{2}\underline{j},\ \underline{b} = \frac{3}{4}\underline{i} - \frac{7}{6}\underline{j}$, then find $\underline{a} + \underline{b}$.

6.4 Difference of Two Vectors

The vector subtraction of two vectors $\underline{a}$ and $\underline{b}$ is represented by $\underline{a}-\underline{b}$ and defined as $\underline{a}+(-\underline{b})$.

If $\underline{a}=x_1\underline{i}+y_1\underline{j},\ \underline{b}=x_2\underline{i}+y_2\underline{j}$

Then $\underline{a}+(-\underline{b})=(x_1\underline{i}+y_1\underline{j})+(-x_2\underline{i}-y_2\underline{j})$

$$=(x_1-x_2)\underline{i}+(y_1-y_2)\underline{j}$$

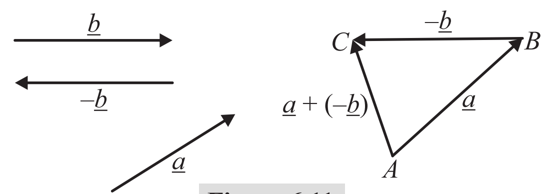


Figure 6.11

Example 7 If $\underline{a}=6\underline{i}+13\underline{j}$ and $\underline{b}=-4\underline{i}+7\underline{j}$, then find $\underline{a}-\underline{b}$.

Solution $\underline{a}-\underline{b}=(6\underline{i}+13\underline{j})-(-4\underline{i}+7\underline{j})$

$$=(6+4)\underline{i}+(13-7)\underline{j}=10\underline{i}+6\underline{j}$$

6.5 Vector in Terms of Two Non-Zero and Non-Parallel Coplanar Vectors

Let $\underline{c}$ be any vector lying in the plane of $\underline{a}$ and $\underline{b}$. Draw $\underline{a}$, $\underline{b}$ and $\underline{c}$ such that their initial points coincide.

Construct lines parallel to the vectors $\underline{a}$ and $\underline{b}$ from the terminal point E of $\underline{c}$. Complete the parallelogram $ODEC$ by extension of $\underline{a}$ and $\underline{b}$ if necessary.

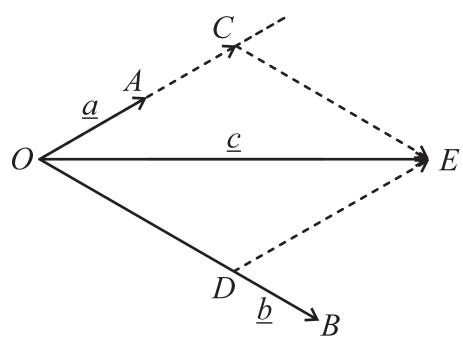


Figure 6.12

$$\overrightarrow{OC}=m\overrightarrow{OA}=m\underline{a},\ \text{where } m \text{ is a scalar.}$$

$$\overrightarrow{OD}=n\overrightarrow{OB}=n\underline{b},\ \text{where } n \text{ is a scalar.}$$

By the parallelogram law of vector addition

$$\overrightarrow{OE}=\overrightarrow{OC}+\overrightarrow{OD}\quad \text{or}\quad \underline{c}=m\underline{a}+n\underline{b}$$

Where $m\underline{a}$ and $n\underline{b}$ are vector components of $\underline{c}$.

> **Remember!**
> Two or more vectors are said to be **coplanar vectors** if they lie in the same plane.

Example 8 If $\underline{a}=2\underline{i}-3\underline{j}$ and $\underline{b}=3\underline{i}+4\underline{j}$, then find the vector components of $3\underline{a}+4\underline{b}$.

Solution $3\underline{a}+4\underline{b}$

$$=3(2\underline{i}-3\underline{j})+4(3\underline{i}+4\underline{j})$$

$$=6\underline{i}-9\underline{j}+12\underline{i}+16\underline{j}=18\underline{i}+7\underline{j}$$

Thus, the vector components of $3\underline{a}+4\underline{b}$ are $18\underline{i}$ and $7\underline{j}$.

EXERCISE 6.2

1. If $\underline{a} = 7\underline{i} - 3\underline{j}$ and $\underline{b} = \underline{i} + 5\underline{j}$, then find the following vectors:

   (i) $\underline{a} + \underline{b}$ (ii) $\underline{a} + 3\underline{b}$ (iii) $3\underline{a} + \frac{1}{2}\underline{b}$

   (iv) $\underline{b} - \underline{a}$ (v) $4\underline{b} - 5\underline{a}$ (vi) $\frac{3}{2}\underline{a} - \underline{b}$

2. If $\underline{a} = 6\underline{i} - \underline{j}$, $\underline{b} = \underline{i} + 5\underline{j}$ and $\underline{c} = 3\underline{i} + 5\underline{j}$, then find the magnitudes of the following vectors:

   (i) $\underline{b} - \underline{c}$ (ii) $\underline{a} - 2\underline{b} + \underline{c}$ (iii) $\underline{c} - \underline{b} - \underline{a}$

3. Find the values of x and y in each of the following equations:

   (i) $(x\underline{i} + y\underline{j}) + (2\underline{i} + 3\underline{j}) = 7\underline{i} + 6\underline{j}$ (ii) $(x\underline{i} - 5\underline{j}) + (3\underline{i} + 5\underline{j}) = -8\underline{i} + y\underline{j}$

   (iii) $(y\underline{i} + 3\underline{j}) + (-5\underline{i} + 2x\underline{j}) = 9\underline{i} + 7\underline{j}$

4. If $\underline{a} = \underline{i} + 3\underline{j}$, $\underline{c} = 2\underline{i} + \underline{j}$ and $\underline{a} + 2\underline{b} = \underline{c}$, then find $|\underline{b}|$.

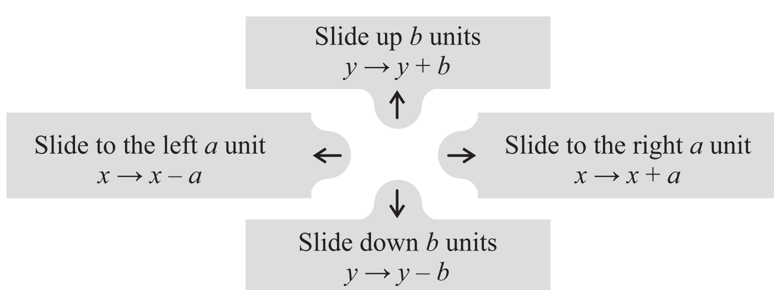
5. If $\underline{a} = -2\underline{i} + 7\underline{j}$ and $\underline{b} = 3\underline{i} - 5\underline{j}$, then find the vector components of $\underline{a} + 5\underline{b}$.

6. If $5\underline{i} - 3\underline{j} = m(\underline{i} - 10\underline{j}) + n(4\underline{i} - 3\underline{j})$, then find the values of m and n.

6.6 Translation by a Vector

A translation vector is a vector that describes the movement of a point, a body, a system from one position to another without rotation or deformation. The translated figure is called the image of original figure. The image is congruent to the original figure.

To translate a figure with translation vector in a coordinate plane, we change the coordinates of its points. If $\underline{v} = a\underline{i} + b\underline{j}$ is the traslation vector and a, b are constants, then we can use the guidelines below:



Let us translate point $P(2, 1)$ by translation vectors $2\underline{j}$, $3\underline{i}$, $-3\underline{j}$ and $-5\underline{i}$.

$P_1 = (2\underline{i} + \underline{j}) + (0\underline{i} + 2\underline{j})$
$= 2\underline{i} + 3\underline{j}$
$P_2 = (2\underline{i} + \underline{j}) + (3\underline{i} + 0\underline{j})$
$= 5\underline{i} + \underline{j}$
$P_3 = (2\underline{i} + \underline{j}) + (0\underline{i} - 3\underline{j})$
$= 2\underline{i} - 2\underline{j}$
$P_4 = (2\underline{i} + \underline{j}) + (-5\underline{i} + 0\underline{j})$
$= -3\underline{i} + \underline{j}$

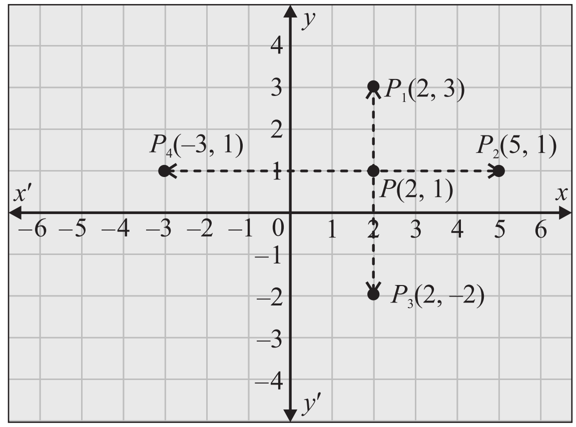


Figure 6.13

Example 9 Plot $A(-4, 2)$, $B(-1, 2)$ and $C(-3, 4)$ to form a triangle ABC. Also translate ΔABC to $\Delta A'B'C'$ by the translation vector $7\underline{i} + 2\underline{j}$.

Solution

New vertex $A' = (-4\underline{i} + 2\underline{j}) + (7\underline{i} + 2\underline{j})$
$= 3\underline{i} + 4\underline{j} = (3, 4)$
New vertex $B' = (-\underline{i} + 2\underline{j}) + (7\underline{i} + 2\underline{j})$
$= 6\underline{i} + 4\underline{j} = (6, 4)$
New vertex $C' = (-3\underline{i} + 4\underline{j}) + (7\underline{i} + 2\underline{j})$
$= 4\underline{i} + 6\underline{j} = (4, 6)$

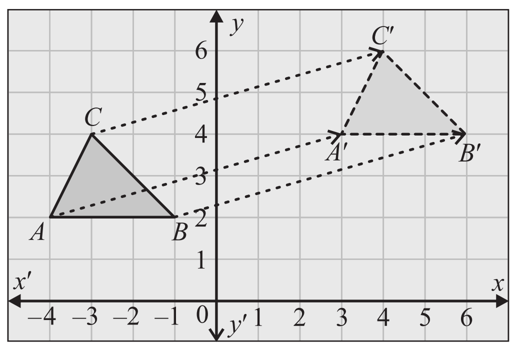


Figure 6.14

6.7 Solution and Application of Geometric Problems Involving Vectors

Example 10 If $A(1, 2)$, $B(4, 6)$ and $C(7, 2)$ are the vertices of triangle. Check whether triangle ABC is an isosceles.

Remember!

A triangle having two sides equal in length is called an isosceles triangle.

Solution

$\overrightarrow{AB} = (4 - 1)\underline{i} + (6 - 2)\underline{j} = 3\underline{i} + 4\underline{j}$

$\overrightarrow{BC} = (7 - 4)\underline{i} + (2 - 6)\underline{j} = 3\underline{i} - 4\underline{j}$

$\overrightarrow{AC} = (7 - 1)\underline{i} + (2 - 2)\underline{j} = 6\underline{i} - 0\underline{j}$

Now $|\overrightarrow{AB}| = \sqrt{(3)^2 + (4)^2} = \sqrt{9+16} = \sqrt{25} = 5$

$|\overrightarrow{BC}| = \sqrt{(3)^2 + (-4)^2} = \sqrt{9+16} = \sqrt{25} = 5$

Shows $|\overrightarrow{AB}| = |\overrightarrow{BC}| = 5$

So, ΔABC is an isosceles.

Example 11 Use vector method, to show that the points $A(1, 2)$, $B(4, 6)$, $C(7, 4)$ and $D(4, 0)$ form a parallelogram.

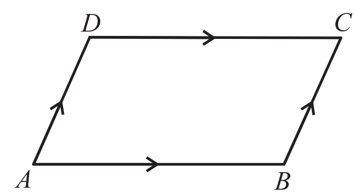
Solution $\overrightarrow{AB} = (4-1)\underline{i} + (6-2)\underline{j} = 3\underline{i} + 4\underline{j}$

$\overrightarrow{DC} = (7-4)\underline{i} + (4-0)\underline{j} = 3\underline{i} + 4\underline{j}$

$\overrightarrow{BC} = (7-4)\underline{i} + (4-6)\underline{j} = 3\underline{i} - 2\underline{j}$

$\overrightarrow{AD} = (4-1)\underline{i} + (0-2)\underline{j} = 3\underline{i} - 2\underline{j}$

$\therefore$ $\overrightarrow{AB} \parallel \overrightarrow{DC}$ and $\overrightarrow{BC} \parallel \overrightarrow{AD}$



Shows the given points form a parallelogram.

Note

Equations of motions are formulas used to describe the movement of objects under uniform acceleration. They help us to calculate the relationship between initial velocity ($\underline{v}_i$), final velocity ($\underline{v}_f$), acceleration ($\underline{a}$), time (t) and displacement ($\underline{s}$). These are very useful in problems involving moving cars, falling objects or thrown balls.

First Equation of Motion $\underline{v}_f = \underline{v}_i + \underline{a}t$

It tells us the final velocity ($\underline{v}_f$) after time t when an object starts with initial velocity $\underline{v}_i$ and moves with acceleration $\underline{a}$.

Second Equation of Motion $\underline{s} = \underline{v}_i t + \frac{1}{2}\underline{a}t^2$

It tells us the distance ($\underline{s}$) travelled by the object in time t.

Third Equation of Motion $\underline{v}_f^2 = \underline{v}_i^2 + 2\underline{a}\underline{s}$

It relates the initial and final velocity, acceleration and displacement.

Example 12 A projectile is launched from ground with initial velocity vector $12\underline{i} + 16\underline{j}$. After how long does it hit a target located 24 metres away horizontally?

Solution Given that $\underline{v}_i = 12\underline{i} + 16\underline{j}$

$\therefore$ Horizontal velocity $= v_x = 12$

Vertical velocity $= v_y = 16$

Horizontal velocity is constant,

therefore $x = v_x t$

$$t = \frac{x}{v_x} = \frac{24}{12} = 2\ s$$

Note

Equation for horizontal motion is $\underline{x} = \underline{v}_x t$

Equation for vertical motion is $\underline{y} = \underline{v}_y t - \frac{1}{2}\underline{g}t^2$

Example 13 A tank fires a shell with velocity vector $\underline{v}_1 = 30\underline{i} + 40\underline{j}$ at a moving target heading east at $\underline{v}_2 = 10\underline{i} + 0\underline{j}$. Find relative velocity and its magnitude.

Solution Relative velocity = Shell velocity – Target velocity

$$\underline{v}_{rel} = (30-10)\underline{i} + (40-0)\underline{j}$$

$$= 20\underline{i} + 40\underline{j}$$

$$|\underline{v}_{rel}| = \sqrt{20^2 + 40^2}$$

$$= \sqrt{400 + 1600}$$

$$|\underline{v}_{rel}| = \sqrt{2000} = 44.72\ m/s$$

Note

Relative velocity is the velocity of an object as observed from another moving object. It tells us how fast and in what direction one object is moving relative to another.

EXERCISE 6.3

1. Plot $A(-5, 4)$, $B(-6, 1)$ and $C(-3, 1)$ to form a ΔABC. Also translate ΔABC to $\Delta A'B'C'$ by translation vector $8\underline{i} - 6\underline{j}$.
2. Plot $A(-6, -2)$, $B(-6, -5)$, $C(-3, -5)$ and $D(-3, -2)$ to form a square $ABCD$. Also translate square $ABCD$ to square $A'B'C'D'$ by translation vector $9\underline{i} + 7\underline{j}$.
3. Plot $A(-6, 3)$, $B(-4, 0)$, $C(-2, 3)$ and $D(-4, 4)$ to form a kite $ABCD$. Also translate kite $ABCD$ to kite $A'B'C'D'$ by translation vector $6\underline{i} - 6\underline{j}$.
4. The coordinates of A, B and D are (1,2), (6,3) and (2,8) respectively. Find the coordinates of C by using vector method if $ABCD$ is a parallelogram.
5. In parallelogram $ABCD$, the vectors representing two opposite sides are $\overrightarrow{AB} = 6\underline{i} + 2\underline{j}$, $\overrightarrow{DC} = -6\underline{i} - 2\underline{j}$. Show that the opposite sides are equal in magnitude and parallel.
6. Points $A(1, 2)$, $B(4, 6)$ and $C(7, 2)$ form a triangle. Check whether triangle ABC is an isosceles by using vector magnitude.
7. Use vectors to show that $PQRS$ is a parallelogram, where the points P, Q, R and S have coordinates (1, 2), (5, 2), (7, 6) and (3, 6) respectively.

8. Use vectors to show that triangle XYZ is an isosceles, where the points X, Y, Z have the coordinates $(0, 0)$, $(2, 0)$ and $(1, 3)$ respectively.

9. A ball is projected with an initial velocity vector $\underline{v}_0 = 10\underline{i} + 20\underline{j}$. The horizontal component is in the x-direction and gravity is $\underline{g} = 0\underline{i} - 10\underline{j}$. Find the maximum height and horizontal range.

10. A car enters a loop with velocity vector $\underline{v} = 30\underline{j}$ and exits with velocity $\underline{v}' = 30\underline{i}$. What is the change in velocity vector?

11. An aeroplane has airspeed $\underline{v}_p = 20\underline{i}$ and there is a crosswind $\underline{v}_w = 50\underline{j}$. Find the resultant velocity and its magnitude.

REVIEW EXERCISE 6

1. Four possible answers are given for the following questions. Choose the correct answer.

(i) x-axis and y-axis divide a coordinate plane into _________ parts.
(a) one (b) two (c) three (d) four

(ii) $P(4,-4)$ lies in _________ quadrant.
(a) first (b) second (c) third (d) fourth

(iii) A vector having magnitude 1, is called:
(a) equal vector (b) parallel vector
(c) unit vector (d) zero vector

(iv) What is the value of $|3\underline{i} + 4\underline{j}|$?
(a) 3 (b) 4 (c) 5 (d) 7

(v) If $\underline{a} = \lambda\underline{b}$, then $\underline{a}$ and $\underline{b}$ are:
(a) equal (b) parallel (c) perpendicular (d) non-parallel

(vi) If $\overrightarrow{OA} = \underline{a}$, $\overrightarrow{OB} = \underline{b}$, then $\overrightarrow{AB}$ is:
(a) $\underline{b} - \underline{a}$ (b) $\underline{a} - \underline{b}$ (c) $\underline{a} + \underline{b}$ (d) $\underline{b} + \underline{a}$

(vii) Translation vector shows:
(a) deformation (b) rotation (c) movement (d) enlargement

(viii) Sum of two vectors is:
(a) a triangle (b) a vector (c) a scalar (d) a length

(ix) The position vector of point P(3, −2) with respect to O is:
(a) $3\underline{i} + 2\underline{j}$ (b) $3\underline{i} - 2\underline{j}$ (c) $-3\underline{i} + 2\underline{j}$ (d) $2\underline{i} - 3\underline{j}$

(x) Vector from point P(3, 4) to origin is:
(a) $3\underline{i} + 4\underline{j}$ (b) $-3\underline{i} + 4\underline{j}$ (c) $-3\underline{i} - 4\underline{j}$ (d) $3\underline{i} - 4\underline{j}$

2. Find magnitude of the $\overrightarrow{AB}$:
(i) $A(7, 7)$, $B(-12, 0)$ (ii) $A(9, 3)$, $B(2, 11)$

3. Find a unit vector in the direction of $\underline{a} = \frac{5}{3}\underline{i} + \frac{1}{3}\underline{j}$.

4. If $\underline{a} = 2\underline{i} - \underline{j}$, $\underline{b} = 3\underline{i} + \underline{j}$ and $\underline{c} = 4\underline{i} + \underline{j}$, then find the following vectors:
(i) $5\underline{b} - \underline{a} + \underline{c}$ (ii) $8\underline{a} + \underline{b} + 5\underline{c}$ (iii) $\underline{c} + \underline{b} - 4\underline{a}$

5. Find the values of x and y in the following equation.
$$(2x\underline{i} + y\underline{j}) + (-\underline{i} + 5\underline{j}) = \frac{1}{4}\underline{i} - 8\underline{j}$$

6. Plot $A(-5, 3)$, $B(-2, 3)$ and $C(-4, 5)$ to form a triangle ABC. Also, translate $\triangle ABC$ to $\triangle A'B'C'$ by translation vector $5\underline{i} - 2\underline{j}$.

7. Use vectors to show that $ABCD$ is a parallelogram, where the points are $A(2, 3)$, $B(6, 3)$, $C(7, 6)$ and $D(3, 6)$.

8. Use vectors to show that triangle ABC is an isosceles triangle, where the points A, B and C have coordinates (1, 2), (4, 6) and (7, 2) respectively.

9. A ball is projected with velocity vector $\underline{v} = 6\underline{i} + 8\underline{j}$. What is the magnitude of velocity?

10. An aircraft is flying due east with an airspeed of 200 km/h. There is a wind blowing due north at 60 km/h. Find the resultant velocity and its magnitude.

Activity

1. Suppose vectors $\underline{a}$ and $\underline{b}$ are equal.
   (i) Can we say they originate from the same point? Why or why not?
   (ii) Do they have equal magnitudes? Explain.
   (iii) Do they have same direction? Why?
2. Suppose vectors $\underline{a}$ and $\underline{b}$ are opposite.
   (i) Can we assume they begin at the same point? Give a reason.
   (ii) Do they have same magnitude? Why?
   (iii) Do they have the same direction? Explain why or why not.

UNIT 7

Trigonometry

Students' Learning Outcomes

After completing this unit, the students will be able to:

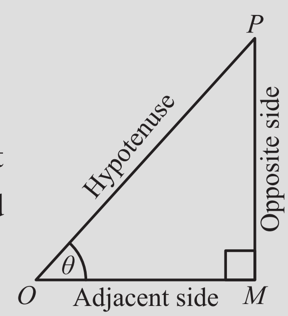
- Extend sine and cosine functions to angles between 90° and 180°.
- Solve problems using the laws of sine, cosine and the area formulas for any triangle.
- Solve simple trigonometric problems in three dimensions.
- Apply concepts of trigonometry to real life world problems (such as video games, flight engineering, navigation and sound waves).
- Interpret and use three figure bearings.
- Solve real life problems involving bearing.
- Apply the concepts of bearing to real world problems.

INTRODUCTION

This unit explores the extended applications of trigonometry beyond right-angled triangles by developing an understanding of sine and cosine functions for angles between 90° and 180°. Students will learn to apply the laws of sine and cosine, as well as area formulas, to solve problems involving any type of triangle. The unit further introduces the use of trigonometric problems in three-dimensional contexts and emphasizes concept of trigonometry in real-world scenarios such as navigation, flight engineering, video game design and sound wave analysis. Students will also gain proficiency in interpreting and using three-figure bearings and applying this knowledge to solve practical bearing-related problems, enhancing spatial reasoning and problem-solving skills.

7.1 Trigonometric Ratios or Functions

Consider an angle θ of right angled triangle POM, with right angle at M, where $\overline{OM}$ is called adjacent side, $\overline{PM}$ is called opposite side and $\overline{PO}$ is called hypotenuse. Then,



(i) sine θ (sin θ) = $\frac{\text{opp.}}{\text{hyp.}}$

(ii) cosine θ (cos θ) = $\frac{\text{adj.}}{\text{hyp.}}$

(iii) cosecant θ (cosec θ) = $\frac{1}{\sin\theta} = \frac{\text{hyp.}}{\text{opp.}}$

(iv) secant θ (sec θ) = $\frac{1}{\cos\theta} = \frac{\text{hyp.}}{\text{adj.}}$

(v) tangent θ (tan θ) = $\frac{\text{opp.}}{\text{adj.}}$

(vi) cotangent θ (cot θ) = $\frac{1}{\tan\theta} = \frac{\text{adj.}}{\text{opp.}}$

The above six trigonometric ratios are also called trigonometric functions.

Remember!

sin θ does not mean the product of sin and θ. The sin θ is correctly read as sin of angle θ.

Note

$\tan\theta = \frac{\sin\theta}{\cos\theta}$, $\cot\theta = \frac{\cos\theta}{\sin\theta}$

7.1.1 Trigonometric Values for Special Angles

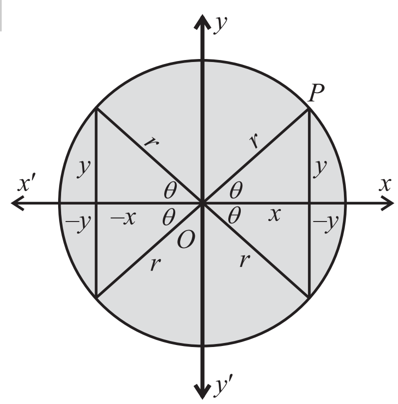
Trigonometric values for 0°, 30°, 45°, 60°, 90° and 180° are given in the following table:

Ratios \ Angle θ	0°	30°	45°	60°	90°	180°
sin θ	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	– 1
tan θ	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined	0

7.1.2 Signs of Trigonometric Functions

The signs depend on the quadrant in which terminal side of the angle lies.

Consider a circle having radius r and centre at point O.



In **1st Quadrant**: $x > 0$, $y > 0$

$$\sin\theta = \frac{y}{r} > 0, \cos\theta = \frac{x}{r} > 0, \tan\theta = \frac{y}{x} > 0,$$

$$\text{cosec}\,\theta = \frac{r}{y} > 0, \sec\theta = \frac{r}{x} > 0, \cot\theta = \frac{x}{y} > 0$$

Hence, in the 1st quadrant all trigonometric functions are positive.

In **2[nd] Quadrant**: $x < 0, y > 0$

$$\sin\theta = \frac{y}{r} > 0, \cos\theta = \frac{-x}{r} < 0, \tan\theta = \frac{y}{-x} < 0, \operatorname{cosec}\theta = \frac{r}{y} > 0, \sec\theta = \frac{r}{-x} < 0, \cot\theta = \frac{-x}{y} < 0$$

Hence, in the 2[nd] quadrant sin and cosec functions are positive and all other trigonometric functions are negative.

In **3[rd] Quadrant**: $x < 0, y < 0$

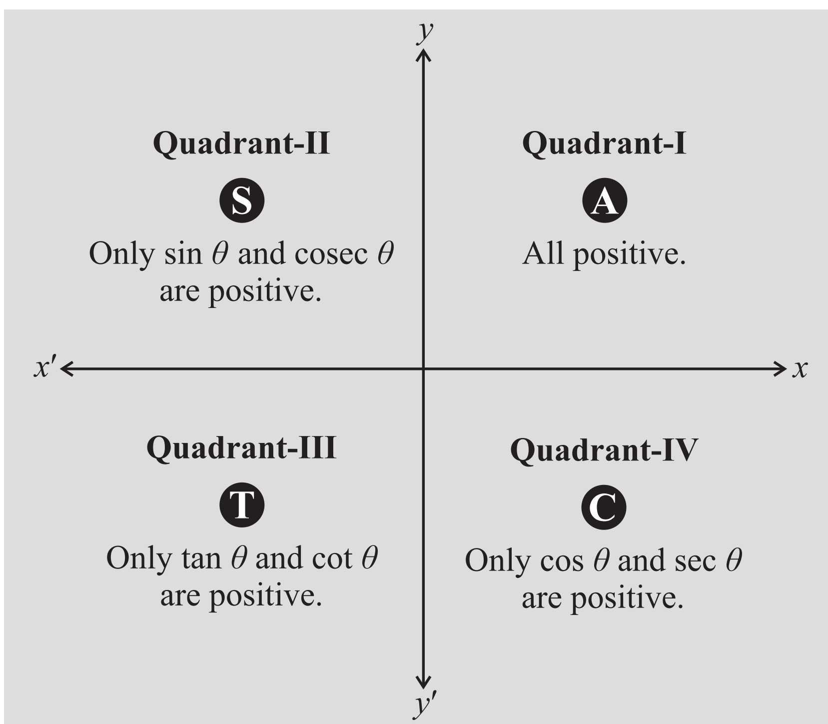
$$\sin\theta = \frac{-y}{r} < 0, \cos\theta = \frac{-x}{r} < 0, \tan\theta = \frac{-y}{-x} > 0, \operatorname{cosec}\theta = \frac{r}{-y} < 0, \sec\theta = \frac{r}{-x} < 0, \cot\theta = \frac{-x}{-y} > 0$$

Hence, in the 3[rd] quadrant tan and cot functions are positive and all other trigonometric functions are negative.

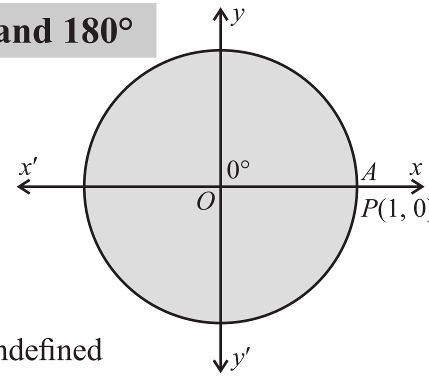
In **4[th] Quadrant**: $x > 0, y < 0$

$$\sin\theta = \frac{-y}{r} < 0, \cos\theta = \frac{x}{r} > 0, \tan\theta = \frac{-y}{x} < 0, \operatorname{cosec}\theta = \frac{r}{-y} < 0, \sec\theta = \frac{r}{x} > 0, \cot\theta = \frac{x}{-y} < 0$$

Hence, in the 4[th] quadrant cos and sec functions are positive and all other trigonometric functions are negative.



7.1.3 Trigonometric Ratios of 0°, 90° and 180°



i Trigonometric Ratios of 0°

Consider a unit circle with centre at O and take any point $P(x, y)$ on it such that $m\angle AOP = 0°$.
As $P(x, y)$ lies on x-axis,

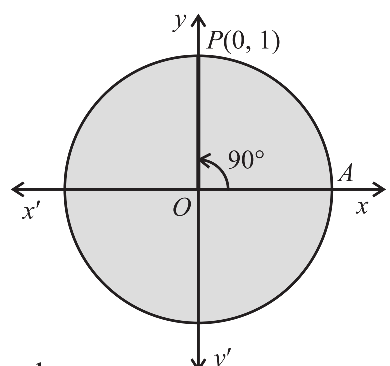
So, $x = 1,\ y = 0,\ r = 1$

$$\sin 0° = \frac{y}{r} = \frac{0}{1} = 0 \quad , \quad \text{cosec } 0° = \frac{1}{\sin 0°} = \frac{1}{0} = \text{undefined}$$

$$\cos 0° = \frac{x}{r} = \frac{1}{1} = 1 \quad , \quad \sec 0° = \frac{1}{\cos 0°} = \frac{1}{1} = 1$$

$$\tan 0° = \frac{y}{x} = \frac{0}{1} = 0 \quad , \quad \cot 0° = \frac{1}{\tan 0°} = \frac{1}{0} = \text{undefined}$$

ii Trigonometric Ratios of 90°



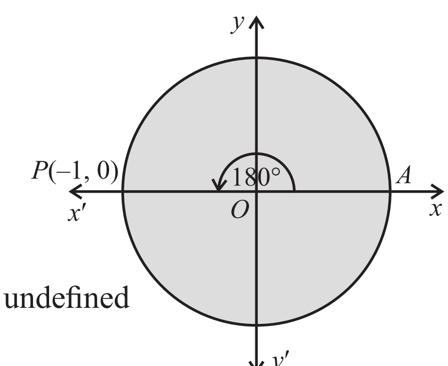
Consider a unit circle with centre at O and take any point $P(x, y)$ on it such that $m\angle AOP = 90°$.
As $P(x, y)$ lies on y-axis,
So, $x = 0,\ y = 1,\ r = 1$

$$\sin 90° = \frac{y}{r} = \frac{1}{1} = 1, \quad \text{cosec } 90° = \frac{1}{\sin 90°} = \frac{1}{1} = 1$$

$$\cos 90° = \frac{x}{r} = \frac{0}{1} = 0, \quad \sec 90° = \frac{1}{\cos 90°} = \frac{1}{0} = \text{undefined}$$

$$\tan 90° = \frac{y}{x} = \frac{1}{0} = \text{undefined} \quad , \quad \cot 90° = \frac{0}{1} = 0$$

iii Trigonometric Ratios of 180°



Consider a unit circle with centre at O and take any point $P(x, y)$ on it such that $m\angle AOP = 180°$.
As $P(x, y)$ lies on x'-axis.
so, $x = -1,\ y = 0,\ r = 1$

$$\sin 180° = \frac{y}{r} = \frac{0}{1} = 0, \quad \text{cosec } 180° = \frac{1}{\sin 180°} = \frac{1}{0} = \text{undefined}$$

$$\cos 180° = \frac{x}{r} = \frac{-1}{1} = -1, \quad \sec 180° = \frac{1}{\cos 180°} = \frac{1}{-1} = -1$$

$$\tan 180° = \frac{y}{x} = \frac{0}{-1} = 0, \quad \cot 180° = \frac{1}{\tan 180°} = \frac{-1}{0} = \text{undefined}$$

7.1.4 Extending sine and cosine Functions to Angles Between 90° and 180°

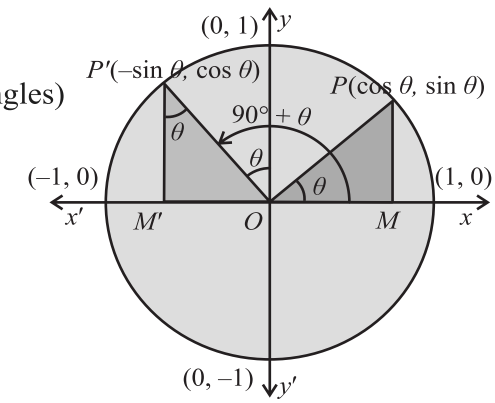
a Trigonometric Ratios of an Angle of the Form (90° + θ)

Consider a unit circle having radius 1 with centre at $O(0, 0)$ in the Cartesian coordinate system. $P(\cos\theta, \sin\theta)$ be any point in quadrant I on the circumference of the circle, where $m\angle MOP = \theta = m\angle YOP'$

$m\angle MOP' = (90° + \theta)$ and $m\angle POP' = 90°$

so, we have $m\angle OP'M' = m\angle YOP'$ (Alternate angles)

That is, the angle $(90° + \theta)$ is obtained by rotating the angle θ counterclockwise by 90°. The coordinates of P' will be $(-\sin\theta, \cos\theta)$, i.e. the new x-coordinate of P' becomes the negative of the y-coordinate of the P and new y-coordinate of P' becomes the x-coordinate of the P.



As we know that in unit circle, cosine and sine of any angle are represented by x-coordinate and y-coordinate respectively. Therefore,

$$\cos(90° + \theta) = x\text{-coordinate of } (90° + \theta) = -\sin\theta$$
$$\sin(90° + \theta) = y\text{-coordinate of } (90° + \theta) = \cos\theta$$

Hence,
$\sin(90° + \theta) = \cos\theta$, $\operatorname{cosec}(90° + \theta) = \sec\theta$
$\cos(90° + \theta) = -\sin\theta$, $\sec(90° + \theta) = -\operatorname{cosec}\theta$
$\tan(90° + \theta) = -\cot\theta$, $\cot(90° + \theta) = -\tan\theta$

Note
$\sin(-\theta) = -\sin\theta$
$\cos(-\theta) = \cos\theta$
$\tan(-\theta) = -\tan\theta$

b Trigonometric Ratios of an Angle of the Form (180° – θ)

Let $m\angle MOP = \theta$ and $m\angle MOP' = (180° - \theta)$

Here ΔPOM and $\Delta P'OM'$ are congruent, such that

$m\overline{OP'} = m\overline{OP}$, $m\overline{OM'} = m\overline{OM}$ and $m\overline{P'M'} = m\overline{PM}$

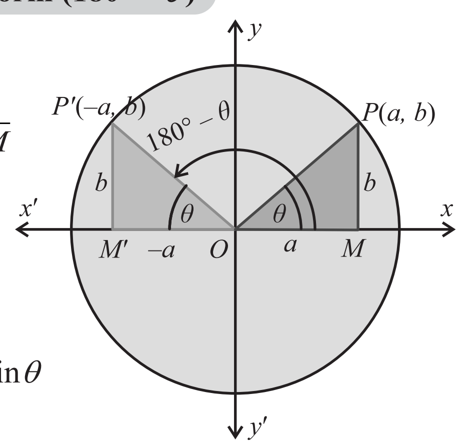
In ΔPOM, we have

$$\sin\theta = \frac{b}{m\overline{OP}} \text{ and } \cos\theta = \frac{a}{m\overline{OP}}$$

In $\Delta P'OM'$, we have

$$\sin(180° - \theta) = \frac{y - \text{coordinate of } P'}{m\overline{OP'}} = \frac{b}{m\overline{OP}} = \sin\theta$$

$$\cos(180° - \theta) = \frac{x\text{–coordinate of } P'}{m\overline{OP'}} = \frac{-a}{m\overline{OP}} = -\cos\theta$$



Hence, $\sin(180^o - \theta) = \sin\theta$	$\text{cosec}(180^o - \theta) = \text{cosec}\,\theta$
$\cos(180^o - \theta) = -\cos\theta$	$\sec(180^o - \theta) = -\sec\theta$
$\tan(180^o - \theta) = -\tan\theta$	$\cot(180^o - \theta) = -\cot\theta$

Example 1 Find the signs of the following:

(i) $\cos 75°$ (ii) $\cot 250°$

Solution (i) 75° lies in first quadrant, where cos is positive.

(ii) 250° lies in third quadrant, where cot is positive.

Example 2 Without using calculator, find the exact values of the following trigonometric functions: (i) $\sin 120^o$ (ii) $\cot 150^o$

Solution (i) $\sin 120°$

120° lies in quadrant II, where sin is positive.

$\therefore$ $\sin 120° = \sin(180^o - 60^o) = \sin 60°$ $\quad\because \sin(180° - \theta) = \sin\theta$

$$= \frac{\sqrt{3}}{2}$$

(ii) $\cot 150°$

150° lies in quadrant II, where cot is negative.

$\therefore$ $\cot 150° = \cot(180^o - 30^o) = -\cot 30°$ $\quad\because \cot(180^o - \theta) = -\cot\theta$

$$= -\sqrt{3}$$

EXERCISE 7.1

1. Find the signs of the following:

(i) $\sin 55^o$ (ii) $\cos 145^o$ (iii) $\tan 111^o$
(iv) $\sec 179^o$ (v) $\text{cosec}\,88^o$ (vi) $\cot 14^o$

2. Fill in the blanks:

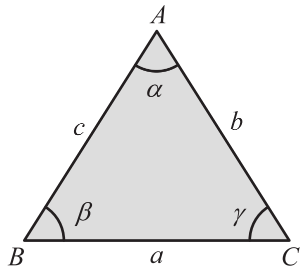
(i) $\tan(180^o - \theta) = \ldots \tan\theta$ (ii) $\sin(180^o - \theta) = \ldots \sin\theta$
(iii) $\tan(90^o + \theta) = \ldots \cot\theta$ (iv) $\cos(90^o + \theta) = \ldots \sin\theta$
(v) $\text{cosec}(180^o - \theta) = \ldots \text{cosec}\,\theta$ (vi) $\sec(90^o - \theta) = \ldots \text{cosec}\,\theta$

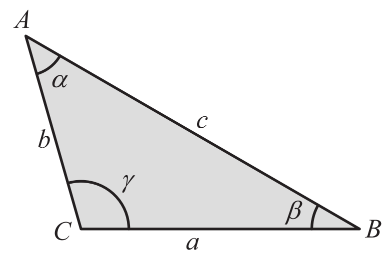
3. Without using calculator, find the exact values of the following trigonometric functions:

(i) $\sin 150°$ (ii) $\tan 150°$ (iii) $\sec 150°$
(iv) $\text{cosec}\,120°$ (v) $\cos 120°$ (vi) $\cot 120°$
(vii) $\sin 135°$ (viii) $\sec 135°$ (ix) $\cot 135°$

7.2 Oblique Triangles

An oblique triangle is any triangle that does not contain a right angle. It can be either acute triangle (all angles less than 90°) or obtuse triangle (one angle greater than 90°). Following triangles are oblique triangles:





7.2.1 The Law of sines

If a, b and c are the sides and α, β and γ are the angles opposite to the sides of any triangle respectively, then the law of sines is:

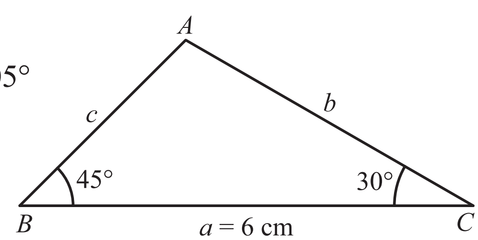
$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

The law of sines is a relationship between the angles and the sides of a triangle. While solving a triangle, the law of sines can be used to solve a triangle, if two angles and one side which is opposite to one of given angles, are given.

Example 3 In a triangle ABC, $\beta = 45°$, $\gamma = 30°$, $a = 6$ cm, calculate b and c.

Solution As we know that

$$\alpha + \beta + \gamma = 180°$$
$$\alpha = 180° - \beta - \gamma$$
$$= 180° - 45° - 30° = 105°$$



Now, by the law of sines

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

$$\frac{a}{\sin 105°} = \frac{b}{\sin 45°} = \frac{c}{\sin 30°}$$

$$\frac{a}{\sin 75°} = \frac{b}{\sin 45°} = \frac{c}{\sin 30°} \quad [\sin 105° = \sin(180° - 75°) = \sin 75°]$$

Taking $\frac{a}{\sin 75°} = \frac{b}{\sin 45°}$

$$b = \frac{a \sin 45°}{\sin 75°} = \frac{6\times(0.71)}{0.97} = \frac{4.26}{0.97} = 4.4 \text{ cm}$$

Now, $\dfrac{a}{\sin 75°} = \dfrac{c}{\sin 30°}$

$$c = \frac{a\times \sin 30°}{\sin 75°} = \frac{6\times(0.5)}{0.97} = \frac{3.0}{0.97} = 3.1 \text{ cm}$$

Skilled Practice!

Solve ΔABC, in which $\alpha = 77°$, $\beta = 66°$, $a = 2$ cm

Challenge!

Can we use law of sines, when two sides and included angle or the three sides of a triangle are given?

7.2.2 The Law of cosines

If a, b and c are the sides and α, β and γ are the angles opposite to the sides of any triangle respectively, then the law of cosines is:

$$a^2 = b^2 + c^2 - 2bc \cos\alpha \quad \text{or} \quad \cos\alpha = \frac{b^2+c^2-a^2}{2bc}$$

$$b^2 = c^2 + a^2 - 2ca \cos\beta \quad \text{or} \quad \cos\beta = \frac{c^2+a^2-b^2}{2ca}$$

$$c^2 = a^2 + b^2 - 2ab \cos\gamma \quad \text{or} \quad \cos\gamma = \frac{a^2+b^2-c^2}{2ab}$$

Example 4 In ΔABC, $a = 3$, $b = 5$, $c = 7$. Find the values of α, β and γ.

Solution Given that $a = 3$, $b = 5$, $c = 7$

By using the law of cosines, we have

$$\cos\alpha = \frac{b^2+c^2-a^2}{2bc} = \frac{5^2+7^2-3^2}{2\times 5\times 7} = \frac{25+49-9}{70} = \frac{13}{14}$$

$$\alpha = \cos^{-1}\left(\frac{13}{14}\right) = 21.8°$$

$$\cos\beta = \frac{a^2+c^2-b^2}{2ac} = \frac{3^2+7^2-5^2}{2\times 3\times 7} = \frac{9+49-25}{42} = \frac{11}{14}$$

$$\beta = \cos^{-1}\left(\frac{11}{14}\right) = 38.2°$$

As $\alpha + \beta + \gamma = 180°$

$$\gamma = 180° - \alpha - \beta = 180° - 21.8° - 38.2° = 120°$$

7.2.3 Area of a Triangle

a When two sides and their included angle of a triangle are given, then

$$\text{Area of } \Delta ABC = \frac{1}{2}bc \sin\alpha = \frac{1}{2}ac \sin\beta = \frac{1}{2}ab \sin\gamma$$

Example 5 Calculate area of the triangle ABC in which $a = 4.2$, $b = 3.2$ and $\gamma = 70°$.

Solution $a = 4.2,\ b = 3.2,\ \gamma = 70°$

$$\text{Area of } \Delta ABC = \frac{1}{2}ab\sin\gamma = \frac{1}{2}(4.2)(3.2)\sin 70^\circ$$

$$= \frac{1}{2}(4.2)(3.2)(0.94) = 6.31 \text{ square units}$$

b When three sides of a triangle are given, then

$$\text{Area of } \Delta ABC = \sqrt{s(s-a)(s-b)(s-c)}, \text{ where } s = \frac{a+b+c}{2}$$

Example 6 Calculate area of ΔABC in which $a = 4$ cm, $b = 3$ cm, $c = 5$ cm.

Solution $a = 4$ cm, $b = 3$ cm, $c = 5$ cm

$$s = \frac{a+b+c}{2} = \frac{4+3+5}{2} = \frac{12}{2} = 6$$

$$s - a = 6 - 4 = 2$$

$$s - b = 6 - 3 = 3$$

$$s - c = 6 - 5 = 1$$

$$\text{Area of } \Delta ABC = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{6(2)(3)(1)}$$

$$= \sqrt{36}$$

$$= 6 \text{ cm}^2$$

Skilled Practice!

Find area of ΔABC, in which

(i) $a = 8$ cm, $b = 9$ cm, $c = 10$ cm

(ii) $a = 6.1$ cm, $b = 2.3$ cm, $\gamma = 80°$

c When two angles and one side of a triangle are given, then

$$\text{Area of } \Delta ABC = \frac{a^2\sin\beta\sin\gamma}{2\sin\alpha} = \frac{b^2\sin\alpha\sin\gamma}{2\sin\beta} = \frac{c^2\sin\alpha\sin\beta}{2\sin\gamma}$$

Example 7 Calculate area of triangle ABC in which $\alpha = 35°, \beta = 65°, a = 2$ cm.

Solution As we know that

$$\alpha + \beta + \gamma = 180°$$

$$35° + 65° + \gamma = 180°$$

$$\gamma = 180° - 100°$$

$$= 80°$$

Skilled Practice!

Calculate area of ΔABC, in which $\beta = 50°$, $\gamma = 40°$, $c = 10$ cm

$$\text{Area of } \Delta ABC = \frac{a^2 \sin\beta \sin\gamma}{2\sin\alpha}$$

$$= \frac{(2)^2 \sin 65° \sin 80°}{2 \sin 35°}$$

$$= \frac{4(0.91)(0.98)}{2(0.57)} = 3.13 \text{ cm}^2$$

EXERCISE 7.2

1. Solve the triangle ABC, in which

(i) $a = 6.1$ cm, $b = 8.4$ cm, $\alpha = 42°$

(ii) $a = 12.2$ cm, $c = 15.8$ cm, $\gamma = 50°$

(iii) $b = 5.2$ cm, $c = 5$ cm, $\gamma = 48°$ (iv) $b = 4.8$ cm, $a = 4$ cm, $\beta = 71°$

(v) $\beta = 70°$, $b = 8$cm, $\alpha = 100°$ (vi) $a = 6$ cm, $\alpha = 55°$, $\gamma = 60°$

(vii) $c = 7$ cm, $\beta = 34°$, $\gamma = 64°$ (viii) $b = 12$ cm, $\alpha = 92°$, $\beta = 77°$

2. Calculate area of each triangle ABC.

(i) $a = 7$ cm, $b = 8$ cm, $\gamma = 38°$ (ii) $a = 11$ cm, $c = 14$ cm, $\beta = 51°$

(iii) $b = 3$ cm, $c = 9$ cm, $\alpha = 78°$ (iv) $a = 10$ cm, $\alpha = 62°$, $\beta = 69°$

(v) $c = 4$ cm, $\beta = 36°$, $\gamma = 80°$ (vi) $c = 6.6$ cm, $\alpha = 23°$, $\gamma = 89°$

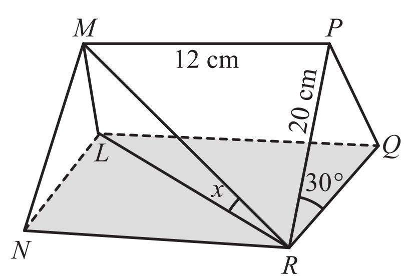
(vii) $a = 5.3$ cm, $b = 4.7$ cm, $c = 8.2$ cm (viii) $a = 6.12$ cm, $b = 8.34$ cm, $c = 7.12$ cm

7.3 Solve Simple Trigonometric Problems in Three Dimensions

Example 8 The diagram shows a prism. Triangle PQR is a cross section of the prism, where

$$m\overline{PR} = 20 \text{ cm}, \; m\overline{MP} = 12 \text{ cm},$$
$$m\angle PRQ = 30° \text{ and } m\angle PQR = 90°.$$

Calculate the size of the angle that the line MR makes with the plane $RQLN$.



Solution In right angled triangle PQR, we have

$$\frac{m\overline{PQ}}{20} = \sin 30°$$

$$m\overline{PQ} = 20 \times \frac{1}{2}$$
$$= 10 \text{ cm}$$

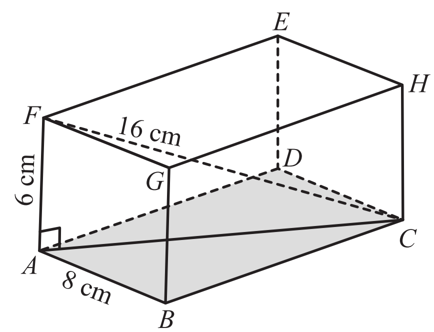
In right angled triangle *MPR,* we have

$$m\overline{MR} = \sqrt{\left(m\overline{PR}\right)^2 + \left(m\overline{MP}\right)^2} = \sqrt{(20)^2 + (12)^2} = \sqrt{400+144} = \sqrt{544}$$
$$= 23.3238 \text{ cm}$$

In right angled triangle *MLR,* we have

$$\sin x = \frac{\text{opp.}}{\text{hyp.}} = \frac{10}{23.3238} = 0.4287$$
$$\therefore \quad x = \sin^{-1}(0.4287)$$
$$= 25.4°$$

Example 9 The diagram shows a cuboid *ABCDEFGH* in which $m\overline{AB} = 8\,cm$, $m\overline{AF} = 6\,cm$ and $m\overline{FC} = 16\ cm$. Find the length of $\overline{BC}$.



Solution In right angled triangle *AFC,* we have

$$m\overline{AC} = \sqrt{\left(m\overline{FC}\right)^2 - \left(m\overline{AF}\right)^2} = \sqrt{(16)^2 - (6)^2}$$
$$= \sqrt{256-36} = \sqrt{220} = 14.83 \text{ cm}$$

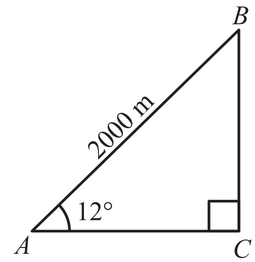
In right angled triangle *ABC,* we have

$$m\overline{BC} = \sqrt{\left(m\overline{AC}\right)^2 - \left(m\overline{AB}\right)^2} = \sqrt{(14.83)^2 - (8)^2}$$
$$= \sqrt{220-64}$$
$$= \sqrt{156} = 12.5 \text{ cm}$$

7.4 Concept of Trigonometry in Real Life

Example 10 A plane takes off at an angle of elevation of 12° and travels 2000 metres along this path. How high is the plane after traveling this distance?

Solution In triangle ABC,



$$\sin 12° = \frac{m\overline{BC}}{2000}$$

$$(0.2079)2000 = m\overline{BC}$$

$$m\overline{BC} = 415.8 \text{ m}$$

The plane is approximately 415.8 m above the ground.

Example 11 In a sound lab, two microphones are placed $c = 3$ m apart and a sound source is $a = 2$ m from one and $b = 2.5$ m from the other. Find the angle between the sound paths a and b.

Solution Here $a = 2$ m, $b = 2.5$ m, $c = 3$ m

Using the law of cosines, we have

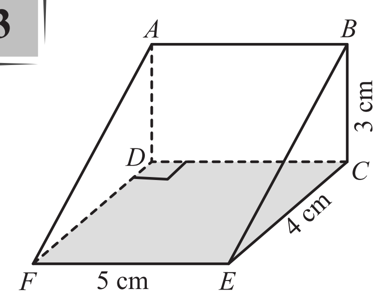
$$\cos\gamma = \frac{a^2 + b^2 - c^2}{2ab} = \frac{2^2 + (2.5)^2 - 3^2}{2 \times 2 \times 2.5} = \frac{4 + 6.25 - 9}{10} = 0.125$$

$$\gamma = \cos^{-1}(0.125)$$

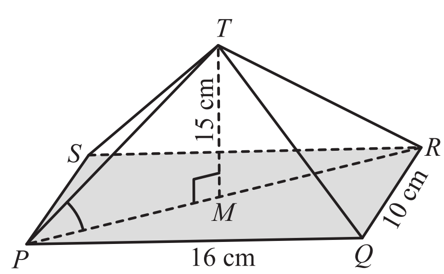
$$= 82.8°$$

EXERCISE 7.3

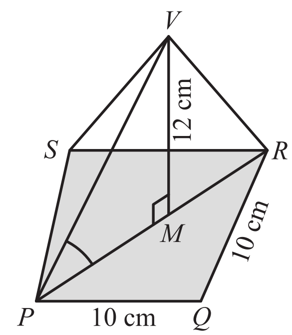
1. In the triangular prism, find
   (i) the length $\overline{CF}$.
   (ii) the length $\overline{BF}$.
   (iii) the angle BFC, correct to one decimal place.



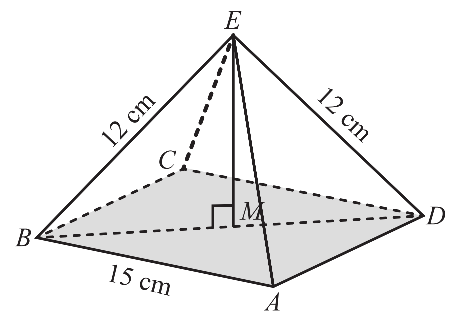
2. The diagram shows a triangular pyramid with a horizontal rectangular base $PQRS$, in which $m\overline{PQ} = 16$ cm, $m\overline{QR} = 10$ cm. M is the midpoint of the line PR. The vertex, T, is vertically above M and $m\overline{MT} = 15$ cm. Calculate the size of the angle between TP and the base $PQRS$. Give your answer correct to 1 decimal place.



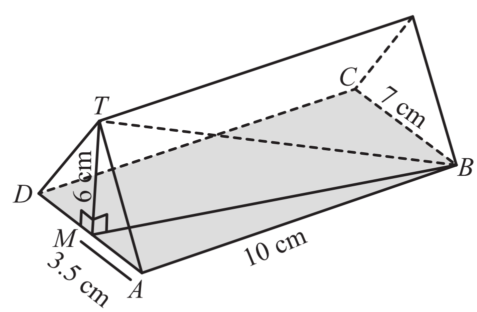
3. The diagram shows a pyramid. The base, $PQRS$, is a horizontal square of side 10 cm. The vertex, V, is vertically above the midpoint, M and $m\overline{VM} = 12$ cm. Calculate the size of angle VPM.



4. $ABCDE$ is a square based pyramid, in which $m\overline{AE} = m\overline{BE} = m\overline{CE} = m\overline{DE} = 12$ cm and $m\overline{AB} = 15$ cm. Calculate the size of angle DEB. Give your answer in degree (whole numbers).



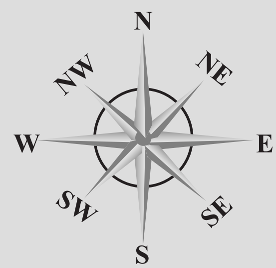
5. The diagram shows a triangular prism with a horizontal rectangular base $ABCD$. $m\overline{AB} = 10$ cm, $m\overline{BC} = 7$ cm, M is the midpoint of AD. The vertex T is vertically above M and $m\overline{MT} = 6$ cm. Calculate the size of the angle between $\overline{TB}$ and the base.



6. In an isometric game, the camera is placed at a 45° angle from the ground. If the player is 10 units in front and 10 units above, what is the direct line of sight distance?
7. A surveyor spots the top of a tower at an elevation angle of 28°. He is standing 60 metres from the base. Find the height of the tower.
8. A boat crosses a river 80 m wide, at a 60° angle to the current. What distance does the boat actually travel?
9. A listener hears a sound from two speakers. One is 6 m directly ahead and the other is at 30° to the side, 6 m away. Find the distance between the speakers.
10. A 10 m ladder leans against a wall making an angle of 75° with the ground. How high does it reach up the wall?

11. A lighthouse is located on a cliff 80 m above sea level. A ship is spotted at an angle of depression of 12°. How far is the ship from the base of the cliff?

Do you know?



Cardinal directions are the four main points on a compass: North, South, East and West. **Ordinal directions** are the four inter-cardinal points of a compass: NE, SE, SW, NW

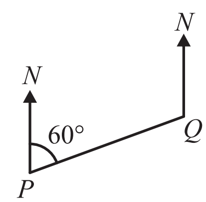
7.5 Three Figure Bearing and its Applications

A bearing is a way of describing the direction or angle from one point to another, measured in degrees from the north direction in a clockwise manner. It is always written as a three-digit angle between 000° and 360°.

It is used in navigation, aviation, surveying and map reading to describe precise direction.

For example, a plane flying east has a bearing of 090°.

Example 12 In the diagram, the bearing of Q from P is 060°. Find the bearing of P from Q.



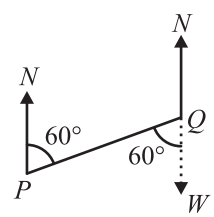
Solution Extend the line NQ, so that angle $NQW = 180°$

As alternate angles are equal, therefore

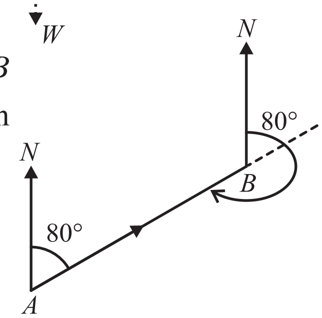
$$m\angle PQW = 060°$$

So, the bearing of P from Q = 180° + 060°

= 240°



Example 13 Abdullah sails a ship from point A to point B on a bearing of 080°. What bearing should he follow to return from point B back to point A?



Solution

The diagram shows the journey from A to B.

Extend the line of the journey and we get an angle of 80° at B.

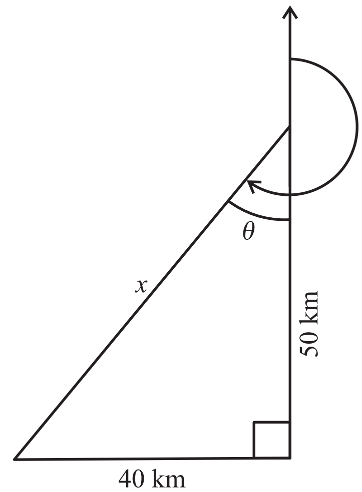
Bearing of A from B = 080° + 180°

= 260°

Example 14 An aeroplane departs from an airport and travels 40 km due east, followed by 50 km due north. The pilot then returns to the airport along the shortest possible route. What is the bearing and the distance of the shortest route?

Solution

To find θ



$$\tan\theta = \frac{40}{50} = 0.8$$

$$\theta = \tan^{-1}(0.8)$$

$$= 38.66^\circ$$

Bearing $= 180^\circ + 38.66^\circ = 218.66^\circ$

Suppose x is the shortest route

$$x^2 = 40^2 + 50^2$$

$$= 1600 + 2500$$

$$= 4100$$

$$x = \sqrt{4100}$$

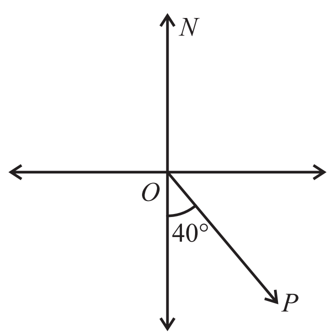
$$= 64.03\text{ km}$$

The pilot will fly on a bearing of 218.66° for 64.03 km to return to the airport.

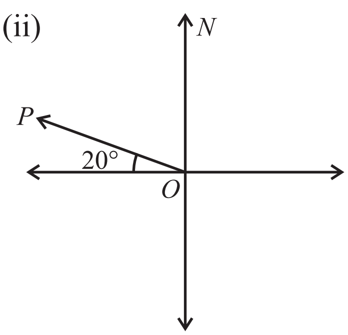
EXERCISE 7.4

1. Find bearing of point P in each of the following:

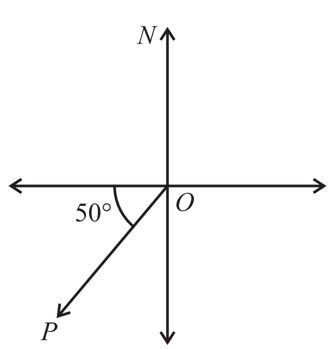
(i)



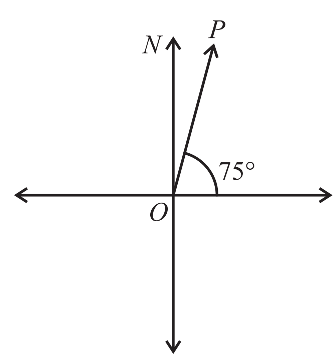
(ii)



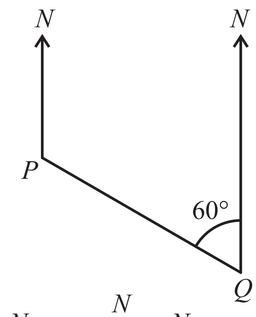
(iii)



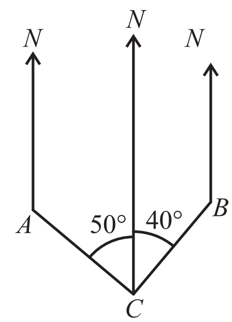
(iv)



2. The diagram shows the positions of two ships P and Q.
   (i) What is the bearing of ship P from ship Q?
   (ii) What is bearing of ship Q from ship P?

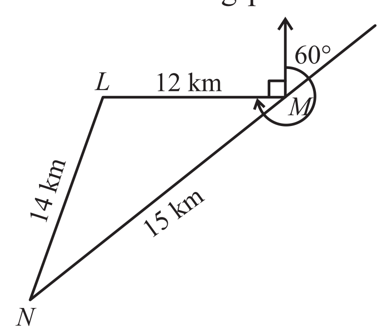


3. The diagram shows 3 places A, B and C. Find the bearing of:
   (i) A from C
   (ii) C from A
   (iii) C from B
   (iv) B from C



4. Abdul Hadi walks 100 m North and then 300 m East.
   (i) How far is he from his starting position?
   (ii) On what bearing should he walk to get back to his starting position?

5. Three ships L, M, N are in the position shown in the diagram. Ship M is North East of ship N. Find the bearing of L from M.



6. A ship sails 20 km on a bearing of 120°. How far to the South has the ship moved from its original position?

7. Fatima walked South for 5.5 km and then turned West for 1.3 km. Calculate Huria's bearing from her starting point.

8. An aircraft flies 150 km east, then 100 km northeast (45° from East). What is the total displacement?

9. A ship sails 100 km on a bearing of 045°, then changes course and sails 120 km on a bearing of 135°. Find the distance between the starting point and the final position.

REVIEW EXERCISE 7

1. Four possible answers are given for the following questions. Choose the correct answer.

(i) What is the value of cot 60°?

(a) 0 (b) 1 (c) $\sqrt{3}$ (d) $\frac{1}{\sqrt{3}}$

(ii) All trigonometric ratios are positive in:

(a) I-quadrant (b) II-quadrant
(c) III-quadrant (d) IV-quadrant

(iii) cosec θ is positive in:

(a) I & III-quadrants (b) II & IV-quadrants
(c) I & II-quadrants (d) I & IV-quadrants

(iv) $\sin(90^{\circ} + \theta) =$

(a) $\sin\theta$ (b) $-\sin\theta$ (c) $\cos\theta$ (d) $-\cos\theta$

(v) $\tan(180^{\circ} - \theta) =$

(a) $\tan\theta$ (b) $\cot\theta$ (c) $-\cot\theta$ (d) $-\tan\theta$

(vi) The law of sines is:

(a) $\frac{b}{\sin\alpha} = \frac{a}{\sin\beta} = \frac{c}{\sin\gamma}$ (b) $\frac{a}{\sin\alpha} = \frac{c}{\sin\beta} = \frac{b}{\sin\gamma}$

(c) $\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$ (d) $\frac{a}{\sin\alpha} = \frac{a}{\sin\beta} = \frac{a}{\sin\gamma}$

(vii) The law of cosines is:

(a) $\cos\alpha = \frac{b^2 + a^2 - c^2}{2ab}$ (b) $\cos\beta = \frac{c^2 + b^2 - a^2}{2cb}$

(c) $\cos\gamma = \frac{a^2 + b^2 - c^2}{2ab}$ (d) $\cos\gamma = \frac{ca^2 + b^2 - c^2}{2ab}$

(viii) Area of $\Delta ABC =$

(a) $\frac{1}{2}ab\sin\alpha$ (b) $\frac{1}{2}bc\sin\beta$ (c) $\frac{1}{2}\sin\gamma$ (d) $\frac{1}{2}ac\sin\beta$

(ix) Bearing is measured from:

(a) East (b) West (c) North (d) South

(x) Bearing is written as a:

(a) 1 figure (b) 2 figures (c) 3 figures (d) 4 figures

2. Calculate the area of ΔABC, in which

(i) $a = 4$ cm, $b = 6$ cm, $c = 8$ cm (ii) $b = 2.1$ cm, $c = 5$ cm, $\gamma = 45^{\circ}$

(iii) $c = 3.1$ cm, $\gamma = 44^{\circ}$, $\alpha = 36^{\circ}$

3. Solve the triangle ABC, in which

(i) $a = 5.4$ cm, $b = 3.4$ cm, $\alpha = 49^{\circ}$ (ii) $\alpha = 32^{\circ}$, $\gamma = 48^{\circ}$, $c = 81$cm

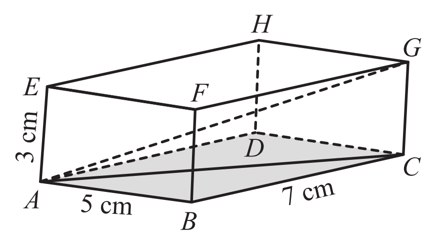
4. The diagram shows a cuboid $ABCDEFGH$ in which $m\overline{AB} = 5$ cm, $m\overline{BC} = 7$ cm and $m\overline{AE} = 3$ cm.

(i) Calculate the length of $\overline{AG}$.

Give your answer correct to 3 significant figures.

(ii) Calculate the size of the angle between $\overline{AG}$ and the plane $ABCD$.

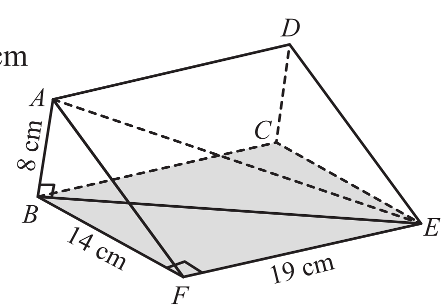
Give your answer correct to 1 decimal place.



5. The diagram shows a triangular prism $ABCDEF$ in which $m\overline{AB} = 8$ cm, $m\overline{BF} = 14$ cm and $m\overline{EF} = 19$ cm.

(i) Calculate the distance between A and F.

(ii) Calculate the angle between $\overline{AF}$ and the plane $BCEF$.



6. Hashim walks 4 km due North, then turns and walks 3km due East. What is the bearing from the starting point to his final position?

7. A pilot flies 200 km on a bearing of 045°, then turns and flies 150 km on a bearing of 135°. How far is the plane from its original position?

UNIT 8

Chords and Arcs of a Circle

Students' Learning Outcomes

After completing this unit, the students will be able to:

- Solve problems by using the properties of a circle:
  - One and only one circle can pass through three non- collinear points.
  - A straight line, drawn from the centre of a circle to bisect a chord (which is not a diameter) is perpendicular to the chord.
  - Perpendicular from the centre of a circle on a chord bisects it.
  - If two chords of a circle are congruent, then they will be equidistant from the centre.
  - Two chords of a circle which are equidistant from the centre are congruent.
  - If two arcs of a circle (or of congruent circles) are congruent, then the corresponding chords are equal.
  - If two chords of a circle (or of congruent circles) are equal, then their corresponding arcs (minor, major or semi-circular) are congruent.
  - Equal chords of a circle (or of congruent circles) subtend equal angles at the centre (at the corresponding centres).
  - If the angles subtended by two chords of a circle (or congruent circles) at the centre (corresponding centres) are equal, then the chords are equal.
- Apply concepts of chords and arcs of a circle to real life world problems (such as decorative features, rainbow, bridges, roller coaster track).

INTRODUCTION

This unit explores the fascinating properties of circles and their practical applications in real-world contexts. Students will learn how to solve problems by understanding key geometric facts, such as the uniqueness of a circle passing through three non-collinear points and the relationships between chords, arcs and the centre of a circle. They will investigate how perpendiculars from the centre affect chords and how congruence in chords and arcs leads to equal distances and angles. These concepts are not only fundamental in geometry but also useful in analyzing and designing real-life structures and patterns like rainbows, bridges, roller coaster tracks and decorative elements. Through engaging activities and problem-solving tasks, students will deepen their understanding of circular geometry and its relevance in everyday life.

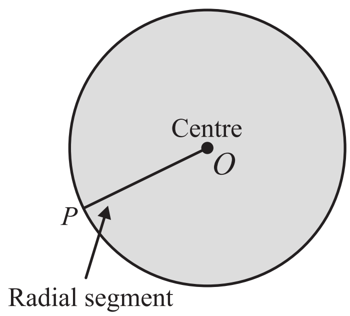
Circle and its Components

Circle A circle is the set of all points that are at the same distance from a fixed point.

We also define circle as:
A circle is the locus of a point in a plane which moves in such a way that its distance from a fixed point is always constant.

Radial segment

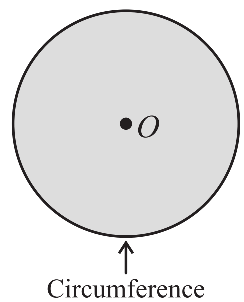
The line segment joining any point of the circle to its centre is called **radial segment** and its measure is called **radius** of the circle.



Circumference

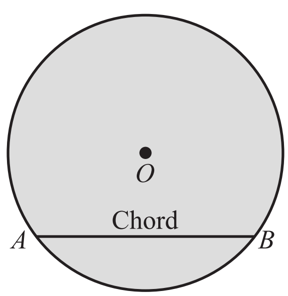
The length of the boundary of a circle is called circumference.

$C = 2\pi r$



Chord

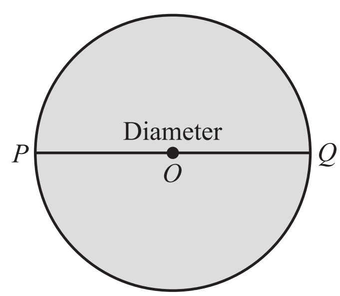
A line segment joining any two distinct points of a circle. In the below figure, $\overline{AB}$ is a chord of the circle.



Diameter

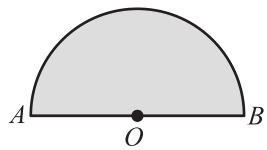
A chord passing through the centre of circle is called diameter.

$\overline{PQ}$ is diameter of the circle.



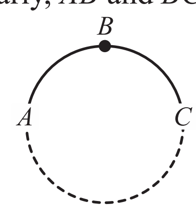
Semi-Circle

A semi-circle is half of the circle, that is a figure bounded by a diameter and circumference cut by the diameter.



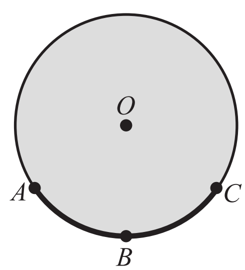
Arc

Any part of the circumference of a circle is called an arc. *ABC* is an arc of the circle. Similarly, *AB* and *BC* are also arcs.



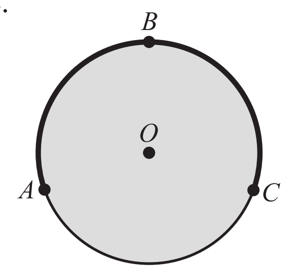
Minor arc

A minor arc is an arc which is smaller than the arc of a semi-circle. *ABC* is a minor arc.



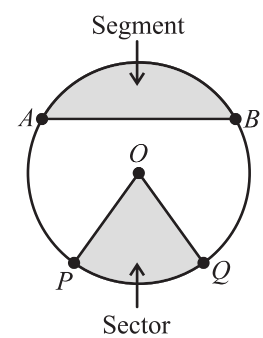
Major arc

A major arc is an arc which is greater than the arc of a semi-circle. *ABC* is a major arc.



Segment

The area (region) enclosed between a chord and the arc which it cuts off is called a segment of the circle.

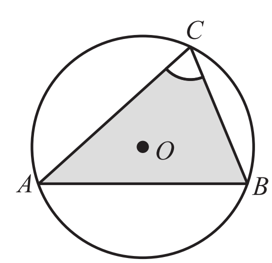


Sector

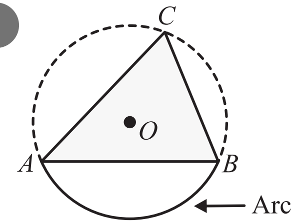
The area (region) enclosed between two radial segments (Radii) and an arc of a circle is called sector of the circle.

Angle in the Segment

If a point on the arc of a segment is joined with the end points of a chord, the angle formed by the line segments so drawn is called angle in the segment. $\angle ACB$ is an angle in the segment *ACB*. This is also called **circum angle.**



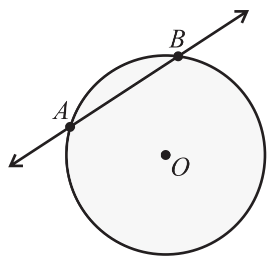
Angle Standing upon an Arc



If a point on the circumference of a circle is joined to the extremities of an arc, the angle so formed is said to angle stand upon the arc.

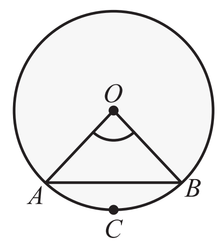
Secant

A straight line cutting the circumference of a circle at two distinct points is called a secant. $\overleftrightarrow{AB}$ is a secant.



Central Angle

If the extremities of an arc or of a chord are joined with the centre of a circle, angle so formed is called a **central angle**. $\angle AOB$ is a central angle of chord AB or of arc ACB.

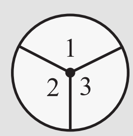


Key Concept

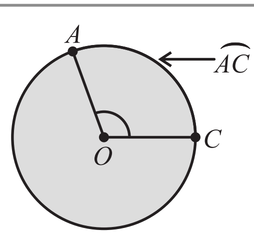
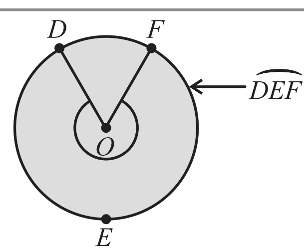
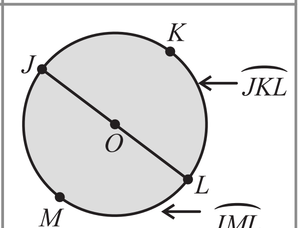
Sum of central angles

The sum of the measures of the central angles of a circle with no interior points in common is 360°.

i.e. $m\angle 1 + m\angle 2 + m\angle 3 = 360°$



Key Concept

Minor arc	Major arc	Semi-circle
An arc whose central angle measures less than 180°.	An arc whose central angle measures greater than 180°.	An arc whose central angle measures 180°.
		

Challenge!

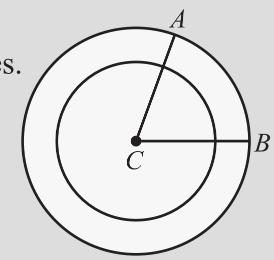
What is the difference between chord and secant?

Do you know?

(i) Circles with same centre are called concentric circles.

(ii) Any two circles are similar.

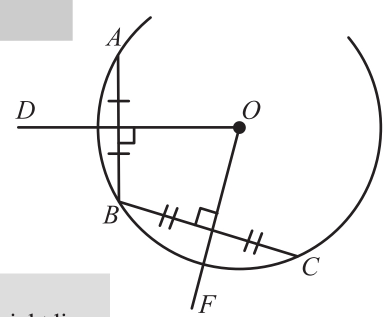
(iii) A cricle whose radius is zero is called point circle.



8.1 Theorems (Properties) of a Circle

8.1.1 One and Only One Circle Can Pass Through Any Three Non-Collinear Points.

If A, B and C are three points not in the same straight line, then one and only one circle can pass through A, B and C.



Note

- A circle cannot be drawn through more than two points in a straight line.
- If two circles have three points common, then they coincide.
- Two circles cannot intersect in more than two points.

Example 1 Prove that a unique circle can pass through the points $A(1, 1)$, $B(4, 2)$ and $C(3, 5)$.

Solution We will check the points are non-collinear by using slope formula.

$$\text{Slope of } \overline{AB} = \frac{2-1}{4-1} = \frac{1}{3}$$

$$\text{Slope of } \overline{BC} = \frac{5-2}{3-4} = \frac{3}{-1} = -3$$

$$\text{Slope of } \overline{AB} \neq \text{Slope of } \overline{BC}$$

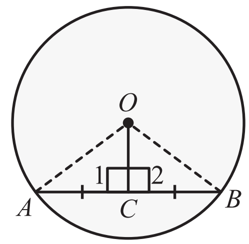
Remember!

If $A(x_1, y_1)$ and $B(x_2, y_2)$,

then Slope of $\overline{AB} = \frac{y_2 - y_1}{x_2 - x_1}$ or $\frac{y_1 - y_2}{x_1 - x_2}$

Hence, the given points are non-collinear and unique circle can pass through them.

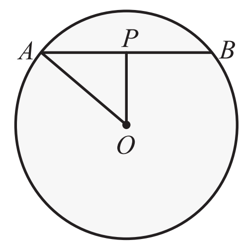
8.1.2 A Straight Line, Drawn from the Centre of a Circle to Bisect a Chord (which is not a diameter) is Perpendicular to the Chord.



If $\overline{AB}$ is any chord (not a diameter) and $\overline{OC}$ meets $\overline{AB}$ at C such that $m\overline{CA} = m\overline{CB}$, then $\overline{OC} \perp \overline{AB}$

Example 2 In a circle, $m\overline{AB} = 8$ cm is bisected at point P by a line segment from the centre O. Find $m\overline{OP}$ if radius is 5 cm.

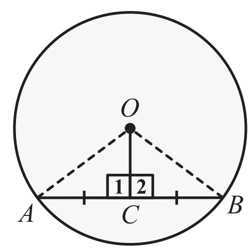
Solution $m\overline{AB} = 8$ cm , $m\overline{OA} = 5$cm , $m\overline{AP} = 4$ cm



Since triangle APO is right angled triangle.
So, by Pythagoras theorem

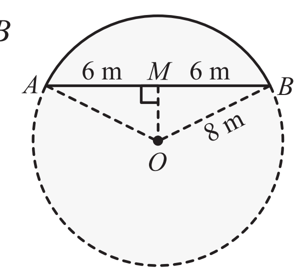
$$\left(m\overline{OA}\right)^2 = \left(m\overline{OP}\right)^2 + \left(m\overline{AP}\right)^2$$
$$5^2 = \left(m\overline{OP}\right)^2 + 4^2$$
$$25 = \left(m\overline{OP}\right)^2 + 16$$
$$\left(m\overline{OP}\right)^2 = 25 - 16 = 9$$
$$m\overline{OP} = 3\,\text{cm}$$

8.1.3 Perpendicular from the Centre of a Circle on a Chord Bisects it.



If $\overline{OC} \perp \overline{AB}$, then $\overline{OC}$ bisects $\overline{AB}$
i.e. $m\overline{CA} = m\overline{CB}$

Example 3 In a circular fountain of radius 8 m, a pipe AB spans across the fountain. If $m\overline{AB} = 12$ m and O is centre of the circle. Find the distance from O to the chord AB.

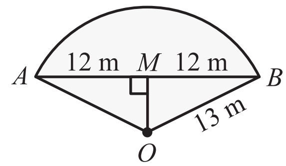


Solution Given that $m\overline{AB} = 12$ m

Distance from O to chord AB is perpendicular so by theorem, M is the mid-point.

$$m\overline{OM} = \sqrt{(m\overline{OB})^2 - (m\overline{BM})^2} \qquad \text{[By Pythagoras theorem]}$$
$$= \sqrt{(8)^2 - (6)^2}$$
$$= \sqrt{64 - 36} = \sqrt{28}$$
$$= 5.29 \text{ m}$$

Example 4 Chord AB of 24 m is across a circular arch of radius 13 m. Find the height of the chord from the centre.



Solution $m\overline{AB} = 24$ m across a circular arch. O is centre of the circle of the arch that $m\overline{OB} = 13$ m.

$$\overline{OM} \perp \overline{AB}, \text{ therefore, } m\overline{MA} = m\overline{MB} = \frac{24}{2} = 12 \text{ m}$$

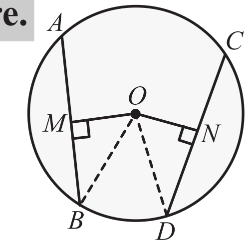
Now, $\left(m\overline{OM}\right)^2 = \left(m\overline{OB}\right)^2 - \left(m\overline{MB}\right)^2$ [By Pythagoras theorem]

$$\left(m\overline{OM}\right)^2 = (13)^2 - (12)^2$$

$$m\overline{OM} = \sqrt{(13)^2 - (12)^2}$$

$$= \sqrt{169 - 144} = \sqrt{25} = 5 \text{ m}$$

8.1.4 If Two Chords of a Circle Are Congruent, then they will be Equidistant from the Centre.

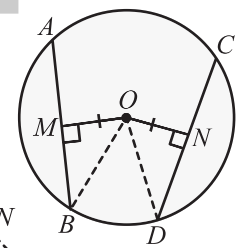


If $\overline{AB}$ and $\overline{CD}$ are congruent (equal in measures) chords of a circle with centre O. i.e,. $m\overline{AB} = m\overline{CD}$, $\overline{OM} \perp \overline{AB}$ and $\overline{ON} \perp \overline{CD}$, then $m\overline{OM} = m\overline{ON}$

Example 5 Two equal-length ropes (chords) are tied in a circular playground. One is 8 m from the centre. Where should the second be placed?

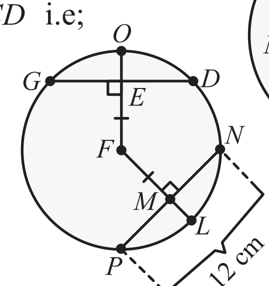
Solution Since chords are congruent, they must be equidistant from the centre. So, the second chord must be 8 m from the centre.

8.1.5 Two Chords of a Circle which are Equidistant from the Centre of a Circle, are Congruent.



If $\overline{AB}$ and $\overline{CD}$ are chords of a circle with centre O, $\overline{OM} \perp \overline{AB}$, $\overline{ON} \perp \overline{CD}$ and $m\overline{OM} = m\overline{ON}$, then $\overline{AB} \cong \overline{CD}$ i.e; $m\overline{AB} = m\overline{CD}$

Example 6 In a circle with centre at F, $m\overline{NP} = 12$ cm and $m\overline{EF} = m\overline{FM}$. Find the length of DE.



Solution As $m\overline{EF} = m\overline{FM}$

then $m\overline{DG} = m\overline{NP}$ (Two chord of a circle which are equidistant from the centre are congruent.)

$m\overline{DG} = 12 \text{ cm} \qquad \because \quad m\overline{NP} = 12 \text{ cm}$

Since the radius bisects $\overline{DG}$, so

$m\overline{GE} = m\overline{DE}$

$\Rightarrow \quad m\overline{DE} = \frac{1}{2} m\overline{DG} \qquad \Rightarrow \quad \frac{1}{2}(12 \text{ cm}) = 6 \text{ cm}$

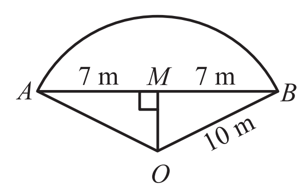
Example 7 In a round table design, two decorations are placed symmetrically 4 cm from the centre. One chord is 14 cm. What is the length of the other?

Solution Since both decorations are equidistant from the centre, therefore chords are congruent.

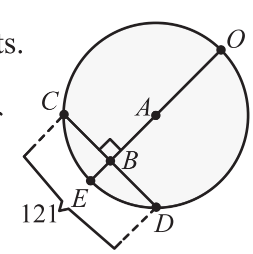
So, the length of second chord is 14 cm.

EXERCISE 8.1

1. Calculate the length of a chord which stands at a distance of 5 cm from the centre of a circle whose radius is 13 cm.
2. In construction, three steel rods are fixed at points A, B, and C (not in a straight line). A circular hoop needs to pass through all three. How many hoops can be used?
3. In a park, lamp posts are 14 m apart on the edge of a circular part of radius 10 m as shown in figure. Find the distance of the chord from the centre of the park.



4. In a circle, chords AB and CD both have length 10 cm. If the distance from the centre to $\overline{AB}$ is 6 cm, what is the distance from centre to $\overline{CD}$?
5. Two holes A and B are drilled 12 cm apart on a circular tabletop of radius 10 cm. Find the perpendicular distance from the centre to AB.
6. A chord 8 cm long is at a distance of 3 cm from the centre. Calculate the radius of the circle.
7. In the given figure, $m\overline{CD} = 121$ units and $m\overline{BC} = 3x$ units. Find the value of x.

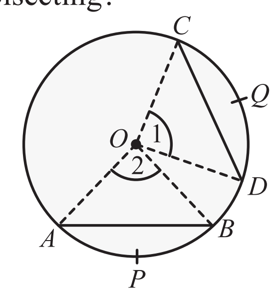


8. In a circle with centre at O, the perpendicular distance of the each chord PQ and RS from the centre is 6 cm. If the length of chord PQ is 18 cm, find the length of the other chord.

9. A line from the centre of a circle cuts a 10 cm chord at right angle where radius of the circle is 6cm. What is the length from the centre to the chord?

10. In a circle, a perpendicular is drawn from the centre to chord AB. If $m\overline{AB} = 12\,\text{cm}$, what is the length of each segment after bisecting?

8.1.6 If Two Arcs of a Circle (or of Congruent Circles) are Congruent, then the Corresponding Chords are Equal.



In the circle with centre O, $\widehat{APB} \cong \widehat{CQD}$ and $\overline{AB}$, $\overline{CD}$ are their corresponding chords, then $\overline{AB} \cong \overline{CD}$

Example 8 In a circular garden, two pair of decorative lights are placed such that the arcs between each pair (A and B, C and D) are equal in the length. If the straight-line distance between points A and B is 8 metres, what is the distance points C and D?

Solution Given that

$$m\overline{AB} = 8 \text{ metres}$$

Since the arcs AB and CD are congruent, so their chords must be equal.

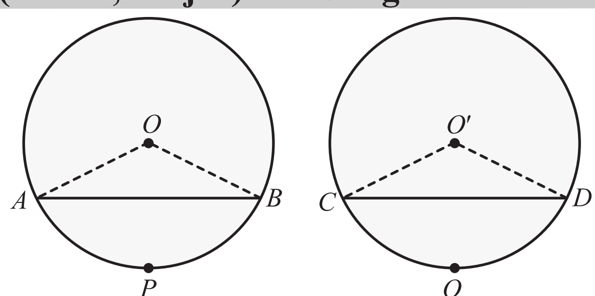
$$m\overline{AB} = m\overline{CD}$$

Hence, $m\overline{CD} = 8$ m

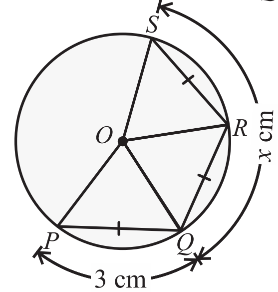
8.1.7 If Two Chords of a Circle (or of Congruent Circles) are Congruent (Equal in Measurement) then their Corresponding Arcs (Minor, Major) are Congruent.

If circles with centre O and O' are congruent and Chord $AB \cong$ Chord CD,

then $\widehat{APB} \cong \widehat{CQD}$



Example 9 Find the value of x in the given figure.



Solution Given that:

Chord $PQ \cong$ Chord $QR \cong$ Chord RS

As, we know that equal chords have equal arcs.

$\therefore$ Arc $PQ \cong$ Arc $QR \cong$ Arc RS

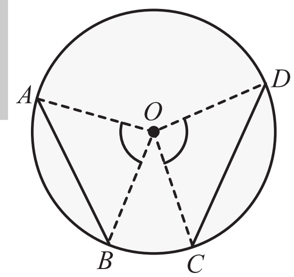
But Arc $PQ = 3$ cm

$\therefore$ Arc $QR = 3$ cm and Arc $RS = 3$ cm

From figure, we have

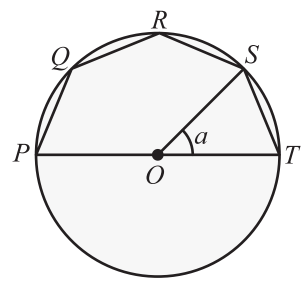
$$\text{Arc } QR + \text{Arc } RS = x$$
$$3 \text{ cm} + 3 \text{ cm} = x$$
$$x = 6 \text{ cm}$$

8.1.8 Equal Chords of a Circle (or of Congruent Circles) Subtend Equal Angles at the Centre (at the corresponding Centres).



In the circle with centre O, $m\overline{AB} = m\overline{CD}$, chord AB and chord CD subtended $\angle AOB$ and $\angle COD$ respectively at the centre O, then $m\angle AOB = m\angle COD$

Example 10 In the given figure, $\overline{PT}$ is a diameter of circle. If $m\overline{PQ} = m\overline{QR} = m\overline{RS} = m\overline{ST}$, then find a.



Solution $m\overline{PQ} = m\overline{QR} = m\overline{RS} = m\overline{ST}$ (given)

$\therefore$ $m\angle POQ = m\angle QOR = m\angle ROS = m\angle SOT = a$

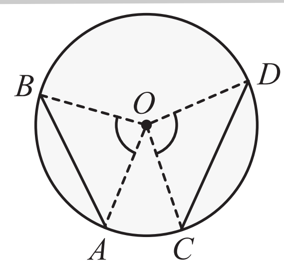
(equal chords have equal angles at the centre)

But

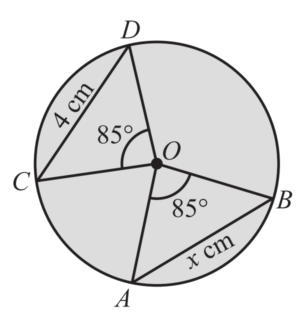
$$m\angle POQ + m\angle QOR + m\angle ROS + m\angle SOT = 180°$$
$$4a = 180°$$
$$a = 45°$$

8.1.9 If the Angles Subtended by Two Chords of a Circle (or Congruent Circles) at the Centre (Corresponding Centres) are Equal, then the Chords are Equal.

In the circle with centre O, $\overline{AB}$ and $\overline{CD}$ are its chords, they subtend $\angle BOA$ and $\angle COD$ respectively at the centre O, of the circle such that $m\angle BOA = m\angle COD$, then $m\overline{AB} = m\overline{CD}$



Example 11 In the given figure, find the value of x.



Solution From the given figure

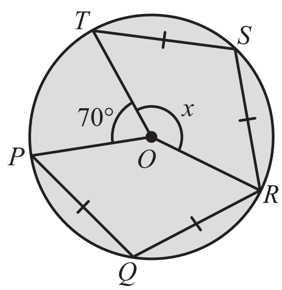
$$m\angle AOB = m\angle COD = 85°$$

$\because$ If central angles are equal, then corresponding chords are also equal.

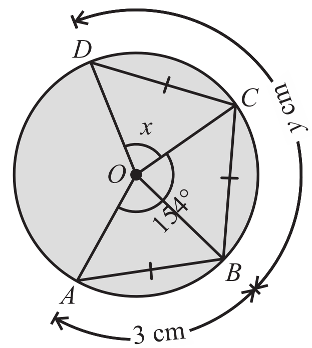
$$m\overline{AB} = m\overline{CD}$$
$$x = 4 \text{ cm}$$

EXERCISE 8.2

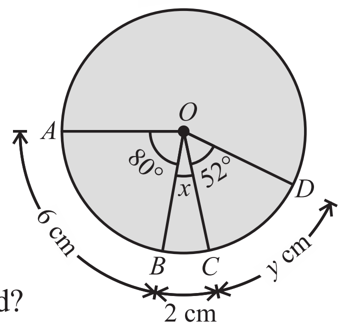
1. In the given figure, if $m\overline{PQ} = m\overline{QR} = m\overline{RS} = m\overline{ST}$ and $m\angle POT = 70°$, then find the value of x.



2. In the given figure, find the values of x and y.



3. Find the values of x and y in the given figure, such that $m\widehat{AB} = m\widehat{BD}$



4. Two congruent arcs in a circular track subtend angles of 60° each at the centre. If the length of one chord of the circle is 10 metres, what is the length of other chord?
5. In a circular fountain, two water jets are installed such that they spray water along arcs of equal length. If one jet sprays between points P and Q and the straight-line distance (chord PQ) is 12 metres, what is the straight-line distance (chord RS) covered by the second jet spraying along arc RS, which is congruent to arc PQ?
6. In a circular park, two walkways $\overline{AB}$ and $\overline{CD}$ are both straight paths (chords) of length 14 metres. What can be said about the minor arcs subtended by these walkways?
7. In two congruent circular clocks, the minute hand points from the centre to the 3 on both. A decorative string connects 3 to 9 on both clocks. Are the arcs from 3 to 9 on both clocks congruent?

REVIEW EXERCISE 8

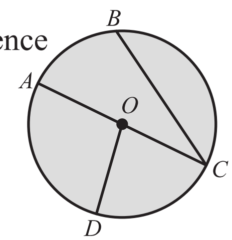
1. Four possible answers are given for the following questions. Choose the correct answer.
   (i) Distance of a point on the circumference to the centre of the circle is called:
   (a) radius (b) arc (c) chord (d) tangent

(ii) Radii of same circles are:
(a) all unequal (b) all equal
(c) half of each chord (d) double of the diameter

(iii) The boundary of the circle is called:
(a) chord (b) segment (c) circumference (d) diameter

(iv) Any part of a circumference is called:
(a) chord (b) diameter (c) radius (d) arc

(v) A chord passing through the centre of the circle is called:
(a) radius (b) diameter (c) secant (d) circumference



(vi) In the given figure, what is AB?
(a) diameter (b) tangent (c) chord (d) arc

(vii) In the given figure, major arc is:
(a) $m\widehat{AB}$ (b) $m\widehat{BC}$ (c) $m\widehat{BDC}$ (d) $m\widehat{AD}$

(viii) _______ circle(s) can pass through the three non-collinear points.
(a) one (b) two (c) three (d) many

(ix) Perpendicular bisector of a chord always passes through the _____ of circle.
(a) arc (b) radius (c) centre (d) circumference

(x) The sum of the measures of central angles of a circle is:
(a) 90° (b) 180° (c) 270° (d) 360°

2. On a circular clock with a radius of 6cm, the points from 2 to 10 form a chord that is 10cm long. Find the perpendicular distance from the centre of clock to the chord.

3. A chord 6 cm long is at a distance of 4 cm from the centre. Calculate the radius of the circle.

4. In a circular park, two benches are placed so that the chords formed by their positions are both 10 m long. What can you say about the angles subtended at the centre by each bench?

5. Jannat is designing a triangular garden with corners at points $A(2, 1)$, $B(4, 4)$ and $C(1, 5)$. Can she install a circular fountain that touches all three corners?

6. Two chords, PQ and RS, each measure 18cm in length. If the distance of PQ from the centre of circle is 7cm. Find the distance of RS from the centre.

7. A tree branch lies across a circular pond, forming a 20 m chord. A measuring rod having length 10 m is perpendicular to chord from the centre. What is the radius of the pond?

8. A steel bar 6 m long lies inside a circular structure with ends on the circle. It is bisected by a perpendicular rod from the centre. Find the radius of the circular structure if the perpendicular distance from the centre to the bar is 4 m.

UNIT 9

Tangent and Angles of a Circle

Students' Learning Outcomes

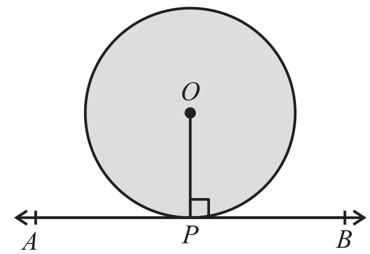
After completing this unit, the students will be able to:

- Solve problems by using the tangent and angle properties of a circle:
  - If a line is drawn perpendicular to a radial segment of a circle at its outer end point, it is tangent to the circle at that point.
  - The tangent to a circle and the radial segment joining the point of contact and the centre are perpendicular to each other.
  - The two tangents drawn to a circle from a point outside it, are equal in length.
  - If two circles touch externally or internally, then the distance between their centres is respectively equal to the sum or difference of their radii.
  - The measure of a central angle of a minor arc of a circle is double that of the angle subtended by the corresponding major arc.
  - Any two angles in the same segment of a circle are equal.
  - The angle in a semi-circle is a right angle, in a segment greater than a semi-circle is less than a right angle, in a segment less than a semi-circle is greater than a right angle.
  - The opposite angles of any quadrilateral inscribed in a circle are supplementary.
- Find the arc length and area of sector of a circle using angle in both degrees and radians.
- Apply concepts of tangents and angles of a circle to real life world problems (such as architecture, monuments, pyramids).
- Apply trigonometry to solve real life problems in arc length and area of the sector of circle.

INTRODUCTION

This unit focuses on solving problems using the tangent and angle properties of a circle, which are essential for understanding the geometry of circular shapes in both mathematical and real-life contexts. Students will explore key concepts such as the perpendicular relationship between a radius and a tangent, the equality of tangents drawn from an external point and the geometric properties of angles formed within and around circles. The unit also examines how the positioning of angles affects their measures and how these properties relate to cyclic quadrilaterals. Practical applications include solving problems involving architecture, engineering structures, monuments and pyramids, where circular and angular relationships are commonly found. Students will also learn how to calculate arc lengths and areas of sectors using angles measured in both degrees and radians and apply trigonometric concepts to real-world scenarios involving sectors of a circle.

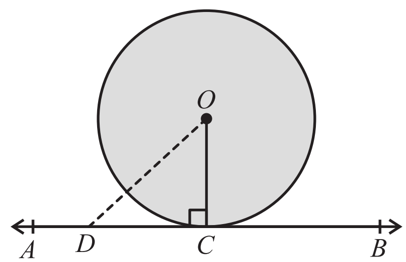
9.1 Tangent to a Circle



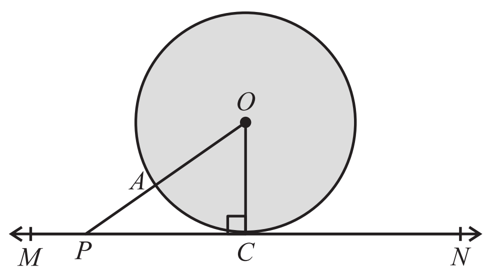
In a plane, a line is a tangent to a circle that touches only one point on the circumference of a circle. Here $\overleftrightarrow{AB}$ is a tangent at point P on the circle. Point P is called **point of tangency** or **point of contact**.

9.1.1 (a) If a Line is Drawn Perpendicular to a Radial Segment of a Circle at its Outer End Point, it is Tangent to the Circle at that Point

If a circle with centre O and $\overline{OC}$ is its radial segment. $\overleftrightarrow{AB} \perp \overline{OC}$ at C, then $\overleftrightarrow{AB}$ is tangent to the circle at point C.

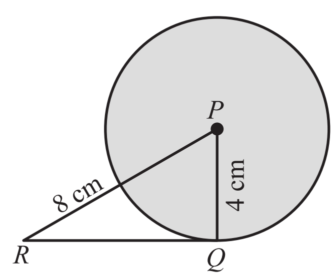


b The Tangent to a Circle and the Radial Segment Joining the Point of Contact and the Centre, are Perpendicular to Each Other.



If a circle with centre O, $\overline{OC}$ is a radial segment. $\overleftrightarrow{MN}$ is a tangent at point C, then $\overline{OC} \perp \overleftrightarrow{MN}$

Example 1 $\overline{QR}$ is a tangent to a circle with centre P at point Q of radius 4 cm. It meets the line segment PR such that $m\overline{PR} = 8$ cm. What is the length of $\overline{QR}$?



Solution Since $\overline{QR}$ is tangent to the circle at Q and it is perpendicular to the $\overline{PQ}$.

ΔPQR is a right-angled triangle and $\overline{PR}$ is its hypotenuse.
By using Pythagoras theorem,

$$\left(m\overline{PR}\right)^2 = \left(m\overline{PQ}\right)^2 + \left(m\overline{QR}\right)^2$$

$$8^2 = 4^2 + \left(m\overline{QR}\right)^2$$
$$64 = 16 + (m\overline{QR})^2$$
$$\left(m\overline{QR}\right)^2 = 64 - 16 = 48$$
$$m\overline{QR} = 6.93\,\text{cm}$$

Real Life Examples of Tangents Perpendicular to Radius

1- The entry or exit roads of a roundabout are often designed tangential to the circular structure to allow smooth transitions and minimize turning effort.

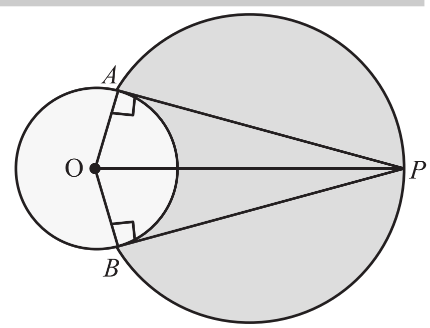
2- Each seating cabin on a Ferris wheel hangs such that its suspension arm is perpendicular to the wheel radius, ensuring the cabin stays tangent to the rotational path.

3- The tip of a clock hand touches the circular perimeter of the clock and if a tangent is drawn at the tip, it is perpendicular to the radius (hand) at that point.

9.1.2 The Two Tangents Drawn to a Circle From a Point Outside it, are Equal in Length.

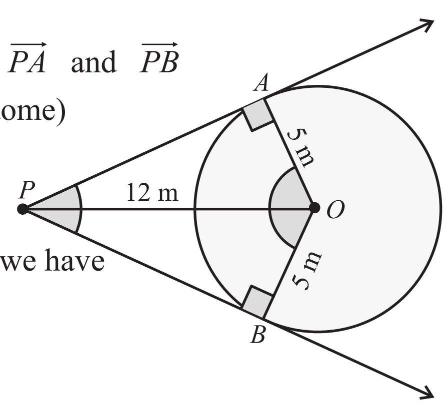
If $\overline{PA}$ and $\overline{PB}$ are two tangents drawn from an external point P to a circle having centre at O, where A and B are points of contact, then

$$m\overline{PA} = m\overline{PB}.$$



Example 2 An architect designs a circular dome on a building. A lighting rod is placed 12 m away from the centre of the dome which has a radius of 5 m. Support cables are attached from the rod to the dome, touching the dome at two points. What is the length of each cable?

Solution Suppose two tangents (cables) $\overrightarrow{PA}$ and $\overrightarrow{PB}$ are drawn to the circumference of the circle (dome) from the point P (on the rod). Here,



$$m\overline{OP} = 12\,m,\ m\overline{OA} = m\overline{OB} = 5\,\text{m}$$

In the right triangle OAP, by Pythagoras theorem, we have

$$\left(m\overline{OP}\right)^2 = \left(m\overline{OA}\right)^2 + \left(m\overline{AP}\right)^2$$
$$12^2 = 5^2 + \left(m\overline{AP}\right)^2$$

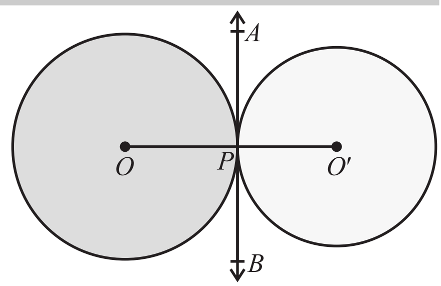
$$144 = 25 + \left(m\overline{AP}\right)^2$$
$$\left(m\overline{AP}\right)^2 = 144 - 25 = 119$$
$$m\overline{AP} = 10.91 \text{ m}$$

So, the length of each cable is 10.91 m

Real-Life Examples of Equal Tangents to a Circle

1. A lamp post is erected at a point outside a circular garden. Two paths are constructed from the lamp post to the edge of the circular garden such that they just touch the garden. These paths represent tangents to the circle.
2. A security camera is placed such that it monitors a circular restricted zone. The lines of sight just touch the boundary of the zone.
3. Two ropes from a peg outside a circular tent are tied to two points on the base circle of the tent.

9.1.3 (a) If Two Circles Touch Externally, then the Distance Between Their Centres is Equal to the Sum of their Radii.



If two circles with centres O and O' touch each other at point P externally, $\overline{PO}$ and $\overline{PO'}$ are their radii respectively, then

$$m\overline{OO'} = m\overline{OP} + m\overline{O'P}$$

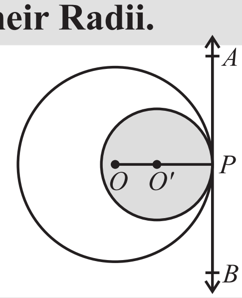
Example 3 The radii of the two circles are 4 cm and 6 cm. What is the distance between their centres if they touch externally?

Solution Since the circles touch externally,

So, the distance between their centres = sum of their radii

= 4 + 6 = 10 cm

b If Two Circles Touch Internally, Then the Distance Between Their Centres is Equal to the Difference of Their Radii.



If two circles with centres O and O' touch each other at point P internally, $\overline{PO}$ and $\overline{PO'}$ are their radii respectively, then $m\overline{OO'} = m\overline{OP} - m\overline{O'P}$.

Example 4 An inner decorative circle touches the outer clock face internally. If the outer circle has radius 12 cm and the inner has radius 7.5 cm, find the distance between their centres.

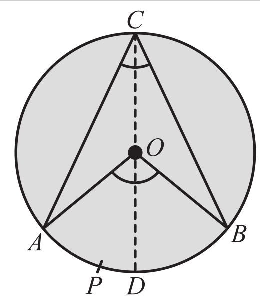
Solution Since, Radius of outer circle = 12 cm

Radius of inner circle = 7.5 cm

Thus, the distance between their centres = 12 – 7.5 = 4.5 cm

9.1.4 The Measure of a Central Angle of a Minor Arc of a Circle is Double that of the Angle Subtended by the Corresponding Major Arc.

In the circle with centre O, $\widehat{APB}$ is a minor arc. It subtends an angle AOB at the centre and an angle ACB is subtended by the major arc, then $m\angle AOB = 2m\angle ACB$



Example 5 A curved running track forms an arc that subtends 60° at the centre of the circular ground. What angle does it subtend at a point on the boundary?

Solution Since curved running track forms an arc that subtends 60° at the centre of the circular ground.

As we know that the measure of an angle which an arc of a circle subtends at the centre is double that which it subtends at any point on the remaining part of the circumference.

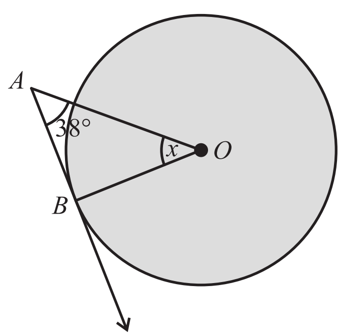
Therefore, angle at circumference $= \frac{60°}{2} = 30°$

Skilled Practice!

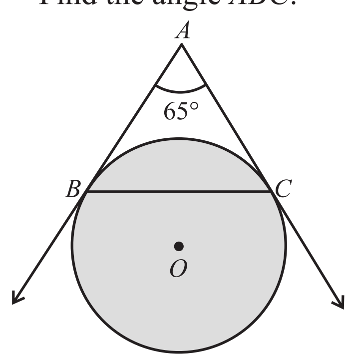
A radar antenna scans an arc subtending 90° at the centre. What is the angle subtended by the arc at the edge of its circular path?

EXERCISE 9.1

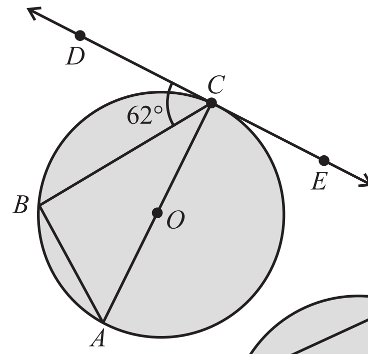
1. Find the value of x.



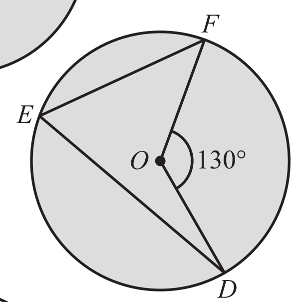
2. Find the angle ABC.



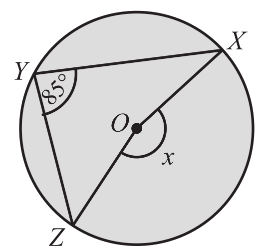
3. A, B and C are points on the circumference of a circle with centre at O. $\overline{AC}$ is the diameter of the circle and $\overleftrightarrow{DE}$ is the tangent to the circle at the point C and $m\angle BCD = 62°$. Find (i) $m\angle BCA$ (ii) $m\angle BAC$



4. D, E and F are points on the circumference of a circle with centre at O and $m\angle DOF = 130°$. Find $m\angle DEF$.



5. X, Y and Z are points on the circumference of a circle with centre at O and $m\angle XYZ = 85°$. Find x.



6. In a historical monument, a circular fountain with a radius of 3 m is built. A flagpole is erected 7 m away from the centre of the fountain. Two ropes from the pole are tied to the edge of the fountain, just touching it. Find the length of each rope.

7. Two circular gears touch each other externally for proper rotation in a machine. The radii of the two circular gears are 5cm and 7cm. What is the distance between their centres if they touch externally?

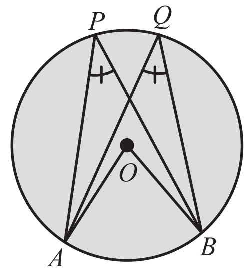
8. A small sensor lies inside a satellite dish and touches its wall internally. If the dish has radius 15 cm and the sensor has radius 2.5 cm, find the distance between their centres.

9. An inner holder touches the outer cylindrical container internally. If the outer container has radius 8 cm and the inner holder has radius 6 cm, find the distance between their centres.

10. A pyramid-shaped sculpture is placed in the center of a circular plaza with a radius of 10 m. A decorative pole stands 26 m from the center of the circle. Two guide wires are attached from the pole to the plaza edge, just touching the circle. Find the length of each wire.

9.1.5 Any Two Angles in the Same Segment of a Circle are Equal.

If $\angle APB$ and $\angle AQB$ be angles in the same segment $APQB$, of a circle whose centre is O, then $m\angle APB = m\angle AQB$

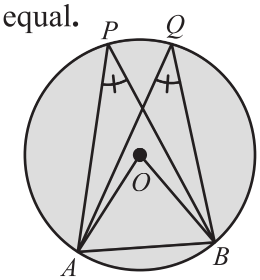


Example 6 In a large circular stained-glass window of a library, two decorative beams are drawn from the same chord AB to two points P and Q on the arc opposite to $\overset{\frown}{AB}$ such that $\angle APB$ and $\angle AQB$ are formed. If $\angle APB$ is measured to be 40°, what is $m\angle AQB$?

Solution According to the article 9.1.5, angles in the same segment are equal.

Since $m\angle APB = 40°$

Therefore, $m\angle AQB = 40°$



Skilled Practice!

A projector casts an image over an arc on a circular wall. Two observers on the arc segment report one observing a 50° angle. What is the other angle in the same segment?

9.1.6

(a) The Angle in a Semi-Circle is a Right Angle.

(b) The Angle in a Segment Greater than a Semi-Circle is Less than a Right-Angle.

(c) The Angle in a Segment Less than a Semi-Circle is Greater than a Right-Angle.

A circle with centre at O and $\overline{AC}$ is a chord of the circle.

(a) In Fig. I, ABC segment is a semi-circle, then $m\angle ABC = 90°$.

(b) In Fig. II, ABC segment is greater than a semi-circle, then $m\angle ABC < 90°$.

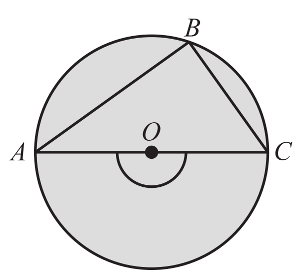


Figure I

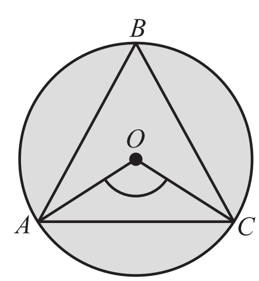


Figure II

(c) In Fig. III, ABC segment is less than a semi-circle, then $m\angle ABC > 90°$.

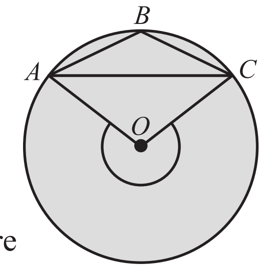
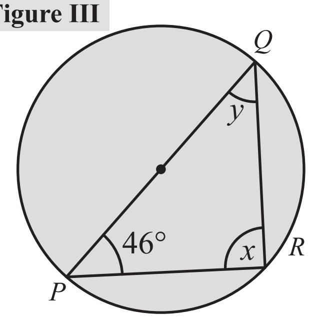


Figure III

Example 7 In the diagram P, Q and R are points on the circumference of a circle and the line segment PQ is the diameter. Find the angles x and y.



Solution Since angle in semi-circle is 90°.

Therefore, $x = 90°$

As we know that the sum of the interior angles of any triangle is 180°.

$$46° + 90° + y = 180°$$

i.e., $$136° + y = 180°$$

$$y = 180° - 136° = 44°$$

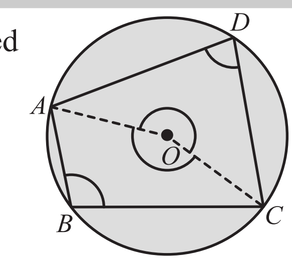
Skilled Practice!

A cable in a semicircular bridge forms a triangle with the diameter of the arch. If the cable meets the endpoints of the diameter and the top of the arch, what is the angle at the top?

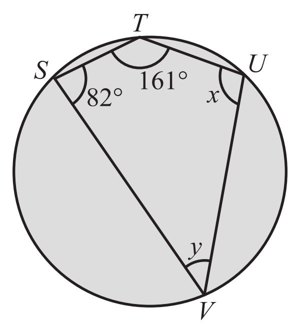
9.1.7 The Opposite Angles of Any Quadrilateral Inscribed in a Circle are Supplementary.

If a circle with centre at O and $ABCD$ is an inscribed quadrilateral, then

$$m\angle CBA + m\angle ADC = 180°, \quad m\angle BAD + m\angle BCD = 180°$$



Example 8 S, T, U and V are points on the circumference of a circle. Find the angles x and y.



Solution As we know that opposite angles of any quadrilateral inscribed in a circle are supplementary.

Therefore,

$x + 82° = 180°$	$y + 161° = 180°$
$x = 98°$	$y = 19°$

Skilled Practice!

A round table has a decorative quadrilateral pattern inscribed on its surface. If one angle of the quadrilateral is 105°, what is the measure of the opposite angle?

9.2 Area of Sector and Arc Length of a Circle

9.2.1 Arc Length and Area of Sector

Let ℓ be the length of a circular arc AB of a circle of radius r and θ be its central angle measure in radians. Then

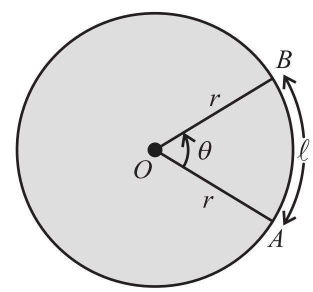
Ratio of ℓ to the circumference $2\pi r$ = Ratio of θ to 2π

or $\quad \ell : 2\pi r = \theta : 2\pi$

or $$\frac{\ell}{2\pi r} = \frac{\theta}{2\pi}$$

$$\ell = \frac{\theta}{2\pi} \times 2\pi r$$

$\ell = r\theta$, where θ is in radian.

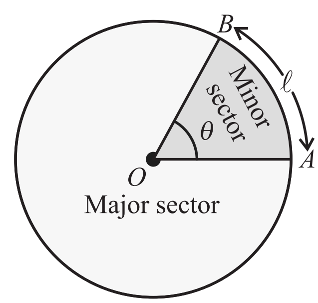


9.2.2 Area of a Sector

A sector of a circle is a region bounded by two radii and corresponding arc.

A slice of Pizza and region between the hands of a watch are examples of a sector of circle.

A pair of radii divides the circle into two regions namely major sector and minor sector.



The area of a sector of a circular region of radius r is $r^2\theta$ where θ is the central angle and it is measured in radians.

Proof: As shown in adjoining figure,

$$\frac{\text{Area of sector}}{\text{Area of circle}} = \frac{\text{Central angle of sector}}{\text{Complete angle in circle}}$$

$$\frac{A}{\pi r^2} = \frac{\theta}{2\pi}$$

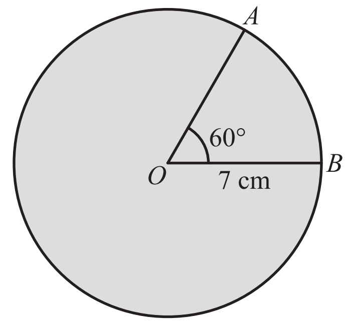
$$A = \frac{\pi r^2 \theta}{2\pi}$$

$\Rightarrow \quad$ Area of sector $= \frac{1}{2} r^2\theta$, where θ is in radians.

> **Note**
>
> $A = \frac{1}{2} r^2\theta$
>
> $\because \quad \ell = r\theta$
>
> So, $A = \frac{1}{2} r(r\theta)$
>
> $A = \frac{1}{2} r\ell$

Example 9 Find the area of sector of a circle having central angle 60° and radius 7 cm. Also find the length of the arc.

Solution Here $\theta = 60° = \frac{60 \times \pi}{180} = 1.047\,\text{rad}, \; r = 7\text{ cm}$



$$\text{Area of sector} = \frac{1}{2}r^2\theta$$
$$= \frac{1}{2}(7)^2(1.047)$$
$$= \frac{1}{2}(49)(1.047) = 25.65\text{ cm}^2$$
$$\text{Length of arc } AB = r\theta$$
$$= (7)(1.047) = 7.33\text{ cm}$$

Example 10 The length of an arc and area of sector of a circle are 5cm and 25cm^2 respectively. Find radius and the central angle of sector.

Solution $\ell = 5\text{cm}, A = 25\text{cm}^2, r = ?, \theta = ?$

As $\ell = r\theta \Rightarrow 5 = r\theta$

$$\theta = \frac{5}{r} \quad \cdots\cdots\cdots \text{(i)}$$

and $A = \frac{1}{2}r^2\theta$

$$25 = \frac{1}{2}r^2\left(\frac{5}{r}\right) \quad \text{using eq. (i)}$$

$$\frac{25 \times 2}{5} = r \quad \Rightarrow \quad r = 10\text{cm}$$

Put value of r in (i), we have

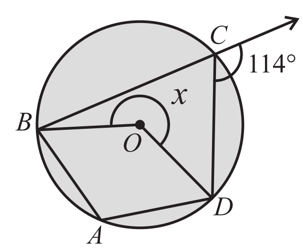
$$\theta = \frac{5}{10} = \frac{1}{2} \text{ radian}$$

Skilled Practice!

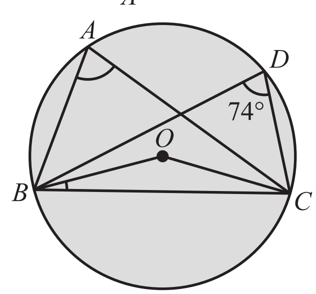
If ℓ is the arc length and r is the radius of the circle where $\ell = 14$m, $r = 4$m, then find the central angle. Also find the area of the sector of the circle.

EXERCISE 9.2

1. A, B, C and D are points on circumference of a circle with centre at O as shown in figure. Find the angle x.

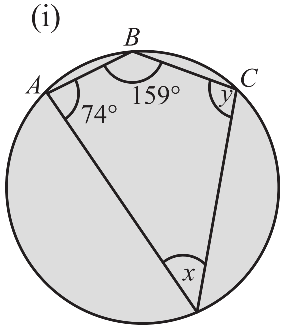


2. In the adjoining figure $m\angle BDC = 74°$, find $m\angle BAC$, $m\angle BOC$ and $m\angle OBC$.

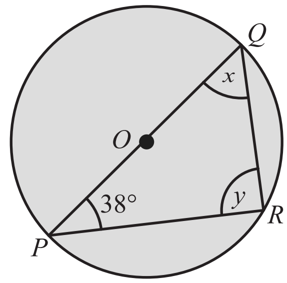


3. Find the angles x and y in the following figures.

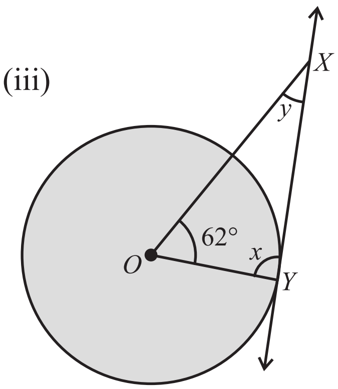
(i)



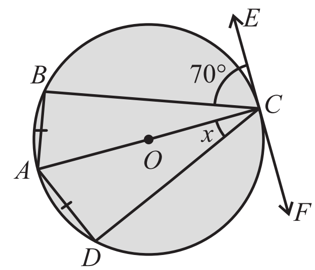
(ii)



(iii)



4. Find the angle x in the given figure.



5. In a circular arch over a monument entrance, two spotlight fixtures are placed such that they each shine from two different points on the arch to the same chord PQ on the base of the arch. If the angle formed at one light is 55°, what is the angle at the second light on the same side of chord PQ?
6. A circular garden has a walking path forming a quadrilateral inscribed in it. If one angle is 87°, what is the opposite angle?
7. A ferris wheel has a radius of 12 m. If a passenger travels a distance of 18 m along the circumference of the ferris wheel, what is the angle (in radians) swept by the passenger's position from the starting point?
8. Find θ, when
   (i) $\ell = 3$ cm, $r = 2.2$ cm (ii) $\ell = 5.6$ cm, $r = 2$ cm
9. Find r, when
   (i) $\ell = 5.5$ cm, $\theta = 40°20'$ (ii) $\ell = 13$ cm, $\theta = 70°$
10. Find ℓ and area of sector, when
    (i) $r = 1.7$ cm, $\theta = 0.25$ radian (ii) $r = 3$ cm, $\theta = 45°$
11. Uzma cut a pizza of radius 14 cm into 8 equal slices. What is the area of one slice (sector).
12. The perimeter and area of a sector are 14 cm and 10 cm^2 respectively. Find the radius of the circle and the central angle of the sector.

REVIEW EXERCISE 9

1. Four possible answers are given for the following questions. Choose the correct answer.

(i) Tangent is a line that touches the circumference of the circle at:
(a) many points (b) three points
(c) two points (d) one point

(ii) ________ tangents can be drawn to a circle from a point outside the circle.
(a) one (b) two (c) three (d) many

(iii) ________ tangents can be drawn to a circle from centre of the circle.
(a) zero (b) one (c) two (d) three

(iv) In circle, radius and tangent are:
(a) parallel (b) equal
(c) perpendicular (d) zero

(v) An angle in a segment greater than a semicircle is ______ angle:
(a) an acute (b) right (c) an obtuse (d) straight

(vi) The angle subtended by the arc at the centre of the circle is called:
(a) acute angle (b) central angle
(c) right angle (d) complete angle

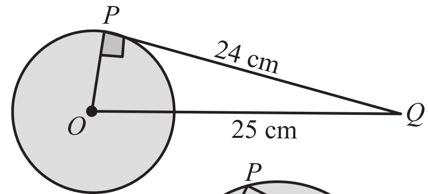
(vii) What is the measure of an angle inscribed in a semi-circle?
(a) 45° (b) 60°
(c) 90° (d) 120°

(viii) Any two angles in the same segment of the circle are:
(a) supplementary (b) complementary
(c) equal (d) zero

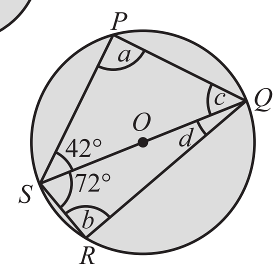
(ix) Area of the sector of the circle = ________
(a) $\frac{1}{2}r\theta$ (b) $\frac{3}{2}r^2\theta$ (c) $\frac{1}{2}r^2\theta$ (d) $\frac{3}{2}r\theta$

(x) If $r = 6$cm and $\theta = 2$ radians, then arc length is:
(a) 12 cm (b) 8 cm (c) 6 cm (d) 3 cm

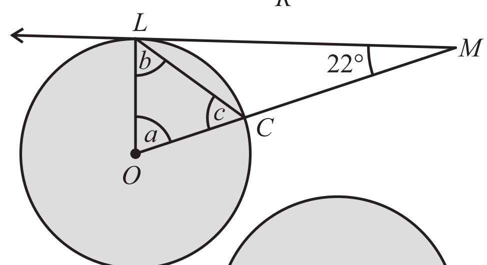
2. In the adjoining figure, find the radius of the circle.



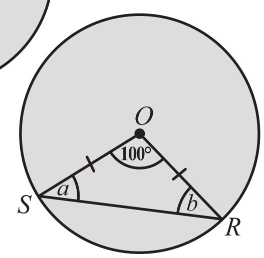
3. Find the angles a, b, c and d in the given figure, if O is the centre of the circle.



4. Find the angles a, b and c, if $\overrightarrow{ML}$ is a tangent line and O is the centre of the circle.



5. Find the angles marked in the given figure, where O is the centre of the circle.



6. Two round dining tables are arranged to touch at their edges. The radii of the two round dining tables are 10 cm and 12 cm. What is the distance between their centres if they touch externally?
7. A circular mosaic pattern is laid on the floor beneath a modern pyramid structure. Two tiles are placed such that they both connect to the same chord EF and reach two points G and H on the arc above EF. If $m\angle EGF = 65°$, find $m\angle EHF$.
8. A triangular frame within a semicircular gate joins the ends of the diameter to the peak. What is the angle formed at the peak?
9. A circular birthday cake has a radius of 15cm. A slice has an arc length of 10cm. What is the area of this slice?
10. A sector cut from a circle of radius 5cm has a perimeter of 16cm. Find area of this sector.

UNIT 10

Practical Geometry of Circles

Students' Learning Outcomes

After completing this unit, the students will be able to:

- Locate the centre of a given circle.
- Draw a circle passing through three given non-collinear points.
- Complete the circle:
  - by finding the centre,
  - without finding the centre,

  when a part of its circumference is given.
- Draw a tangent to a given arc, without using the centre, through a given point *P* when *P* is
  - the middle point of the arc.
  - at the end of the arc.
  - outside the arc.
- Draw a tangent to a given circle from a point *P* when *P* lies
  - on the circumference.
  - outside the circle.
- Draw two tangents to a circle meeting each other at a given angle.
- Apply concepts of practical geometry of a circle to real-life world problems (such as athletic tracks, recreational parks, Ferris wheels and mechanical machines).

INTRODUCTION

This unit focuses on the key geometric properties and constructions related to circles and their practical applications. Students will learn how to accurately locate the centre of a circle and draw a circle passing through three given non-collinear points. The unit further guides students in completing a circle using various methods, whether or not the centre is known and even when only a part of the circumference is given. A major focus is placed on constructing tangents to arcs and circles from different positions of a given point, both with and without using the centre. Additionally, students will gain hands-on experience in drawing two tangents that meet at a specific angle. To bridge theory with real-world relevance, this unit highlights how these geometric concepts are applied in everyday structures such as athletic tracks, Ferris wheels and engineering designs. Through these activities, students will enhance their precision and spatial reasoning of geometry in the physical world.

10.1 Construction of a Circle

The construction of a circle is a fundamental concept in practical geometry. A circle can be drawn using a compass when its centre and radius are known. However, in more advanced constructions, a circle can also be constructed through three given non-collinear points, as these points uniquely determine a circle. In some cases, a circle may need to be completed when only a part of its circumference is given, either by finding the centre or using geometric techniques without it.

10.1.1 Locate the Centre of a Given Circle

Given: *A* circle.

Required: To locate the centre of the circle.

Steps of Construction:

1. Take any two chords $\overline{AB}$ and $\overline{CD}$.
2. Draw $\overleftrightarrow{PQ}$ as right bisector of $\overline{AB}$.
3. Draw $\overleftrightarrow{RS}$ as right bisector of $\overline{CD}$.
4. $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect each other at point O.
5. Point O is the centre of the circle.

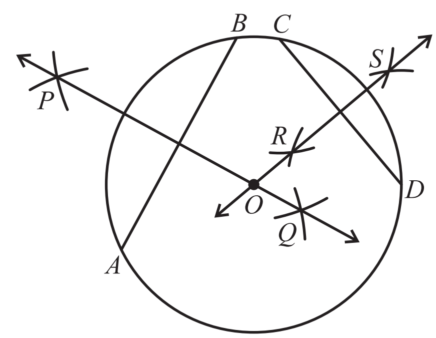


Fig. 10.1.1

> **Note**
> - Points that lie on the same line are collinear.
> - Points that do not lie on the same line are non-collinear.

10.1.2 Draw a Circle Passing through Three Given Non-collinear Points

Given: Points A, B and C are non-collinear.

Required: Draw a circle passing through points A, B and C.

Steps of Construction:

1. Join B to A and C.
2. Draw $\overleftrightarrow{DE}$ as right bisector of $\overline{AB}$.
3. Draw $\overleftrightarrow{FG}$ as right bisector of $\overline{BC}$.

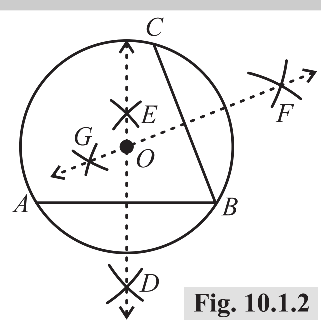


Fig. 10.1.2

4. $\overleftrightarrow{DE}$ and $\overleftrightarrow{FG}$ intersect each other at point O. Now O, is centre of the required circle.
5. Take O as centre and draw a circle of radius $m\overline{OA} = m\overline{OB} = m\overline{OC}$, which is the required circle.

10.1.3 Complete a Circle by Finding its Centre, When a Part of Its Circumference is Given

Given: PQR is an arc of a circle.

Required: To complete the circle by finding its centre.

Steps of Construction:

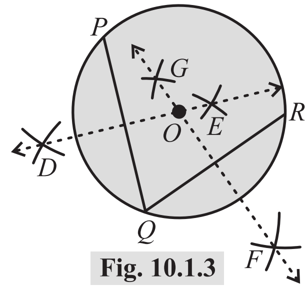


Fig. 10.1.3

1. Join Q to P and R.
2. Draw $\overleftrightarrow{DE}$ as right bisector of $\overline{PQ}$.
3. Draw $\overleftrightarrow{FG}$ as right bisector of $\overline{QR}$.
4. $\overleftrightarrow{DE}$ and $\overleftrightarrow{FG}$ intersect each other at point O. So, O is the centre of the circle.
5. Take O as centre and draw a circle of radius $m\overline{OP} = m\overline{OQ} = m\overline{OR}$.

   Thus, this circle passes through points P, Q and R on the given part of the circumference.

10.1.4 Complete a Circle without Finding its Centre, When a Part of its Circumference (Arc) is Given

Given: $\overparen{ABC}$ is a part of circumference of a circle.

Required: Complete a circle of which $\overparen{ABC}$ is a part without finding the centre of the circle.

Steps of Construction:

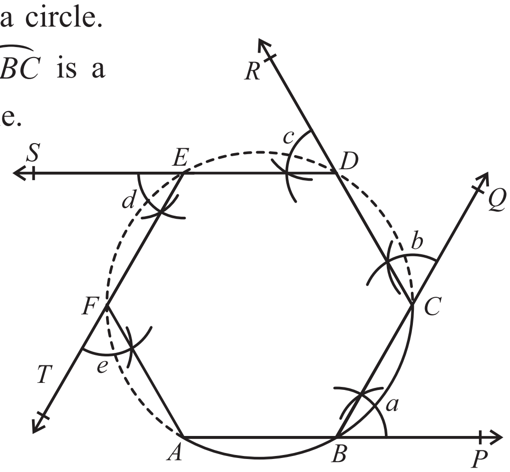


Fig. 10.1.4

1. Take $\overline{AB}$, $\overline{BC}$ two chords of the given arc ABC such that $m\overline{AB} = m\overline{BC}$.
2. Extend $\overline{AB}$ and $\overline{BC}$ to form rays $\overrightarrow{AP}$ and $\overrightarrow{BQ}$ respectively.
3. Construct $m\angle b = m\angle a$.
4. Take $m\overline{CD} = m\overline{AB} = m\overline{BC}$ and construct $m\angle c = m\angle a = m\angle b$.

5. Similarly, construct $\angle d$ and $\angle e$.
6. Now, draw arcs on $\overline{CD}$, $\overline{DE}$, $\overline{EF}$ and $\overline{FA}$ with smooth hand.
   Thus, the required circle is completed.

EXERCISE 10.1

1. Construct a circle with the help of given radius and verify its centre by construction:
   (i) $r = 1.5$ cm (ii) $r = 1.7$ cm (iii) $r = 2$ cm
2. Take any three non-collinear points P, Q, R and construct a circle passing through these points.
3. Draw an arc ABC and complete a circle by finding its centre.
4. Draw an arc PQR and complete a circle without finding its centre.
5. Take any three non-collinear points (locations of the lamp posts in the park), construct a circle passing through all these points.
6. A part of the Ferris wheel rim is visible as an arc. Using any three points on the arc, construct the circle by finding its centre.
7. ABC an arc of a fountain, complete a circle without finding its centre.

10.2 Tangent to a Circle

10.2.1 To Draw a Tangent to a Given Arc without using the Centre Through a Given Point *P*

Case I When point *P* is the middle point of the arc

Given: $\overparen{APB}$ is an arc where P is midpoint of the arc.

Required: Draw tangent to the arc at point P.

Steps of Construction:

1. Join end points A and B of the arc.
2. Draw right bisector $\overleftrightarrow{LM}$ of $\overline{AB}$, it passes through point P.

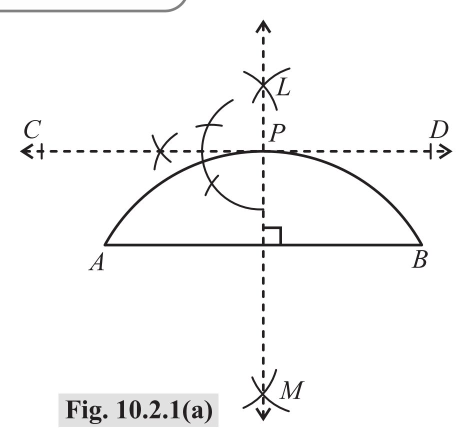


Fig. 10.2.1(a)

3. Construct right angle MPC and produce $\overline{CP}$ in the direction of P.

So, $\overleftrightarrow{CPD}$ is the required tangent.

Case II **When point P is an endpoint of the arc.**

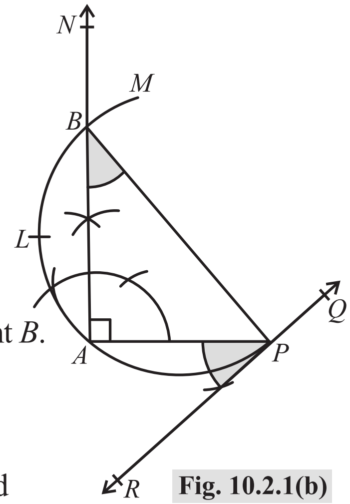


Fig. 10.2.1(b)

Given: $\overset{\frown}{PLM}$ is an arc, where P is its endpoint.

Required: Draw tangent to the arc PLM at point P.

Steps of Construction:

1. Take any point A on the arc and join P to A.
2. Draw right angle PAN so that $\overrightarrow{AN}$ cuts the arc at point B.
3. Join point B to P.
4. Construct $m\angle APR = m\angle ABP$.
5. Produce $\overline{RP}$ towards P, so $\overleftrightarrow{RPQ}$ is the required tangent.

Case III **When Point P is outside the arc.**

Given: $\overset{\frown}{ABC}$ is an arc and a point P is outside it.

Required: To draw a tangent to the arc from point P.

Steps of Construction:

1. Join A to P, $\overline{AP}$ cuts the arc at Q.
2. Find the midpoint M of $\overline{AP}$
3. Take $\overline{AMP}$ as diameter and draw a semi-circle.
4. Draw perpendicular $\overrightarrow{QS}$ on $\overline{AP}$ that cuts the semi-circle at R.
5. Take $m\overline{RP}$ as radius and draw an arc that cuts $\overset{\frown}{ABC}$ at D.
6. Join P to D and produce towards D, so $\overrightarrow{PD}$ is the required tangent at D.

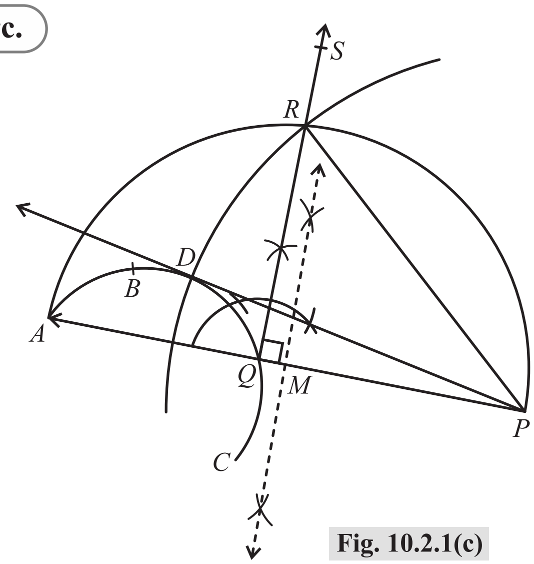


Fig. 10.2.1(c)

10.2.2 (a) Draw a Tangent to a given Circle from a Point P when P Lies on the Circumference.

Given: A circle with centre D and a point P on its circumference.

Required: Draw a tangent to the circle at point P.

Steps of Construction:

1. Join centre D with the point P. Now, $\overline{DP}$ is the radial segment.
2. Draw a perpendicular $\overleftrightarrow{AB}$ at point P to the radial segment $\overline{DP}$.

So, $\overleftrightarrow{APB}$ is the required tangent.

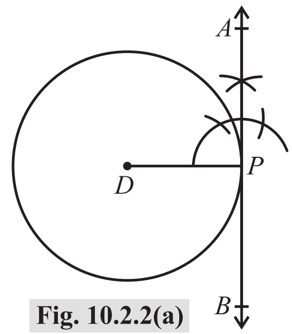


Fig. 10.2.2(a)

b Draw a tangent to the circle from a point *P* when *P* lies outside the circle.

Given: A circle with centre O and a point P that lies outside the circle.

Required: To draw a tangent to the circle from point P.

Steps of Construction:

1. Join centre O to point P
2. Find M, the mid-point of $\overline{OP}$.
3. Take M as centre and draw a semi-circle with radius as $m\overline{OM} = m\overline{MP}$.
   This semi-circle cuts the circumference at point T.
4. Join P to T and produce towards T.

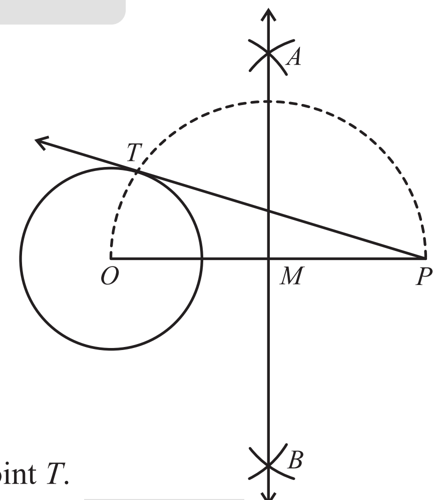


Fig. 10.2.2(b)

Thus, $\overrightarrow{PT}$ is the required tangent and length of tangent is measured as $m\overline{PT}$.

10.2.3 Draw Two Tangents to a Given Circle Meeting each other at a Given Angle

Example 1 Draw a circle with radius 1.5 cm. Draw two tangents to this circle that meet an angle of 30°.

Solution

Given: A circle with centre O, of radius 1.5 cm.

Required: Draw two tangent to the given circle that should meet at 30°.

Steps of Construction:

1. Draw a diameter $\overline{AOB}$.
2. Draw a perpendicular at B on $\overline{AOB}$, this is one tangent.

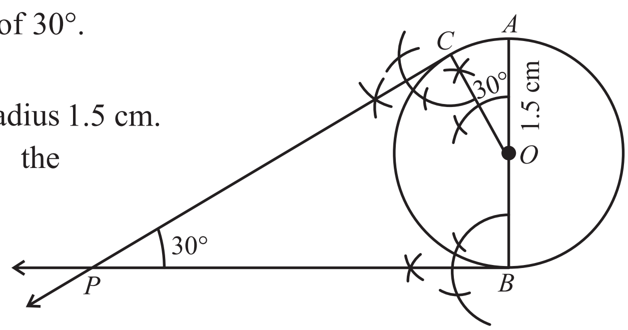


Fig. 10.2.3

3. Draw an angle of 30° at O, as $m\angle AOC = 30°$, where C is a point on the circumference of the given circle.

4. Draw a perpendicular at C on $\overline{OC}$ as $\overrightarrow{CP}$ that meets the already drawn tangent at B.

Hence, tangents $\overrightarrow{BP}$ and $\overrightarrow{CP}$ are the required tangents that meet at $m\angle CPB = 30°$.

EXERCISE 10.2

1. Draw tangent to $\widehat{APB}$ at point P, when P is midpoint of the arc.
2. Draw tangent to $\widehat{PQR}$ at point P, when P is endpoint of the arc.
3. Draw tangent to $\widehat{ABC}$ from a point P, when P is outside the arc.
4. Draw a circle of radius 1.3 cm and draw a tangent at point P, when P lies on its circumference.
5. Draw a circle of radius 1.5 cm and draw a tangent at point P, when P is at a distance of 8 cm from its centre.
6. Draw a circle of radius 1.6 cm. Draw two tangents that meet an angle of 30°.
7. Draw a tangent to any point on a circular part of the track having radius = 2 cm.
8. From a pulley point, construct two tangents to a machine wheel having $r = 2.1$cm, such that the angle between them is 30°.

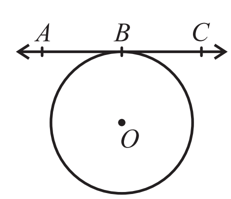
REVIEW EXERCISE 10

1. Four possible answers are given for the following questions. Choose the correct answer:

(i) In the adjacent figure, $\overleftrightarrow{ABC}$ is:

(a) a chord (b) an arc

(c) a tangent (d) a secant



(ii) There are __________ types of arcs.

(a) 2 (b) 3

(c) 4 (d) 5

(iii) Right bisector of the chord of a circle always passes through the:
(a) diameter (b) non-collinear points (c) radius (d) centre

(iv) A circle has only one:
(a) chord (b) centre (c) diameter (d) secant

(v) The point where two tangents meet outside a circle forms:
(a) a semicircle (b) a diameter (c) a radius (d) an angle

(vi) Two equal tangents from a point to a circle can be drawn when the point is ________ the circle.
(a) on (b) inside (c) outside (d) at centre of

(vii) Tangents drawn from a single external point to a circle are:
(a) unequal in length (b) perpendicular to each other
(c) equal in length (d) inside the circle

(viii) At least how many chords are needed to locate the centre of the circle?
(a) 1 (b) 2 (c) 3 (d) 4

(ix) To draw a tangent at middle point of an arc, first step is to:
(a) join endpoints of an arc
(b) draw radius
(c) draw a line perpendicular to chord
(d) draw an angle

(x) The angle between the radius and a tangent at the point of contact is:
(a) 30° (b) 60° (c) 90° (d) 120°

2. Draw a circle of radius 1.4 cm and draw a tangent at point *P*, when *P* lies on its circumference.

3. Draw a circle of radius 1.2 cm and draw a tangent at point *P,* when *P* is at a distance of 5 cm from the centre.

4. Draw a circle of radius 1.7 cm. Draw two tangents that meet an angle of 30°.

5. Take a part of the circular track, use chords and perpendicular bisectors to complete a circular track.

6. Two decorative fences touch the circular flower bed of radius 2.1 cm and meet outside it at an angle of 30° to form an entrance arch. Draw two tangents to the flower bed from the point where the fences meet at an angle of 30°.

UNIT 11

Information Handling

Students' Learning Outcomes

After completing this unit, the students will be able to:

- Construct cumulative frequency table and cumulative frequency polygon or Ogive.
- Interpret the median, quartiles, deciles, percentiles and inter quartile range from cumulative frequency curve.
- Interpret and analyze box and whisker plots.
- Construct and interpret data from scatter diagrams and also draw lines of best fit.
- Measure correlation using scatter diagram.
- Calculate the range, standard deviation and variance for grouped data.
- Use the mean and standard deviation to compare two sets of data.
- Solve real life situations involving variance and standard deviation for grouped data.
- Apply concepts from measures of dispersion to solve real life situations (determining the consistency of data, checking variability in forecasting, manufacturing, finance and economics).

INTRODUCTION

This unit focuses on essential statistical tools and techniques for analyzing and interpreting data. Students will learn to organize data using frequency tables and visualize it through cumulative frequency polygons (Ogives) and box-and-whisker plots. They will interpret key statistical measures such as median, quartiles, deciles, percentiles and interquartile range to understand data distribution. The unit also includes constructing and analyzing scatter diagrams to assess correlation and draw lines of best fit. Students will calculate and interpret measures of dispersion (range, standard deviation and variance) for grouped data and apply these to compare datasets. Real-life applications in fields such as forecasting, manufacturing, finance and economics are integrated to highlight the importance of statistical analysis in decision-making and assessing consistency in data.

In class IX, we learned that a frequency distribution organizes raw data into a table using class intervals and corresponding frequencies, allowing us to summarize and interpret large sets of data more effectively.

11.1 Construction of Cumulative Frequencies

Cumulative frequency is the running total of the frequencies, where each frequency is added to the sum of all frequencies before it.

A table of cumulative frequencies, also known as a cumulative frequencies distribution, presents the running total of frequencies for each data point or class interval showing how many observations fall at or below a specific value.

Example 1 The marks of 50 college students out of 100 are given below. Construct cumulative frequency table.

85 , 66 , 76 , 45 , 66 , 91 , 77 , 64 , 71 , 74
47 , 78 , 76 , 42 , 70 , 58 , 71 , 67 , 80 , 78
73 , 48 , 68 , 87 , 81 , 72 , 65 , 69 , 78 , 84
75 , 53 , 58 , 87 , 56 , 72 , 62 , 92 , 73 , 83
97 , 81 , 51 , 61 , 58 , 72 , 62 , 79 , 88 , 74

Solution Highest marks = 97, Lowest marks = 42

The same 50 scores grouped into a frequency distribution.

Table 11.1 Cumulative Frequency Table

Class Intervals	Tally Marks	Frequencies	Cumulative Frequencies
40 − 44	|	1	1
45 − 49	|||	3	1 + 3 = 4
50 − 54	||	2	4 + 2 = 6
55 − 59	||||	4	6 + 4 = 10
60 − 64	||||	4	10 + 4 = 14
65 − 69	卌 |	6	14 + 6 = 20
70 − 74	卌 卌	10	20 + 10 = 30
75 − 79	卌 |||	8	30 + 8 = 38
80 − 84	卌	5	38 + 5 = 43
85 − 89	||||	4	43 + 4 = 47
90 − 94	||	2	47 + 2 = 49
95 − 99	|	1	49 + 1 = 50
Total		$n = \sum f = 50$	

11.1.1 Cumulative Frequency Polygon or Ogive

A cumulative frequency polygon, also known as an ogive, is a graphical representation that shows the cumulative total of frequencies up to each class boundary in a frequency distribution.

It helps in understanding how data values accumulate over intervals and is especially useful for estimating medians, quartiles, deciles and percentiles. The curve is typically drawn by plotting the cumulative frequencies against the upper class boundaries or lower class boundaries and joining the points with a smooth or straight-line curve. It provides a clear visual insight into the distribution and spread of the data.

There are two methods for construction of a cumulative frequency polygon or ogive:

(a) Less than method (b) More than method

11.1.2 Steps for Construction of Cumulative Frequency Polygon or Ogive

Step I Prepare a cumulative frequency table from the given data.

Step II Mark the upper class boundaries on the x-axis for less than method. Mark the lower class boundaries on the x-axis for more than method.

Step III Mark the corresponding cumulative frequencies on the y-axis.

Step IV Plot each point using the upper class boundary or lower class boundary on the x-axis and its cumulative frequency on the y-axis.

Step V Connect the plotted points using a smooth freehand curve or use straight line segments to form the cumulative frequency polygon or ogive.

11.1.3 Important Definitions

Median

The median is the middle value in a set of data when the values are arranged in ascending or descending order. It divides the data into two equal parts.

Quartiles

Quartiles divide an arranged data set into four equal parts, each containing 25% of the data.

> **Remember!**
> Q_1 is the lower quartile, Q_2 is the median and Q_3 is the upper quartile.

Q_1 (First Quartile)

25% of the data falls below this value.

Q_2 (Second Quartile)

50% of the data falls below this value (also called the median).

Q_3 (Third Quartile)
75% of the data falls below this value.

Deciles

Deciles divide an arranged data set into ten equal parts, each representing 10% of the data. In other words, there are nine decile values from D_1 to D_9.

For example, D_3 is the value below which 30% of the data lies and D_7 is the value below which 70% of the data falls.

Remember!
P_{25} is the same as Q_1, P_{50} is the same as the median (Q_2), and P_{75} is the same as Q_3.

Percentiles

Percentiles divide an arranged data set into 100 equal parts, each containing 1% of the data. There are ninety nine percentile values from P_1 to P_{99}.

For example, P_{90} means 90% of the data lies below that value.

Interquartile Range (IQR)

The interquartile range is a measure of spread that represents the distance between the first quartile Q_1 and the third quartile Q_3. It is calculated as $Q_3 - Q_1$

Example 2 The table provides the information about the speeds, in km/h of 90 vehicles passing a speed checkpoint. Construct the cumulative frequency polygon by less than method and locate median, Q_1, Q_3, D_4, P_{95} and IQR on it.

Speed (km/h)	40 – 50	50 – 60	60 – 70	70 – 80	80 – 90	90 – 100
f	10	18	30	17	9	6

Solution

Speed (km/h)	f	$c.f.$
40 – 50	10	10
50 – 60	18	10 + 18 = 28
60 – 70	30	28 + 30 = 58
70 – 80	17	58 + 17 = 75
80 – 90	9	75 + 9 = 84
90 – 100	6	84 + 6 = 90
Total	$\Sigma f = 90$	

Cumulative Frequency Polygon

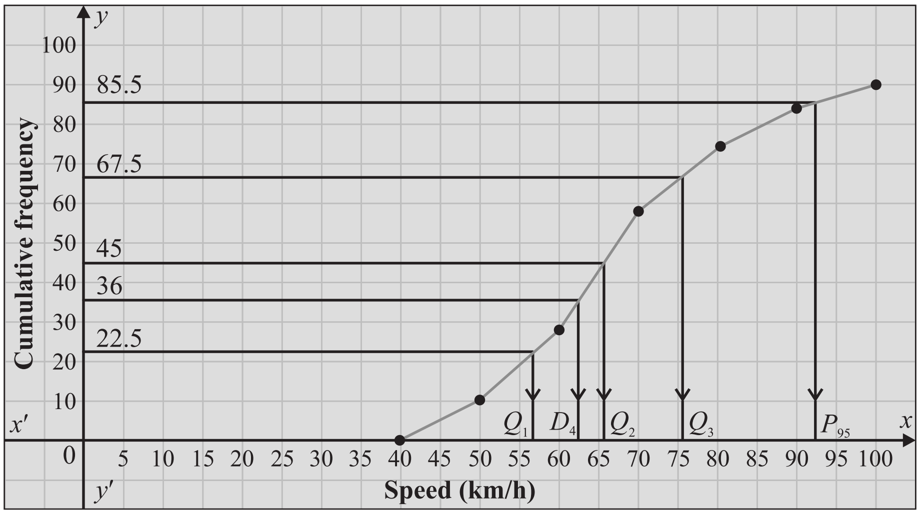


Fig. 11.1

The position of median is $\frac{n}{2} = \frac{90}{2} = 45$. Draw a horizontal line from 45 on the y-axis until it meets the curve and then draw vertical line from this point to x-axis. Now, take the reading on the x-axis where the vertical line hits the x-axis. So,

$$\text{Median} = 65.7$$

By the same way, we can find Q_1, Q_3, D_4, P_{95} and semi interquartile range.

The position of Q_1 is $\frac{n}{4} = \frac{90}{4} = 22.5$. So, from the curve $Q_1 = 56.9$.

The position of Q_3 is $\frac{3n}{4} = \frac{3(90)}{4} = 67.5$. So, from the curve $Q_3 = 75.6$.

The position of D_4 is $\frac{4n}{10} = \frac{4(90)}{10} = 36$. So, from the curve $D_4 = 62.7$.

The position of P_{95} is $\frac{95n}{100} = \frac{95(90)}{100} = 85.5$. So, from the curve $P_{95} = 92.5$.

$$IQR = Q_3 - Q_1 = 75.6 - 56.9 = 18.7$$

Example 3 Draw the cumulative frequency polygon for the following frequency distribution of the weekly wages (rupees in thousands) of number of workers by more than method. Also locate Q_2 on it.

Weekly wages (rupees in thousands)	0 – 20	20 – 40	40 – 60	60 – 80	80 – 100
No. of workers	30	48	61	40	21

Solution

Weekly wages (rupees in thousands)	No. of workers (f)	$c.f.$
0 – 20	30	200
20 – 40	48	200 – 30 = 170
40 – 60	61	170 – 48 = 122
60 – 80	40	122 – 61 = 61
80 – 100	21	61 – 40 = 21
Total	$\Sigma f = 200$	

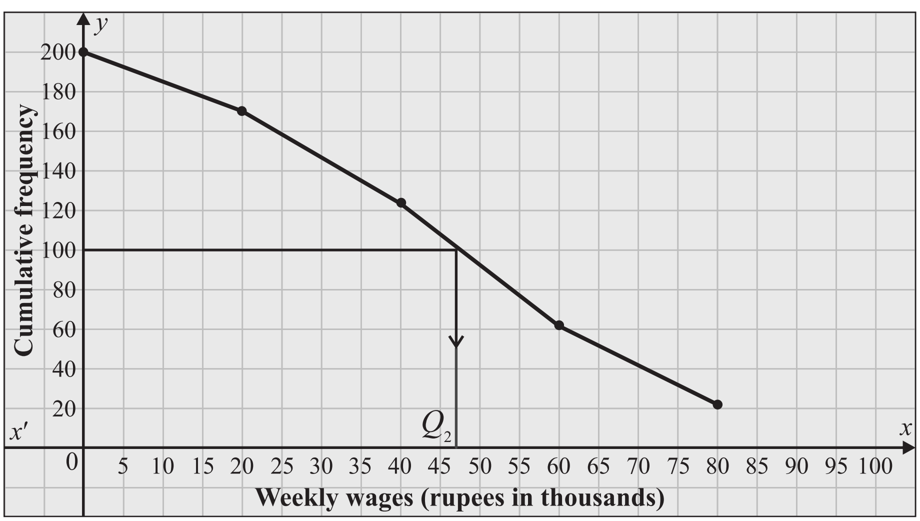


Fig. 11.2

The position of Q_2 is $\frac{n}{2} = \frac{200}{2} = 100$. So, from the curve $Q_2 = 47.2$

11.2 Box-and-Whisker Plot

A box-and-whisker plot or box plot is a graphical representation of a data set five-number summary (minimum, first quartile, median, third quartile and maximum). It is useful for visualizing data distribution and comparing data sets.

There are following key components of box plot:

The Box: It represents the interquartile range, which is middle 50% of the data with the box edges marking the first quartile (Q_1) and third quartile (Q_3).

The median: A line within the box representing the second quartile (Q_2) or the middle of the data set.

Whiskers: Lines extending from the box to the minimum and maximum values, showing the overall spread of the data.

We can show this information on a box-and-whisker plot. To begin with, we draw a horizontal number line using a suitable scale. On the number line, the minimum value, the maximum value and the quartiles are indicated as shown in the figure given below:

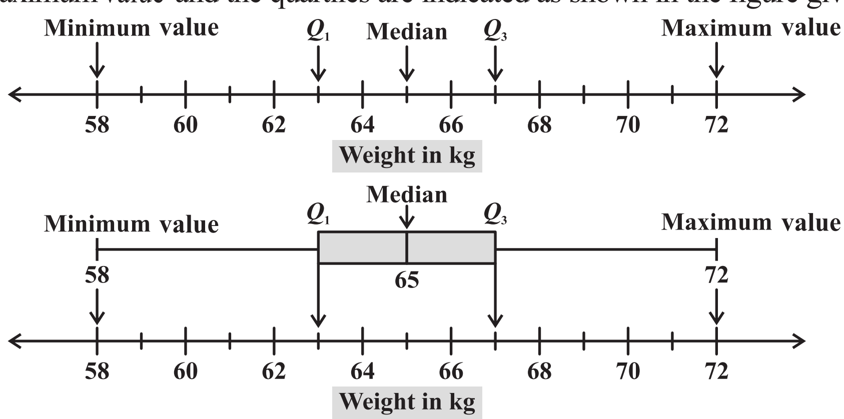


Figure 11.3

Example 4 For the given set of data,

2, 10, 3, 16, 14, 13, 4, 7, 11, 5, 17, 14, 18

(i) What is the median of the data set?
(ii) What are the upper and lower quartiles?
(iii) What are the extreme values of the data set?
(iv) Draw a box-and-whisker plot to display the data.

Solution (i) Arrange the given data in ascending order.

2, 3, 4, 5, 7, 10, 11, 13, 14, 14, 16, 17, 18

Here $n = 13$,

$$\text{Median} = \left(\frac{n+1}{2}\right)^{\text{th}} \text{value} = \left(\frac{13+1}{2}\right)^{\text{th}} \text{value} = 7^{\text{th}} \text{ value} = 11$$

(ii) Lower quartile $= \left(\frac{n+1}{4}\right)^{th}$ value $= \left(\frac{13+1}{4}\right)^{th}$ value $= 3.5^{th}$

As the lower quartile is 3.5^{th} value, so we find average of 3^{rd} and 4^{th} value.

Lower quartile $= \frac{4+5}{2} = 4.5$

Upper quartile $= 3\left(\frac{n+1}{4}\right)^{th}$ value $= 3\left(\frac{13+1}{4}\right)^{th}$ value $= 10.5^{th}$

As the upper quartileis 10.5^{th} value, so we find average of 10^{th} and 11^{th} values.

Upper quartile $= \frac{14+16}{2} = 15$

(iii) Minimum value = 2, Maximum value = 18

(iv) The box-and whisker plot is drawn below:

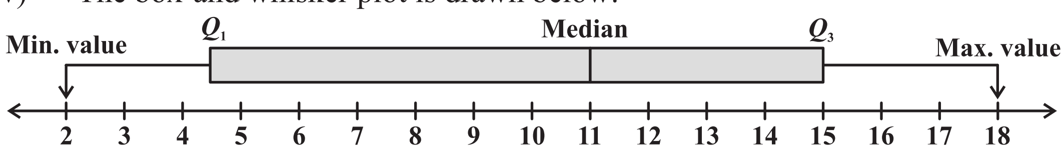
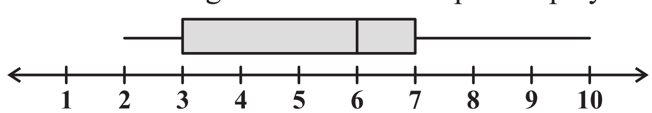


Fig. 11.4

Example 5 Kiran recorded the number of days it rained or snowed each month during the last year. The following box-and-whisker plot displays her data:



(i) What do the extremes tells about the number of days it rained or snowed?

(ii) What is the median number of days it rained or snowed?

(iii) What are the upper and lower quartiles for the data?

(iv) Where the most data is clustered?

Solution

(i) The least number of days it rained or snowed in a month was 2 and the greatest number of days it rained or snowed in a month was 10.

(ii) Median = 6 days (iii) Lower quartile = 3 days, Upper quartile = 7 days

(iv) Most of the data is clustered in the box.

11.3 Scatter Diagram

A scatter diagram (or scatter plot) is a graphical tool used to display the relationship between two numerical variables. Each point on the graph represents a pair of values. One variable is plotted on the horizontal axis and the other is plotted on the vertical axis.

Line of Best Fit

A line of best fit (also called a trend line) is a straight line drawn through a scatter diagram that best represents the overall direction of the data.

It is called an estimated line of best fit because it is drawn manually, based on general guidelines to best represent the overall trend of the data. It should go through the middle of the points. Ideally, there should be about the same number of points above and below the line.

Correlation

It is the relationship between two variables, where a change in one variable is associated with a change in the other.

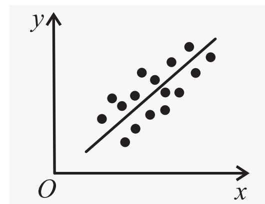
According to the correlation, scatter diagrams are divided into following three categories:

Do you know?

Scatter diagram visually shows the correlation between two variables.

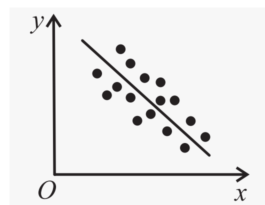
Positive Correlation

If one variable increases as the other increases, then it is called positive correlation. If you draw a straight line along the data points, the slope of the line will go up. For example, height and weight.



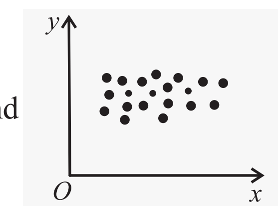
Negative Correlation

If one variable increases as the other decreases, then it is called negative correlation. If you draw a straight line along the data points, the slope of the line will go down. For example, speed and travel time.



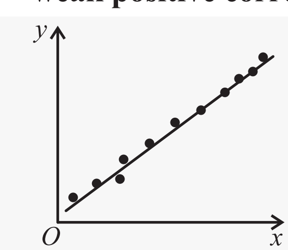
Zero Correlation

If there is not consistent pattern between the two variables, then it is called zero correlation. For example, shoe size and test scores.

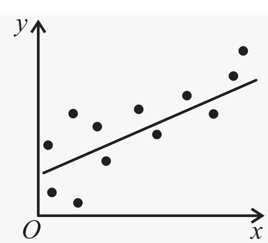


Correlation Pattern of Scatter Diagram

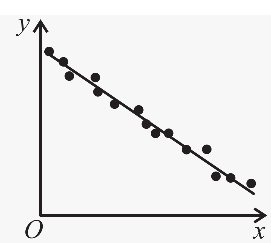
- When the value of Y increases as the X value increases, it is called **strong positive correlation**.
- When the value of Y decreases as the X value increases, it is called **strong negative correlation**.
- When the value of Y increases slightly as the X value increases, it is called **weak positive correlation**.



Strong positive correlation

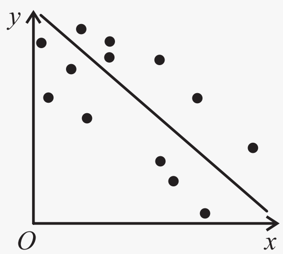


Weak positive correlation

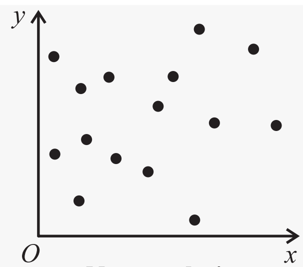


Strong negative correlation

- When the value of Y decreases slightly as the X value increases, it is called **weak negative correlation**.
- When there is no connection between the two variables, it is called **no-correlation**.



Weak negative correlation



No correlation

Example 6 The table below displays the amount of time spent on revision and the corresponding test scores for ten students.

Time spent on revision (hours)	0.5	1	4	6	2	3	7	5	8	9
Test score (%)	22	38	62	68	30	46	70	60	86	90

(i) Construct a scatter diagram and draw a line of best fit on it.

(ii) Describe the correlation shown in the scatter diagram.

(iii) Saleem spent 5.5 hours on revision. Use line of best fit to estimate his test score.

Solution

(i)

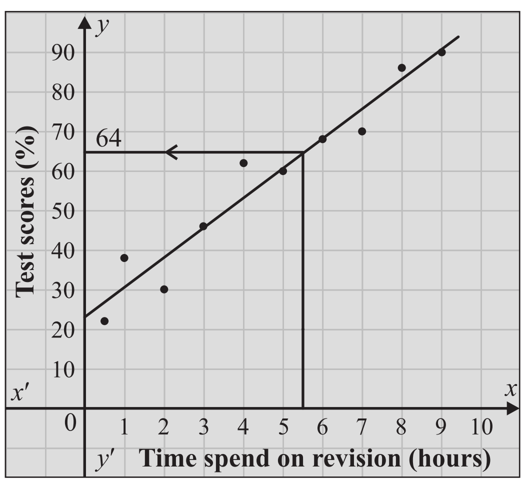


Fig. 11.5

Steps:

- Draw x-axis for "Time spend on revision" and y-axis for "Test scores".
- For each pair from the table, mark a point on the graph.

- Observe the pattern of points and draw a line that should be close as possible to all points.

(ii) When the time spent on revision increases, the test score increases. Therefore, strong positive correlation is between given two variables.

(iii) At 5.5 hours, test score of Saleem will be 64%.

EXERCISE 11.1

1. The following frequency distribution represents the achievement of a student's group upon a memory test:
   (i) Plot cumulative frequency graph of group scores.
   (ii) Determine median, lower and upper quartiles, D_3, P_{90} and interquartile range graphically.

   Note: Write your answers in whole numbers.

Scores	44 – 48	49 – 53	54 – 58	59 – 63	64 – 68	69 – 73	74 – 78
No. of Students	3	12	15	23	12	8	7

2. Construct an ogive for the following distribution of scores:

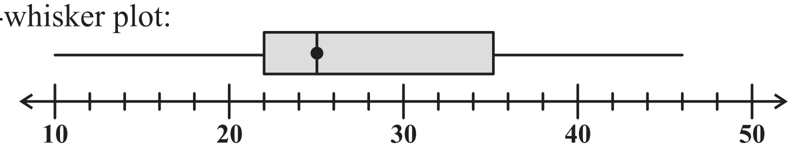
Scores	69 – 79	79 – 89	89 – 99	99 – 109	109 – 119
f	4	10	13	8	5

   Determine median, lower and upper quartiles, D_6, P_{80} and interquartile range graphically.

3. Find Q_1, Q_3, median, range and IQR for the dataset given below:
   3, 6, 8, 4, 7, 5, 10, 11, 13, 9, 14, 12, 15, 16, 17
   Also draw a box-and-whisker plot.

4. Find Q_1, Q_3, median, range, IQR and extreme values plot for the following data:
   102, 98, 95, 100, 93, 110, 108, 104, 97, 96, 92, 101, 99, 105, 107
   Also draw a box-and-whisker plot.

5. Find Q_1, Q_3, median, minimum and maximum values for the following box-and-whisker plot:



6. Draw the box-and-whisker plot, if min. value = 12, max. value = 26, median = 19, Q_1 = 14 and Q_3 = 24.

7. Construct a scatter diagram and draw a line of best fit for the following quiz scores for 8 students in a class:

1st quiz	14	28	34	36	43	45	46	49
2nd quiz	16	20	33	35	38	39	42	43

Describe correlation between 1st and 2nd quiz scores also.

8. The following table shows the test scores (one measuring speed and the other measuring strength) of football players out of 200:

1st Test	20	38	70	40	100	110	50	136	150	80
2nd Test	160	142	130	120	70	50	140	44	36	100

(i) Construct a scatter diagram and draw a line of best fit.
(ii) Describe correlation between the 1st and 2nd tests.
(iii) Abdullah scores 60 in the 2nd test. Estimate his score in the 1st test.

9. The following table shows the number of daily social media hours and GPA of university students.

Social Media Hours	0.5	1	2	3	4	5	6	7	8	9
GPA	4.0	3.9	3.7	3.5	3.3	3.1	2.9	2.8	2.6	2.5

(i) Construct a scatter diagram and draw a line of best fit.
(ii) Identify the nature of correlation between social media use and GPA.
(iii) Estimate the GPA for a student who spends 6.5 hours on social media.

11.4 Measures of Dispersion

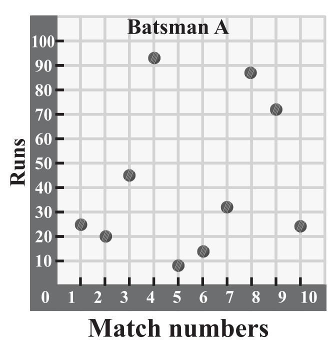
The following data shows the runs scored by two batsmen in the last 10 matches:

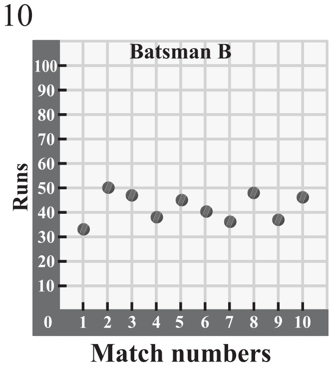
Batsman *A*: 25, 20, 45, 93, 8, 14, 32, 87, 72, 24

Batsman *B*: 33, 50, 47, 38, 45, 40, 36, 48, 37, 46

$$\text{Mean of Batsman } A = \frac{25 + 20 + 45 + 93 + 8 + 14 + 32 + 87 + 72 + 24}{10} = \frac{420}{10} = 42$$

$$\text{Mean of Batsman } B = \frac{33 + 50 + 47 + 38 + 45 + 40 + 36 + 48 + 37 + 46}{10} = \frac{420}{10} = 42$$





Although both batsmen have the same mean score, batsman B's runs are closely clustered around the mean, while batsman A's scores are widely spread from 0 to 100. Therefore, to understand how the data values are spread out, we need more statistical information. For this purpose, we will study measures of dispersion.

Measures of dispersion are statistical tools used to describe how spread out or scattered the values in a data set are around a central value.

Different measures of dispersion are

1. Range
2. Variance
3. Standard deviation
4. Interquartile range
5. Mean deviation
6. Quartile deviation
7. Coefficient of Variations

In this unit, we will discuss only a few measures of dispersion.

11.4.1 Range

The difference between the highest value (H) and the lowest value (L) is called range.

$$\text{Range } (R) = H - L$$

where H is the highest value and L is the lowest value.

Example 7 Find the range of the following data:

15, 22, 18, 30, 17, 25

Solution Here $H = 30$; $L = 15$

Range $(R) = H - L$

$= 30 - 15 = 15$

Skilled Practice!

What is the range of first 20 prime numbers?

Example 8 The table below shows the weekly incomes (in thousands of rupees) of a group of employees in a company. Find the range of the weekly income.

Weekly Income (Rs 1000)	Number of Employees
25 − 35	4
35 − 45	6
45 − 55	5
55 − 65	3

Solution Here $H = 65$

$L = 25$

Range $(R) = H - L$

$= 65 - 25 = 40$

Remember!

If the frequency of initial class is zero, then the next class will be considered for the calculation of range.

Thus, the difference between the lowest and highest weekly incomes among employees is Rs. 40,000.

11.4.2 Variance

The average of the squares of the deviations of the values from their mean is called variance. It is denoted by σ^2 (read as sigma square).

If $x_1, x_2, \ldots, x_n$, are n observations, then the variance is given by

$$\sigma^2 = \frac{\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2}{n} \qquad \text{(For ungrouped data)}$$

or

$$\sigma^2 = \frac{\sum_{i=1}^{n} x_i^2}{n} - \left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^2 \qquad \text{(For ungrouped data)}$$

Let $x_1, x_2, x_3, \ldots, x_n$ be the n values with the frequencies $f_1, f_2, f_3, \ldots, f_n$ respectively.

Then,

$$\sigma^2 = \frac{\sum_{i=1}^{n} f_i\left(x_i - \bar{x}\right)^2}{\sum_{i=1}^{n} f_i} \qquad \text{(For grouped data)}$$

or

$$\sigma^2 = \frac{\sum_{i=1}^{n} f_i x_i^2}{\sum_{i=1}^{n} f_i} - \left(\frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i}\right)^2 \qquad \text{(For grouped data)}$$

Note

The greater the variance, the more spread out the data; the smaller the variance, the more consistent or tightly clustered the values are around the mean.

11.4.3 Standard Deviation

Standard deviation is the positive square root of the average of the squares of deviations of the given values from their mean. It is denoted by σ.

Standard deviation helps us understand the extent to which values differ from the mean.

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2}{n}} \qquad \text{(For ungrouped data)}$$

or $\sigma = \sqrt{\dfrac{\sum_{i=1}^{n} x_i^2}{n} - \left(\dfrac{\sum_{i=1}^{n} x_i}{n}\right)^2}$ (For ungrouped data)

$\sigma = \sqrt{\dfrac{\sum_{i=1}^{n} f_i\left(x_i - \bar{x}\right)^2}{\sum_{i=1}^{n} f_i}}$ (For grouped data)

or $\sigma = \sqrt{\dfrac{\sum_{i=1}^{n} f_i x_i^2}{\sum_{i=1}^{n} f_i} - \left(\dfrac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i}\right)^2}$ (For grouped data)

History!

Karl Pearson was the first person to use the word standard deviation.

Challenge!

Can variance be negative?

Example 9 A factory produces rods with lengths (in cm):

50.1, 49.9, 50.0, 50.2, 49.8

Check the consistency using standard deviation.

Solution $\bar{x} = \dfrac{50.1+49.9+50+50.2+49.8}{5} = \dfrac{250}{5} = 50$

x	$x-\bar{x}$	$(x-\bar{x})^2$
50.1	0.1	0.01
49.9	–0.1	0.01
50.0	0	0.00
50.2	0.2	0.04
49.8	–0.2	0.04
		$\Sigma\left(x-\bar{x}\right)^2 = 0.10$

$\sigma^2 = \dfrac{\Sigma\left(x-\bar{x}\right)^2}{n} = \dfrac{0.10}{5} = 0.02$

$\sigma = \sqrt{0.02} = 0.141\,cm$

As the standard deviation is very low, this shows high consistency in rod lengths.

Example 10 The number of carpets sold in each day of a week is 13, 8, 4, 9, 7, 12, 10. Find its variance.

Solution

x	13	8	4	9	7	12	10	$\Sigma x = 63$
x^2	169	64	16	81	49	144	100	$\Sigma x^2 = 623$

$$\sigma^2 = \frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2$$

$$= \frac{623}{7} - \left(\frac{63}{7}\right)^2 = 89 - 81 = 8$$

Challenge!
Can the standard deviation be more than the variance?

Example 11 A financial analyst recorded the closing share prices of several companies on a specific day and grouped them as follows:

Share prices (Rs.)	40 – 50	50 – 60	60 –70	70 – 80	80 – 90	90 – 100
Frequency	2	3	6	4	2	1

Find the mean, variance and standard deviation. What can you conclude about the volatility in share prices?

Solution

Price Range (Rs.)	f	x	fx	fx^2
40 – 50	2	45	90	4050
50 – 60	3	55	165	9075
60 – 70	6	65	390	25350
70 – 80	4	75	300	22500
80 – 90	2	85	170	14450
90 –100	1	95	95	9025
	$\Sigma f = 18$		$\Sigma fx = 1210$	$\Sigma fx^2 = 84450$

$$\bar{x} = \frac{\Sigma fx}{\Sigma f} = \frac{1210}{18} = 67.22$$

$$\sigma^2 = \frac{\Sigma fx^2}{\Sigma f} - \left(\frac{\Sigma fx}{\Sigma f}\right)^2 = \frac{84450}{18} - (67.22)^2$$

$$= 4691.67 - 4518.52 = 173.15$$

$$\sigma = \sqrt{173.15} = 13.16$$

Standard deviation of Rs. 13.16 shows moderate price volatility.

Example 12 Two students, Ali and Rooma, appeared in 5 Mathematics tests. Their score (out of 100) were as follows:

Ali: 72, 68, 74, 70, 76

Rooma: 72, 65, 80, 68, 75

(i) Who performed better on average?

(ii) Who was more consistent?

Remember!

When comparing two data sets:

- The mean tells us about the average performance.
- The standard deviation tells us about the consistency.

Solution

$$\bar{x}(\text{Ali}) = \frac{\Sigma x}{n} = \frac{72+68+74+70+76}{5} = \frac{360}{5} = 72$$

$$\sigma^2(\text{Ali}) = \frac{\Sigma(x-\bar{x})^2}{n}$$

$$= \frac{(72-72)^2+(68-72)^2+(74-72)^2+(70-72)^2+(76-72)^2}{5}$$

$$= \frac{(0)^2+(-4)^2+(2)^2+(-2)^2+(4)^2}{5}$$

$$= \frac{0+16+4+4+16}{5} = \frac{40}{5} = 8$$

$$\sigma(\text{Ali}) = \sqrt{8} = 2.83$$

$$\bar{x}(\text{Rooma}) = \frac{\Sigma x}{n} = \frac{72+65+80+68+75}{5} = \frac{360}{5} = 72$$

$$\sigma^2(\text{Rooma}) = \frac{\Sigma(x-\bar{x})^2}{n}$$

$$= \frac{(72-72)^2+(65-72)^2+(80-72)^2+(68-72)^2+(75-72)^2}{5}$$

$$= \frac{(0)^2+(-7)^2+(8)^2+(-4)^2+(3)^2}{5}$$

$$= \frac{0+49+64+16+9}{5} = \frac{138}{5} = 27.6$$

$$\sigma(\text{Rooma}) = \sqrt{27.6} = 5.25$$

(i) Since $\bar{x}(\text{Ali}) = 72$, $\bar{x}(\text{Rooma}) = 72$. Both performed equally on average.

(ii) $\sigma(\text{Ali}) = 2.83$, $\sigma(\text{Rooma}) = 5.25$

Hence, Ali was more consistent.

EXERCISE 11.2

1. Find the range of the following data sets:

   (i) 63, 89, 98, 125, 79, 108, 117, 60 (ii) 43.5, 13.6, 18.9, 38.4, 61.4, 29.4

2. If the range and the lowest value of a set of data are 46.7 and 13.4

respectively, then find the highest value.

3. Calculate the range of the following data:

Income (in Rs.)	4000 – 4500	4500 – 5000	5000 – 5500	5500 – 6000	6000 – 6500
No. of workers	8	12	30	21	6

4. A group of 7 workers reported the number of items they assembled in a day as:

52, 55, 50, 53, 54, 56, 52

Find the standard deviation and variance of the items assembled.

5. A librarian recorded the number of visitors during 5 days of a week.

120, 135, 130, 125, 140

Calculate the variance and standard deviation of visitors.

6. Find the range, variance and standard deviation of first 23 odd numbers.

7. The rainfall recorded in various places of five districts in a week is given below. Find its variance and standard deviation.

Rainfall (in mm)	42	51	54	61	63	71
Number of places	5	13	4	9	5	4

8. Machine A Output (units): 98, 100, 102, 101, 99
   Machine B Output (units): 95, 100, 105, 90, 110
   (i) Which machine has better performance?
   (ii) Which machine is more consistent?

9. The monthly sales (rupees in lacs) for two salespersons over 6 months are:

   Person A: 5.5, 5.7, 5.4, 5.6, 5.8, 5.6

   Person B: 5.5, 6.5, 4.5, 6.0, 5.0, 6.0

   Compare their performance and consistency.

10. The table given below shows the daily wages of workers in a textile mill, grouped into six income brackets:

Daily Wage (Rs)	800 – 1000	1000 – 1200	1200 – 1400	1400 – 1600	1600 – 1800	1800 – 2000
Frequency	2	4	6	8	2	1

Calculate the mean, variance and standard deviation of the wages.

11. A company forecasts monthly sales (rupees in millions): 15, 18, 14, 20, 13. Find variability in sales predictions.

12. Unemployment rates (%) in five provinces are 5.2, 6.0, 4.8, 5.5, 6.2. Calculate standard deviation and describe is there balanced unemployment rate?

13. Find variance and standard deviation:

(i) $\Sigma x = 45, \Sigma x^2 = 421, n = 5$ (ii) $\Sigma fx = 210, \Sigma fx^2 = 7560, \Sigma f = 6$

(iii) $\bar{x} = 18, \Sigma fx^2 = 1670, \Sigma f = 5$

REVIEW EXERCISE 11

1. Four possible answers are given for the following questions. Choose the correct answer.

(i) ________ is used to get the cumulative frequencies.

(a) Addition (b) square root (c) multiplication (d) division

(ii) Second quartile represents:

(a) mean (b) mode (c) median (d) variance

(iii) First quartile divides the data into ________ equal parts.

(a) Two (b) Three (c) four (d) ten

(iv) Difference between the highest and the lowest values is called:

(a) mean (b) variance (c) range (d) standard deviation

(v) Scatter diagram represents a relationship between ________ variables.

(a) five (b) two (c) three (d) four

(vi) ________ is measure of dispersion:

(a) mean (b) median (c) mode (d) variance

(vii) Positive square root of variance is called:

(a) mean (b) median

(c) standard deviation (d) range

(viii) ________ is not measure of dispersion.

(a) range (b) arithmetic mean

(c) variance (d) standard deviation

(ix) Variance of the data 8, 8, 8, 8, 8, 8 is:

(a) 0 (b) 16 (c) 8 (d) 48

(x) Range of first 20 natural numbers is:

(a) 20 (b) 10 (c) 19 (d) 30

2. The following results for the long jump were recorded:

Distance (in cm)	170 – 180	180 – 190	190 – 200	200 – 210	210 – 220	220 – 230
f	2	6	9	12	8	3

Construct the cumulative frequency polygon and locate median, Q_1, Q_3, D_7, P_{90} and interquartile range on it.

3. The summary statistics for a data set is given below. Show it with a box-and-whisker plot.

Min. Value	Max. Value	Q_1	Median	Q_3
5	70	12.6	43	55.6

4. The following table shows the weekly hours spent exercising and resting heart rate (beats per minute) for ten individuals:

Exercise Hours	0	1	2	3	4	5	6	7	8	9
Heart Rate	90	85	83	80	76	74	72	70	68	65

(i) Plot the data on a scatter diagram and draw a line of best fit.

(ii) State the type of correlation observed.

(iii) Predict the heart rate of Sakeena who exercises for 6.5 hours per week.

5. Find the range for the given data, 25 , 30 , 35 , 40 , 50 , 60 , 65 , 75

6. Calculate range, variance and standard deviation for the following data set:

Class Interval	0 – 5	5 – 10	10 – 15	15 – 20	20 – 25	25 – 30
f	4	12	20	24	16	4

7. The sum of 5 numbers is 45 and the sum of their squares is 421. Find the mean and standard deviation of the data.

8. The monthly household expenses (rupees in thousands) of two families for 6 months were:

Family A: 45, 47, 46, 48, 46, 47

Family B: 38, 52, 40, 50, 42, 49

Calculate the mean and standard deviation of the monthly expenses. Which family spends more on average? Which family has more stable expenses?

9. The daily wages of 40 workers in a factory are grouped as follows:

Daily Wages (Rs.)	1000-1200	1200-1400	1400-1600	1600-1800	1800-2000
No. of Workers	6	10	12	8	4

Find the mean, variance and standard deviation of the daily wages.

UNIT 12

Probability

Students' Learning Outcomes

After completing this unit, the students will be able to:

- Calculate the probability of combined events using, where appropriate: sample space diagrams, possibility diagram, tree diagrams and Venn diagrams.
- Apply addition law of probability to solve problems involving mutually exclusive events (such as left and right-hand turns, tossing a coin, even and odd numbers on a dice, winning and losing a game)
- Apply the Multiplication law of probability to solve problems involving independent and dependent events (trading, flipping a coin, such as 2 cards being drawn one by one with replacement and without replacement etc.)

INTRODUCTION

This unit explores the foundational principles and practical methods for calculating the probability of combined events, a critical concept in understanding uncertainty and decision-making. Students will learn to visualize and solve probability problems using sample space diagrams, possibility diagrams, tree diagrams and Venn diagrams. Emphasis will be placed on applying the addition law of probability to mutually exclusive events, situations where two outcomes cannot occur simultaneously, such as flipping a coin or choosing between left and right-hand turns. Additionally, the unit will cover the multiplication law of probability for both independent and dependent events, helping students analyze real-world scenarios like drawing cards with or without replacement and understanding outcomes in games or trade activities.

History!

The earliest exploration of probability as a science was undertaken by Girolamo Cardano (1501–1576), an Italian physician and mathematician.

12.1 Probability

Probability helps us understand how likely something is to happen. We use it every day without even realizing it like guessing if it will rain, deciding whether to carry an umbrella or thinking about the chances of winning a game. It gives us a way to make smart guesses when we don't know for sure what will happen. Simply, probability is useful whenever we deal with uncertainty.

Do you know?

A statistical experiment is a type of random experiment where the outcomes are numerically measured and analyzed using statistical methods.

All statistical experiments are random experiments, but all random experiments are not statistical experiments.

12.1.1 Random Experiment

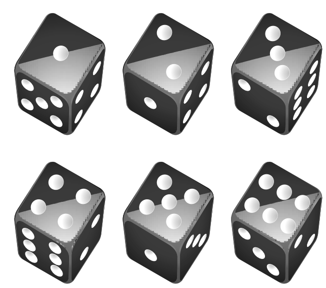
A random experiment is an experiment in which:

(i) The set of all possible outcomes are known.

(ii) Exact outcome is not known.

For example, tossing a coin, rolling a dice.

12.1.2 Outcomes and Sample Space



The possible result of a random experiment is known as **outcome**.

The set of all possible outcomes in a random experiment is called a **sample space**. It is generally denoted by S.

For example, when we roll a dice, the possible outcomes are the face numbers 1, 2, 3, 4, 5, 6 of the dice. Therefore the sample space is $S = \{1, 2, 3, 4, 5, 6\}$.

Each element of a sample space is called a **sample point**.

12.1.3 Event

In a random experiment, each possible outcome is called an event. Thus, an event will be a subset of the sample space.

For example, getting two heads is an event when we toss two coins.

Remember!

Performing an experiment once is called a **trial**.

When we toss a coin thrice, then each toss of a coin is a trial.

Events	Explanation	Example
Equally likely events	Two or more events are said to be equally likely if each one of them has an equal chance of occurring.	Numbers 1, 2, 3, 4, 5, 6 are equally likely events, when we roll a dice.
Certain events	In an experiment, the event which surely occurs is called certain event.	When we roll a dice, the event of getting any natural number from 1 to 6 is a certain event.
Impossible events	An impossible event is an event that cannot happen under any circumstances in a given situation.	When we toss two coins, the event of getting three heads is an impossible event.
Mutually exclusive events	Two or more events are said to be mutually exclusive if they don't have common sample points. i.e., events A and B are said to be mutually exclusive if $A \cap B = \phi$	When we roll a dice the events of getting odd numbers and even numbers are mutually exclusive events.
Exhaustive events	The collection of events whose union is the whole sample space is called exhaustive events.	When we toss a coin twice, the collection of events of getting two heads, exactly one head and no head are exhaustive events.
Complementary events	The complement of an event A is the event representing a collection of sample points not in A. It is denoted A' or A^c or $\overline{A}$. The event A and its complement A' are mutually exclusive and exhaustive.	When we toss a coin, the event 'Head' and the event 'Tail' are complementary events.

Remember!

If an event E consists of only one outcome, then it is called an elementary event.

History!

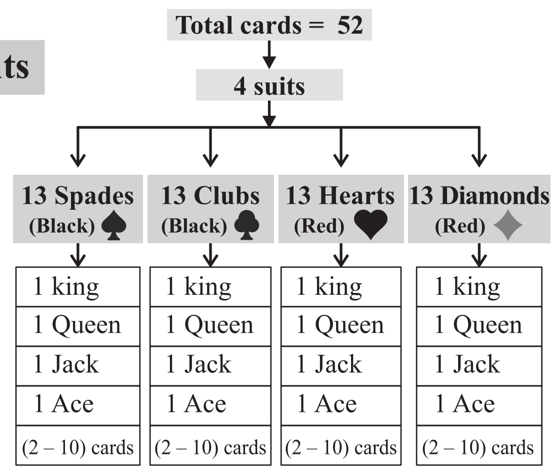
In 1713, Bernoulli was the first to recognize the wide-range applicability of probability in fields outside gambling.

12.1.4 Combined Events

A combined event (or compound event) is an event that consists of two or more simple events happening together. Remember simple event is an event that has a single point of the sample space.

For example, flipping a coin twice and getting at least one head is a combined event.

12.1.5 Types of Combined Events



a Independent Events

Two or more events are said to be independent, if the outcome of one does not affect the outcome of the other.

For example, flipping a coin multiple times, each flip is independent of the others.

b Dependent Events

Two or more events are said to be dependent, if the outcome of one does affect the outcome of the other.

For example, drawing cards from a deck without replacement; the outcome of the first draw changes the probabilities for subsequent draws.

c Mutually Exclusive Events

Two or more events are called mutually exclusive events if they cannot happen at the same time**.** If one event occurs, the other cannot.

For example, when we roll a single dice, getting a 3 and a 5 at the same time is not possible.

d Non-Mutually Exclusive Events

Two or more events are called non-mutually exclusive events if they can happen at the same time. These events may overlap.

For example, drawing a card that is red or a king from a deck

12.1.6 Tree Diagram

A **tree diagram** is a visual representation used in probability to show all possible outcomes of one or more events. Each branch in a tree diagram represents a possible outcome.

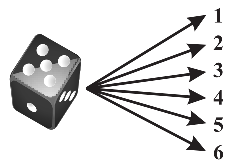


Figure 12.2

(i) When we throw a dice, then from the tree diagram (Figure 12.2), the sample space can be written as

$$S = \{1, 2, 3, 4, 5, 6\}$$

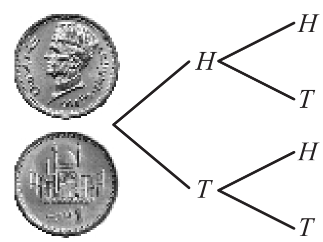


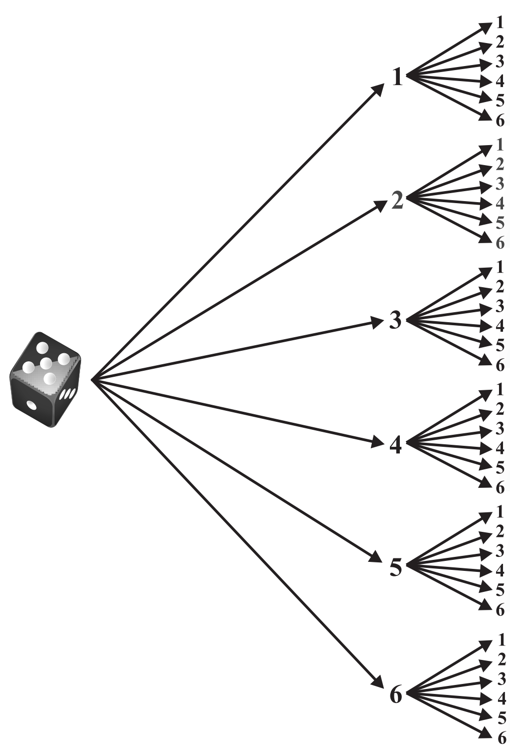
Figure 12.3

(ii) When we toss two coins, then from the tree diagram (Figure 12.3), the sample space can be written as $S = \{HH, HT, TH, TT\}$

Example 1 Express the sample space for rolling two dice using tree diagram.

Solution When we roll two dice, since each dice contains 6 faces marked with 1, 2, 3, 4, 5, 6, the tree diagram will look like.

Hence, the sample space can be written as



$$S = \left\{ \begin{array}{l} (1, 1), (1, 2), (1, 3) \\ (1, 4), (1, 5), (1, 6) \\ (2, 1), (2, 2), (2, 3) \\ (2, 4), (2, 5), (2, 6) \\ (3, 1), (3, 2), (3, 3) \\ (3, 4), (3, 5), (3, 6) \\ (4, 1), (4, 2), (4, 3) \\ (4, 4), (4, 5), (4, 6) \\ (5, 1), (5, 2), (5, 3) \\ (5, 4), (5, 5), (5, 6) \\ (6, 1), (6, 2), (6, 3) \\ (6, 4), (6, 5), (6, 6) \end{array} \right.$$

12.1.7 Sample Space Diagram

Sample space diagram is a representation that shows all possible outcomes of an experiment in list or table form.

When we toss two coins, then the sample space can be written as

$$S = \{HH, HT, TH, TT\}$$

12.1.8 Possibility Diagram

Possibility diagram is a representation in grid or table that shows all possible combinations of two events.

When we toss two coins, then the possibility diagram will look like

Coin 2 \ Coin 1	***H***	***T***
H	*HH*	*HT*
T	*TH*	*TT*

Example 2 Hashim rolled a dice and tossed a coin simultaneously. Draw a possibility diagram.

Solution

Coin \ Dice	**1**	**2**	**3**	**4**	**5**	**6**
H	$(H,1)$	$(H,2)$	$(H,3)$	$(H,4)$	$(H,5)$	$(H,6)$
T	$(T,1)$	$(T,2)$	$(T,3)$	$(T,4)$	$(T,5)$	$(T,6)$

12.1.9 Probability of an Event

In a random experiment, let S be the sample space and $E \subseteq S$. Then if E is an event, the probability of occurrence of E is defined as:

$$P(E) = \frac{\text{Number of outcomes favourable to occurence of } E}{\text{Number of all possible outcomes}} = \frac{n(E)}{n(S)}$$

This way of defining the probability is applicable only to finite sample spaces. So, in this unit, we will be dealing problems only with finite sample spaces.

Remember!

- $0 \leq P(E) \leq 1$ (The probability value always lies from 0 to 1).
- If $P(E) = 1$, then the event is certain.
- If $P(E) = 0$, then the event is impossible.
- The complement event of E is $\overline{E}$.

$$P(E) + P(\overline{E}) = 1$$

Example 3 A bag contains 5 blue balls and 4 green balls. Abu Bakar draws a ball at random from the bag. Find the probability that the ball drawn is (i) blue (ii) not blue.

Solution Total number of possible outcomes $n(S) = 5 + 4 = 9$

(i) Let A be the event of getting a blue ball.

$n(A) = 5$

Therefore, $P(A) = \frac{n(A)}{n(S)} = \frac{5}{9}$

(ii) $\overline{A}$ be the event of not getting a blue ball.

So $P(\overline{A}) = 1 - P(A) = 1 - \frac{5}{9} = \frac{4}{9}$

Activity

- Divide students into groups.
- Ask the students to take a coin, toss it 5 times, 10 times, 15 times or 20 times and record their observations.
- What proportion of the number of tosses shows heads? a tails? What is the probability that the outcome is head?

Example 4 Two dice are rolled. Find the probability that the sum of outcomes is (i) equal to 4 (ii) greater than 10 (iii) less than 13.

Solution When we roll two dice, the sample space is given by:

$$S = \{(1,1\}, (1,2), (1,3), (1,4), (1,5), (1,6)$$
$$(2,1), (2,2), (2,3), (2,4), (2,5), (2,6)$$
$$(3,1), (3,2), (3,3), (3,4), (3,5), (3,6)$$
$$(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)$$
$$(5,1), (5,2), (5,3), (5,4), (5,5), (5,6)$$
$$(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\};\ n(S) = 36$$

Challenge!
What is the complement event of an impossible event?

(i) Let A be the event of getting the sum of outcomes equal to 4.

Then $A = \{(1, 3), (2, 2), (3, 1)\};\ n(A) = 3$

Probability of getting the sum of outcomes equal to 4 is

$$P(A) = \frac{n(A)}{n(S)} = \frac{3}{36} = \frac{1}{12}$$

(ii) Let B be the event of getting the sum of outcomes greater than 10.

Then $B = \{(5, 6), (6, 5), (6, 6)\};\ n(B) = 3$

Probability of getting the sum of outcomes greater than 10 is

$$P(B) = \frac{n(B)}{n(S)} = \frac{3}{36} = \frac{1}{12}$$

(iii) Let C be the event of getting the sum of outcomes less than 13.
Here all the 36 outcomes have the sum value less than 13. Hence $C = S$.
Therefore, $n(C) = n(S) = 36$
Probability of getting the total value less than 13 is $P(C) = \frac{n(C)}{n(S)} = \frac{36}{36} = 1$

Example 5 Two coins are tossed together. What is the probability of getting different faces on the coins?

Solution When two coins are tossed together, the sample space is

$$S = \{HH, HT, TH, TT\}; n(S) = 4$$

Let A be the event of getting different faces on the coins.

$$A = \{HT, TH\}; \quad n(A) = 2$$

Probability of getting different faces on the coins is $P(A) = \frac{n(A)}{n(S)} = \frac{2}{4} = \frac{1}{2}$

Example 6 A dice is rolled and a coin is tossed simultaneously. Find the probability that the dice shows an odd number and the coin shows a head.

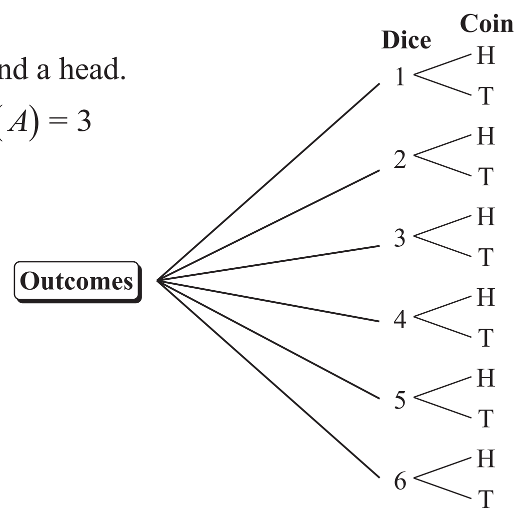
Solution Sample space

$S = \{(1, H), (1, T), (2, H), (2, T), (3, H), (3, T), (4, H), (4, T), (5, H), (5, T), (6, H), (6, T)\}$;
$n(S) = 12$

Let A be the event of getting an odd number and a head.

$$A = \{(1, H), (3, H), (5, H)\}; \ n(A) = 3$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{3}{12} = \frac{1}{4}$$



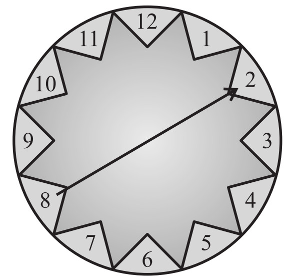
Activity

Ask students to collect the details and find the probabilities of
(i) selecting a boy from your class.
(ii) selecting a girl from your class.
(iii) selecting a boy from 10^{th} class in your school.
(iv) selecting a girl from 10^{th} class in your school.

Example 7 A game of chance consists of spinning an arrow which is equally likely to come to rest pointing to one of the numbers 1, 2, 3, …, 12. What is the probability that it will point to:

(i) 7 (ii) a prime number (iii) a composite number?

Solution



Sample space $S = \{1,2,3,4,5,6,7,8,9,10,11,12\}; n(S) = 12$

(i) Let A be the event of resting in 7, $n(A) = 1$

$$P(A) = \frac{n(A)}{n(S)} = \frac{1}{12}$$

(ii) Let B be the event that the arrow will come to rest in a prime number.

$$B = \{2, 3, 5, 7, 11\}\ ;\ n(B) = 5$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{5}{12}$$

(iii) Let C be the event that arrow will come to rest in a composite number.

$$C = \{4, 6, 8, 9, 10, 12\}\ ;\ n(C) = 6$$

$$P(C) = \frac{n(C)}{n(S)} = \frac{6}{12} = \frac{1}{2}$$

Activity

There are four routes R_1, R_2, R_3 and R_4 from Huria's home to her place of work. There are three parking lots P_1, P_2, P_3 and two entrances B_1, B_2 into the office building. There are three elevators E_1, E_2, E_3 to her floor. Using the tree diagram explain how many ways can she reach her office.

EXERCISE 12.1

1. Find the sample space for tossing three coins using tree diagram.
2. A teacher randomly selects one boy and one girl from a group of 2 boys (Hamid, Ahmad) and 2 girls (Zainab, Samia). Draw a tree diagram and list the sample space for all possible outcomes.
3. If A is an event of a random experiment such that $P(A) : P(\overline{A}) = 17 : 15$ and $n(S) = 640$, then find (i) $P(\overline{A})$ (ii) $n(A)$
4. A coin is tossed thrice. What is the probability of getting two consecutive tails?
5. A dice is rolled and coin is tossed together.
   (i) Find sample space by drawing possibility diagram.
   (ii) Find sample space by sketching tree diagram.
   (iii) What is a probability of getting a tail and an even number?

6. Two unbiased dice are rolled once.
   (i) Find sample space by sketching tree diagram.
   (ii) Find sample space by drawing possibility diagram.
   (iii) Find the probability of getting
      (a) same number on both dice.
      (b) the product as a prime number.
      (c) the sum as an even number.
      (d) the sum as 13.
7. Three fair coins are tossed together. Find the probability of getting
   (i) all tails (ii) at least one head
   (iii) at most two tails (iv) 2 heads
   (v) at most 2 heads (vi) no head
8. A bag contains 4 red balls, 5 white balls, 6 green balls and 3 black balls. Ali draws a ball at random from the bag. Find the probability that the ball drawn is
   (i) white (ii) red
   (iii) not white (iv) not black
9. A number is selected at random from the set of whole numbers 1 to 15, both inclusive. Find the probability that the number selected is:
   (i) odd (ii) a multiple of 5 (iii) the square of 2
   (iv) prime (v) 20
10. If the probability of an event A is $\frac{7}{10}$, then find the probability of the event "not A".
11. A dice is rolled twice. Find the probability of having a number greater than 4 on each roll.

12.2 Addition Law of Probability

(i) If A and B are any two non mutually exclusive events then

$$P(A\cup B) = P(A)+P(B)-P(A\cap B)$$

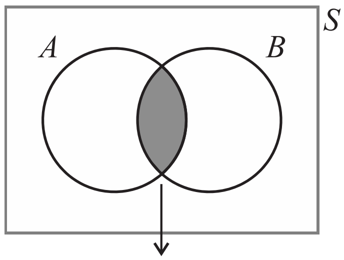
(ii) If A and B are any two mutually exclusive events then

$$P(A\cup B) = P(A) + P(B)$$

12.2.1 Venn Diagram

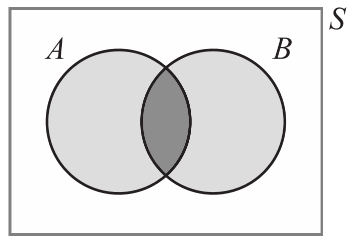
In a random experiment, let S be the sample space. Let $A \subseteq S$ and $B \subseteq S$ be the events in S. we say that

(i) $(A \cap B)$ is an event that occurs only when both A and B occurs.



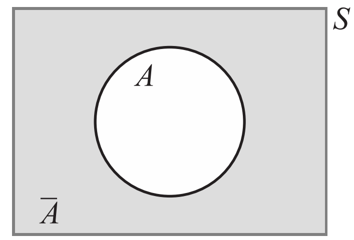
$A \cap B$

(ii) $(A \cup B)$ is an event that occurs when either one of A or B occurs.



$A \cup B$

(iii) $\overline{A}$ is an event that occurs only when A doesn't occur.



Example 8 If $P(A) = 0.37$, $P(B) = 0.42$, $P(A \cap B) = 0.09$, then find $P(A \cup B)$.

Solution Given that: $P(A) = 0.37$, $P(B) = 0.42$, $P(A \cap B) = 0.09$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cup B) = 0.37 + 0.42 - 0.09 = 0.7$$

Challenge!
What is addition of $P(A \cup B)$ and $P(A \cap B)$?

Example 9 A flower is selected at random from a basket containing 50 yellow, 70 red and 80 white flowers. Find the probability of selecting a yellow or red flower?

Solution Total number of flowers, $n(S) = 50 + 70 + 80 = 200$

No. of yellow flowers, $n(Y) = 50$

$$\therefore \quad P(Y) = \frac{n(Y)}{n(S)} = \frac{50}{200}$$

No. of red flowers, $n(R) = 70$

$$\therefore \quad P(R) = \frac{n(R)}{n(S)} = \frac{70}{200}$$

Y and R are mutually exclusive events, so probability of drawing either a yellow or red flower is

$$P(Y \cup R) = P(Y) + P(R)$$

$$\therefore \quad P(Y \cup R) = \frac{50}{200} + \frac{70}{200} = \frac{120}{200} = \frac{3}{5}$$

Example 10 Two dice are rolled together. Find the probability of getting a same number on the both dice or sum of faces as 10.

Solution Let S be the sample space, then $n(S) = 36$

Let A be the event of getting a same number on the both dice and B be the event of getting face sum 10.

$$\text{Then} \quad A = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$$

$$B = \{(4, 6), (5, 5), (6, 4)\}$$

$$\therefore \quad A \cap B = \{(5, 5)\}$$

$$n(A) = 6,\ n(B) = 3,\ n(A \cap B) = 1$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{6}{36}$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{3}{36}$$

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{1}{36}$$

$\therefore$ P(getting a same number on the both dice or a total of 10) = $P(A \cup B)$

$$= P(A) + P(B) - P(A \cap B)$$

$$= \frac{6}{36} + \frac{3}{36} - \frac{1}{36} = \frac{8}{36} = \frac{2}{9}$$

Hence, the required probability is $\frac{2}{9}$.

EXERCISE 12.2

1. If A and B are two events such that $P(A) = \frac{1}{4}$, $P(B) = \frac{1}{2}$ and $P(A \cap B) = \frac{1}{8}$, find $P(A \cup B)$.
2. In an apartment, selecting a house from door numbers 1 to 50 randomly, find the probability of getting the door number of the house to be an even number or a perfect square number.
3. The probability of a team winning any match is $\frac{3}{10}$ and the probability of losing any match is $\frac{2}{10}$. What is the probability that

   (i) the team wins or loses a particular match.

(ii) the team neither wins nor loses a match.

4. In a single throw of two dice, find the probability of having sum of 7 or 11.
5. Find the probability of getting a sum of 5 or 7 in a throw of two dice.
6. A card is taken out at random from a standard pack of 52 cards. Find the probability of taking out.
   (i) A king or a Jack. (ii) Neither a king nor a Jack.
7. A dice is thrown twice. What is the probability that at least one of the two throws comes up with number 3.
8. There are 15 cards in a bag marked as 1, 2, 3, …., 15. Find the probability of picking a card at random, the number written on which is a multiple of 5 or of 7.
9. Two fair coins are tossed once. What is the probability of getting at least one head or two heads.
10. At a busy intersection, 50% of vehicles turn right, 30% turn left and 20% go straight. What is the probability that a randomly selected vehicle turn left or right?
11. Two fair coins are tossed. What is the probability of getting either two heads or two tails?
12. If A and B are two mutually exclusive events of a random experiment and $P(\text{not } A) = 0.45$, $P(A \cup B) = 0.65$, then find $P(B)$.
13. If $P(A) = \frac{2}{3}$, $P(B) = \frac{2}{5}$, $P(A \cup B) = \frac{1}{3}$ then find $P(A \cap B)$.
14. A and B are two events such that, $P(A) = 0.42$, $P(B) = 0.48$, and $P(A \cap B) = 0.16$.
   Find: (i) $P(\bar{A})$ (ii) $P(\bar{B})$ (iii) $P(A \cup B)$

12.3 Multiplication Law of Probability

(i) If A and B are independent events, then

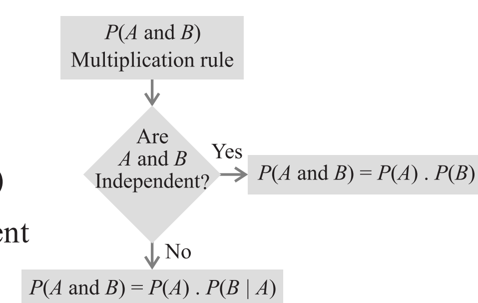
$$P(A \text{ and } B) = P(A \cap B) = P(A) \times P(B)$$

(ii) **Conditional Probability**

If A and B are dependent events, then

$$P(A \text{ and } B) = P(A \cap B) = P(A) \times P(B \mid A)$$

where $P(B \mid A)$ represents the probability of event B occurring after it is assumed that event A has already occurred (read $B \mid A$ as "B given A").



Example 11 A bag contains 4 red and 3 green balls. Kabeer draw two balls one after the other with replacement (the second is drawn after the first is replaced). Find the probability that the first ball is red and the second ball is also red.

Solution Let A be the event that first ball is red and B be the event that second ball is also red.

Then $P(A)=\frac{4}{7}$ and $P(B)=\frac{4}{7}$ (Since the ball is replaced, the sample space is not affected).

$$P(A \text{ and } B) = P(A \cap B) = P(A) \times P(B)$$
$$= \frac{4}{7} \times \frac{4}{7} = \frac{16}{49}$$

Example 12 A bag contains 3 white and 2 black marbles. Two marbles are drawn one after the other without replacement. Find the probability that the first marble is white and the second marble is also white.

Solution Let A be the event that first marble is white and B be the event that second marble is also white.

Total marbles = 3 + 2 = 5

Then $P(A)=\frac{3}{5}$ and $P(B\,|\,A)=\frac{2}{4}$ (Since the marble is not replaced, the sample space is affected).

$$P(A \text{ and } B) = P(A) \times P(B\,|\,A)$$
$$= \frac{3}{5} \times \frac{2}{4} = \frac{6}{20} = \frac{3}{10}$$

Example 13 Majid dealt two cards successively (without replacement) from a well shuffled deck of 52 playing cards. Find the probability that the first card is a king and the second card is a queen.

Solution Let A be the event that drawing card is a king and B be the event that drawing card is a queen. Remember that we must take care for the sample space of event B, that event A has already occurred.

$$P(A) = \frac{4}{52} = \frac{1}{13} \quad ; \quad P(B\,|\,A) = \frac{4}{51}$$
$$P(A \text{ and } B) = P(A) \times P(B\,|\,A)$$
$$= \frac{1}{13} \times \frac{4}{51} = \frac{4}{663}$$

EXERCISE 12.3

1. Two numbers are randomly chosen from 1 to 10 with replacement. Find the probability that:
   (i) both are prime (ii) their product is even
2. One letter is chosen from the word PUNJAB and another from LAHORE. What is the probability that:
   (i) both are vowels
   (ii) one is consonant and other is vowel
3. A single dice is rolled twice. Find the probability that one roll is a multiple of 3 and the other is a 5.
4. Two dice are rolled. Find the probability of getting an odd number on one and a multiple of 2 on other.
5. From a pack of well shuffled cards, two cards are drawn at random one by one with replacement. Find the probability that the first is heart and second is king.
6. If two cards are selected from a standard deck of 52 cards without replacement, find the probability that
   (i) Both are black. (ii) Both are queens.
   (iii) Both are spades. (iv) Both are diamonds.
7. Saleem draws two cards one by one without replacement from a well shuffled pack of 52 playing cards. What is the probability that first card is jack and the second card is queen.

REVIEW EXERCISE 12

1. Four possible answers are given for the following questions. Choose the correct answer.
   (i) The probability of impossible event is:
   (a) 0 (b) 1 (c) 2 (d) –1
   (ii) What is the probability of getting a head when a fair coin is tossed once?
   (a) 0 (b) 0.25 (c) 0.5 (d) 1
   (iii) The sum of probabilities of all possible outcomes of an experiment is:
   (a) 0.5 (b) 0.25 (c) 1 (d) 0.4

(iv) If $P(A) = 0.6$, then the probability of event A not happening is:
(a) 0.4 (b) 0.6 (c) 1 (d) 1.6

(v) Two events are said to be mutually exclusive if:
(a) they can happen at the same time.
(b) one affects the other.
(c) they cannot happen together.
(d) they are always equal.

(vi) If one coin is tossed and one dice is rolled, then the number of sample point are:
(a) 3 (b) 6 (c) 2 (d) 12

(vii) What is the probability of sure event?
(a) 1 (b) 0 (c) $\frac{2}{3}$ (d) $\frac{4}{5}$

(viii) The probability of getting a number greater than 6 on dice is:
(a) 0.33 (b) 1 (c) 0 (d) 0.5

(ix) ________ outcomes are possible when we draw a card from deck of cards.
(a) 13 (b) 1 (c) 52 (d) 26

(x) If $P(E) = 0.07$, then the probability of 'not E' is:
(a) 0.95 (b) 0.89 (c) 0.93 (d) 0.90

2. Arshia selects a day from (Tuesday, Wednesday) and a time from (Morning, Evening). Draw a possibility diagram. Also sketch a tree diagram.

3. A card is drawn from a deck of cards. Find the probability of getting an ace or a spade card.

4. Fatima dealt two cards successively (without replacement) from a shuffled deck of 52 playing cards. Find the probability that both cards are red.

5. If two dice are rolled, then find the probability of getting the product of face values 6 or the difference of face values 5.

6. A bag contains 7 orange and 5 purple marbles. Two marbles are drawn one after the other without replacement. Find the probability that the first marble is orange and the second marble is also orange.

Answers

EXERCISE 1.1

1. (i) i (ii) 1 (iii) $-i$ (iv) -27
   (v) 0 (vi) 0 (vii) i (viii) -1
2. (i) $2+2i$ (ii) $3-i\sqrt{7}$ (iii) $\frac{2}{5}+\frac{4i}{5}$ (iv) $\sqrt{2}-i\sqrt{3}$
3. (i) $x=-2, y=5$ (ii) $x=1, y=7$ (iii) $x=2, y=-1$
   (iv) $x=2, y=1$ (v) $x=3, y=5$

EXERCISE 1.2

1. (i) $5+(5-z)i$ (ii) $25-i$ (iii) $2+i$ (iv) $2+16i$
   (v) $18-i$ (vi) $7-26i$ (vii) $\frac{13}{10}+\frac{1}{10}i$ (viii) $\frac{8}{15}+\frac{3}{5}i$
2. (i) $-3-2i$ (ii) $-4+3i$ (iii) $-5+7i$ (iv) $\frac{2}{3}-\frac{5}{4}i$
3. (i) $\frac{4}{41}-\frac{5}{41}i$ (ii) $\frac{3}{20}-\frac{1}{20}i$ (iii) $\frac{7}{58}+\frac{3}{58}i$ (iv) $\frac{\sqrt{5}}{21}+\frac{4}{21}i$
5. $x=\frac{-2}{5}, y=\frac{4}{5}$ 6. $x=\frac{1}{2}, y=3$ 7. $a=\frac{5}{2}, b=\frac{7}{2}$

EXERCISE 1.3

1. (i) 5 (ii) $\sqrt{41}$ (iii) 1 (iv) $\sqrt{5}$ 5. (i) -5 (ii) -1

EXERCISE 1.4

1. (i) Real part = 55 , Imag. part = 48 (ii) Real part = $\frac{5}{34}$, Imag. part = $\frac{-3}{34}$
   (iii) Real part = $\frac{4}{41}$, Imag. part = $\frac{5}{41}$ (iv) Real part = $\frac{7}{625}$, Imag. part = $\frac{24}{625}$
   (v) Real part = $\frac{18}{13}$, Imag. part = $\frac{1}{13}$ (vi) Real part = $\frac{-7}{25}$, Imag. part = $\frac{24}{25}$
   (vii) Real part = -2 , Imag. part = $\frac{3}{2}$
2. (i) $z=1, w=3-2i$ (ii) $z=-7+4i, w=6-5i$ (iii) $z=4-i, w=1-i$
   (iv) $z=-2+9i, w=2-6i$ (v) $z=-1+2i, w=\frac{1}{2}-\frac{3}{2}i$

REVIEW EXERCISE 1

1. (i) b (ii) d (iii) a (iv) b (v) d
   (vi) c (vii) d (viii) b (ix) a (x) d
3. (i) i (ii) 1 (iii) i

(iv) $-9-38i$ **(v)** $-\frac{13}{74}+\frac{41}{74}i$ **4.** $-8-9i, \frac{8}{145}-\frac{9}{145}i$

6. **(i)** $7+22i$ **(ii)** $\frac{23}{13}+\frac{2}{13}i$ **(iii)** $7-22i$ **(iv)** $\sqrt{533}$

7. Real part = $\frac{2}{53}$, Imag. part = $\frac{-7}{53}$ **8.** $z=-2i$, $w=\frac{3}{5}+\frac{4}{5}i$ **9.** $a=\frac{3}{25}, b=\frac{4}{25}$ **10.** $x=-1, y=0$

EXERCISE 2.1

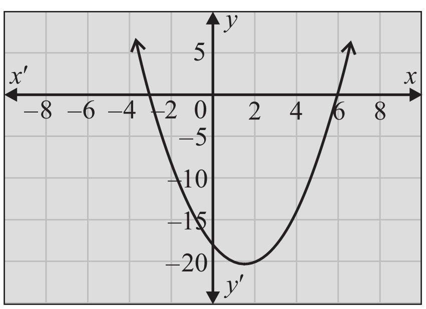
1. **(i)** $2x^2-3x+1=0$ **(ii)** $x^2-3x-6=0$ **(iii)** $2x^2-8x-7=0$
(iv) $9x^2-12x+8=0$ **(v)** $2x^2-5x+2=0$ **(vi)** $6x^2+7x-20=0$

2. **(i)** $\{3,-2\}$ **(ii)** $\{4,-7\}$ **(iii)** $\left\{\frac{1}{3},-\frac{5}{2}\right\}$ **(iv)** $\left\{3,-\frac{3}{2}\right\}$
(v) $\left\{4,\frac{11}{2}\right\}$ **(vi)** $\{11,-13\}$

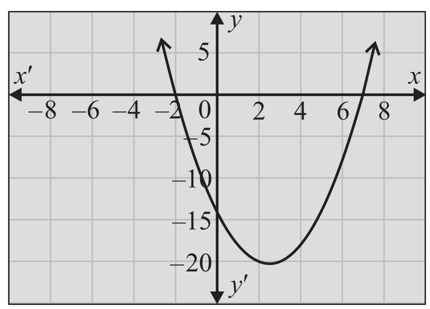
3. **(i)** $\left\{-\frac{1}{2},-2\right\}$ **(ii)** $\{6,-7\}$ **(iii)** $\left\{\frac{3}{4},-\frac{4}{3}\right\}$ **(iv)** $\left\{4,\frac{4}{3}\right\}$
(v) $\left\{\frac{2}{3},13\right\}$ **(vi)** $\{2,7\}$

4. **(i)** $\left\{\frac{3}{2},1\right\}$ **(ii)** $\left\{5,\frac{-3}{2}\right\}$ **(iii)** $\left\{\frac{3}{2},-5\right\}$ **(iv)** $\left\{\frac{7\pm\sqrt{5}}{2}\right\}$
(v) $\left\{\frac{40}{13},6\right\}$ **(vi)** $\left\{\frac{7\pm\sqrt{33}}{2}\right\}$

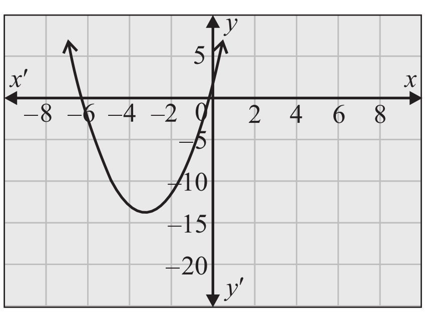
5. **(i)** Solution set = $\{-3, 6\}$



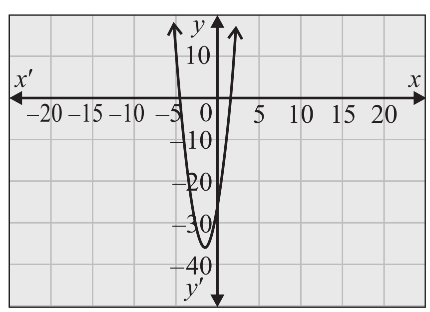
(ii) Solution set = $\{-2, 7\}$



(iii) Solution set = $\{-6, -0.5\}$

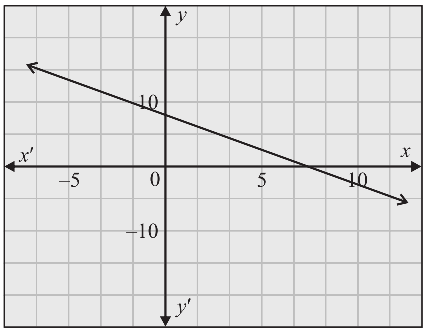


(iv) Solution set = $\{-4.5, 1.5\}$

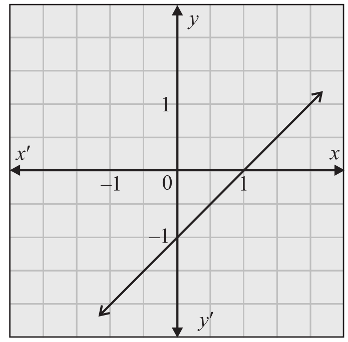


EXERCISE 2.2

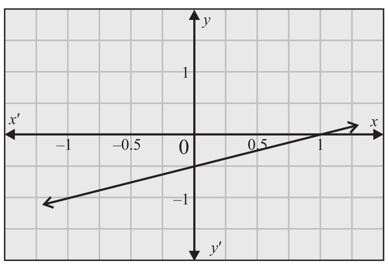
1. **(i)** (8, 0) and (0, 8)



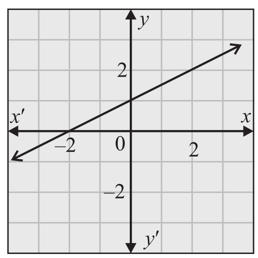
(ii) (1, 0) and (0, –1)



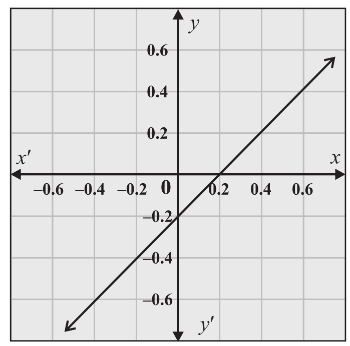
(iii) (1, 0) and (0, –0.5)



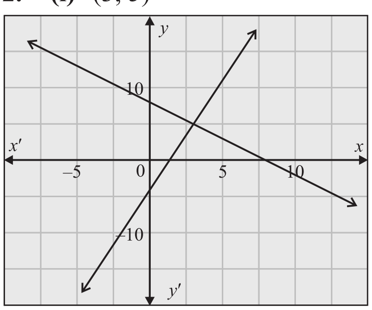
(iv) (–2, 0) and (0, 1)



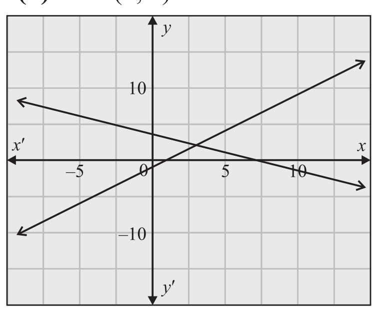
(v) (0.2, 0) and (0, –0.2)



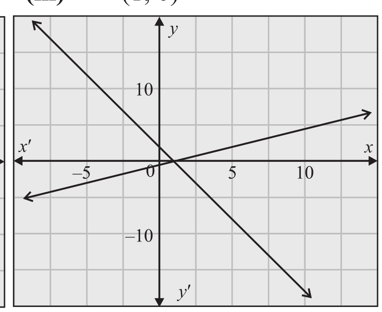
2. **(i)** (3, 5)



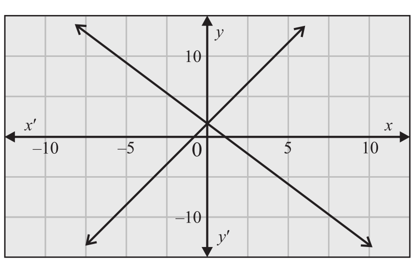
(ii) (3, 2)



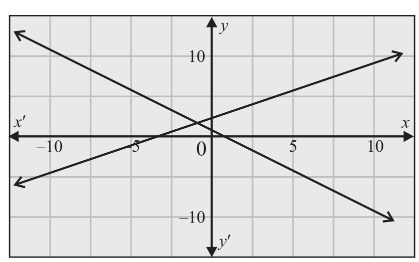
(iii) (1, 0)



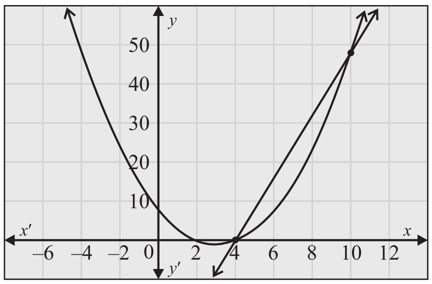
(iv) (0, 2)

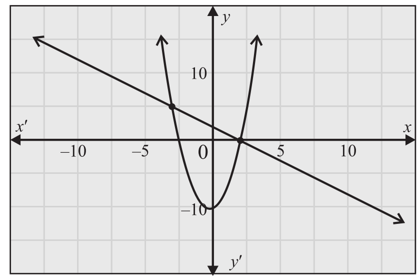


(v) (–1, 2)



3. **(i)** (4, 0) and (10, 48) **(ii)** (2, 0) and (–3, 5)





EXERCISE 2.3

1. **(i)** $x^2 - 5x - 36 = 0$ **(ii)** $x^2 + 2x - 35 = 0$ **(iii)** $25x^2 + 65x + 42 = 0$

(iv) $4x^2 - 8x - 21 = 0$ **(v)** $x^2 - 6x + 4 = 0$ **(vi)** $x^2 + 4x + 1 = 0$

2. $x^2 + 5x + 6 = 0$ **3.** $x^2 - 2px + 4q = 0$

4. **(i)** $4x^2 + 2x + 1 = 0$ **(ii)** $x^2 + x + 1 = 0$ **(iii)** $4x^2 + 14x + 49 = 0$

(iv) $x^2 + 4x + 16 = 0$ **(v)** $x^2 + 6x + 21 = 0$ **5.** $a = c$

6. $k = 2$ **7.** $m = -\frac{3}{2}$ **9.** (i) $p^2 = 4q$ (ii) $q = 1$

EXERCISE 2.4

1. **(i)** Irrational and unequal **(ii)** Real and equal **(iii)** Imaginary **(iv)** Rational and unequal
(v) Imaginary **(vi)** Real and equal

2. $t < \frac{1}{108}$ **3.** $p = \pm 8$ **4.** $q < 4$ **5.** $m = 16$

EXERCISE 2.5

1. **(i)** $(-\infty, -4) \cup (1, \infty)$ **(ii)** $(-\infty, 1) \cup (3, \infty)$ **(iii)** $(-3, 2)$ **(iv)** No solution **(v)** $\left[\frac{3}{2}, \frac{5}{2}\right]$ **(vi)** $[1, 2]$

EXERCISE 2.6

1. $F^\circ = \frac{9C^\circ}{5} + 32$ **2.** **(a)** $P = \frac{I}{RT}$ **(b)** $T = \frac{I}{PR}$ **3.** $a = \frac{S - (n-1)d}{2}$

4. $h = \frac{V}{\pi r^2}$ **5.** $h = \frac{2A}{b_1 + b_2}$ **6.** $x = \frac{y - c}{m}$ **7.** $\ell = \frac{P - 2w}{2}$

8. $x = \frac{y^2}{4a}$ **9.** $S = P + C$ **10.** $h = \frac{3V}{\pi r^2}$

EXERCISE 2.7

1. Year 2030 **2.** Rs. 15 **3.** 3.79 seconds **4.** $1 \le t \le 5$ **5.** 80 metres **6.** 10 hours per week

REVIEW EXERCISE 2

1. **(i)** a **(ii)** d **(iii)** c **(iv)** b **(v)** a
(vi) a **(vii)** c **(viii)** b **(ix)** d **(x)** c

2. **(i)** $\left\{-\frac{7}{8}, 1\right\}$ **(ii)** $\left\{-2, \frac{5}{2}\right\}$ **3.** $2x^2 - 15x + 18 = 0$

4. **(i)** Rational and unequal **(ii)** Irrational and unequal **5.** $v = \sqrt{2gh}$ **6.** $k = -\frac{1}{2}$

EXERCISE 3.1

1. Number of rows in $A = 2$, Number of columns in $A = 2$, Number of rows in $B = 3$,
Number of columns in $B = 1$, Number of rows in $C = 1$, Number of columns in $C = 3$,
Number of rows in $D = 3$, Number of columns in $D = 3$, Number of rows in $E = 2$,
Number of columns in $E = 3$

2. Order of A = 2-by-1, Order of B = 1-by-2, Order of C = 2-by-2, Order of D = 2-by-3,
Order of E = 3-by-2, Order of F = 1-by-1

3. $A = F,\ B = D,\ C = E$ **4.** $a = 3, b = 7, c = 11, d = 0$

5. $x = 4, y = 5$ **6.** $a = 7, b = 3, c = 4, d = 3$ **7.** $p = 4, q = 2$

EXERCISE 3.2

1. A = row matrix, B = null matrix, C = unit matrix, D = null matrix,
E = column matrix, F = column matrix

2. A = column matrix, B = square matrix, C = rectangular matrix,
D = square matrix, E = rectangular matrix, F = row matrix,
G = square matrix, H = rectangular matrix

3. A = unit matrix, B = scalar matrix, C = diagonal matrix,
D = diagonal matrix, E = scalar matrix

4. $A^t = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$, $B^t = \begin{bmatrix} 3 & 7 & 6 \end{bmatrix}$, $C^t = \begin{bmatrix} 5 \\ -2 \\ 4 \end{bmatrix}$, $D^t = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}$

5. $-A = \begin{bmatrix} 3 & 0 \\ -5 & -6 \end{bmatrix}$, $-B = \begin{bmatrix} 3 & -3 \\ 2 & -2 \end{bmatrix}$, $-C = \begin{bmatrix} 9 & -1 \\ -1 & 7 \end{bmatrix}$

EXERCISE 3.3

1. A and D are conformable for addition and subtraction.
B, G and M are conformable for addition and subtraction.
C and F are conformable for addition and subtraction.
E and H are conformable for addition and subtraction.

2. **(i)** $\begin{bmatrix} 2 & 1 \\ 1 & 6 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 22 & 2 \\ 3 & -10 \end{bmatrix}$ **(iii)** $\begin{bmatrix} 1 & -4 \\ -8 & 10 \end{bmatrix}$ **(iv)** $\begin{bmatrix} 12 & 3 \\ 4 & 4 \end{bmatrix}$ **(v)** $\begin{bmatrix} 0 & -9 \\ -14 & -16 \end{bmatrix}$ **(vi)** $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

3. **(i)** $\begin{bmatrix} -5 \\ 7 \end{bmatrix}$ **(ii)** $\begin{bmatrix} -9 & 3 \end{bmatrix}$ **(iii)** $\begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix}$ **(iv)** $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

EXERCISE 3.4

1. (i) $AB=\begin{bmatrix}5 & 0\\ -3 & -2\end{bmatrix}, BA=\begin{bmatrix}1 & 6\\ 2 & 2\end{bmatrix}$ (ii) $AB=\begin{bmatrix}11\end{bmatrix}, BA=\begin{bmatrix}3 & -6\\ -4 & 8\end{bmatrix}$

(iii) $AB=\begin{bmatrix}8 & 20\\ 8 & 20\end{bmatrix}, BA=\begin{bmatrix}28\end{bmatrix}$ (iv) $AB=\begin{bmatrix}3 & 10 & 3\\ 3 & -1 & 0\\ -3 & 12 & 3\end{bmatrix}, BA=\begin{bmatrix}-2 & 2\\ 2 & 7\end{bmatrix}$

(v) $AB=\begin{bmatrix}7 & 25\\ 6 & 25\end{bmatrix}, BA=\begin{bmatrix}16 & 7 & 7\\ 14 & 8 & 8\\ 19 & 8 & 8\end{bmatrix}$ 3. $a=-3, b=\dfrac{-5}{2}$ 4. $x=4, y=-2$

EXERCISE 3.5

1. (i) 40 (ii) 59 (iii) 6 2. Singular = B, D; Non-singular = A, C

3. $x=2$ 4. $\text{Adj}A=\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}, \text{Adj}B=\begin{bmatrix}7 & 2\\ -3 & 5\end{bmatrix}, \text{Adj}C=\begin{bmatrix}2 & -5\\ 3 & -3\end{bmatrix}$

5. (i) $\begin{bmatrix}\frac{1}{5} & 0\\ 0 & \frac{1}{5}\end{bmatrix}$ (ii) $\begin{bmatrix}\frac{-1}{32} & \frac{1}{8}\\ \frac{7}{64} & \frac{1}{16}\end{bmatrix}$ (iii) $\begin{bmatrix}\frac{1}{20} & \frac{-1}{5}\\ \frac{-1}{8} & 1\end{bmatrix}$ (iv) $\begin{bmatrix}\frac{3}{34} & \frac{5}{34}\\ \frac{5}{34} & \frac{-3}{34}\end{bmatrix}$

(v) $\begin{bmatrix}\frac{1}{2} & \frac{-4}{3}\\ \frac{-1}{2} & \frac{5}{3}\end{bmatrix}$ (vi) $\begin{bmatrix}\frac{5}{2} & \frac{3}{2}\\ -2 & -1\end{bmatrix}$ 6. $A^{-1}=\begin{bmatrix}-1 & 3\\ -2 & 5\end{bmatrix}$

EXERCISE 3.6

1. (i) $x=2, y=3$ (ii) $x=3, y=-1$ (iii) $x=5, y=-2$ (iv) $x=0, y=1$

2. (i) $x=\dfrac{8}{3}, y=\dfrac{1}{3}$ (ii) $x=1, y=3$ (iii) $x=-3, y=0$ (iv) $x=-2, y=\dfrac{5}{2}$

3. $x=3, y=4$ 4. $x=8, y=-6$ 5. Load of beam A = 60 kN, load of beam B = 40 kN

6. (2, 1) 7. Son's present age = 10 years, Man's present age = 42 years.

8. Speed of the slower cyclist = 8.07 km/h, speed of the faster cyclist = 13.93 km/h 9. $\dfrac{9}{16}$

REVIEW EXERCISE 3

1. (i) b (ii) a (iii) d (iv) b (v) d
(vi) c (vii) c (viii) b (ix) a (x) b

2. (i) $\begin{bmatrix}-1 & -1\\ -3 & -2\end{bmatrix}$ (ii) $\begin{bmatrix}1 & 1\\ 3 & 2\end{bmatrix}$ (iii) $\begin{bmatrix}23 & 19\\ 38 & 36\end{bmatrix}$

4. **(i)** 14 **(ii)** $\begin{bmatrix} 4 & -3 \\ -2 & 5 \end{bmatrix}$ **(iii)** $\begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **(v)** $\begin{bmatrix} 41 & 12 \\ 33 & 10 \end{bmatrix}$ **(vi)** $\begin{bmatrix} 5 & 3 \\ 2 & 4 \end{bmatrix}$

5. **(i)** $x = \frac{15}{23}, y = \frac{29}{23}$ **(ii)** $x = -6, y = \frac{3}{2}$ **(iii)** $x = \frac{12}{5}, y = \frac{1}{5}$

6. 5, 11 **7.** Cost of knife = Rs. 19, cost of fork = Rs. 10

8. Daily wage of a man = Rs. 250 ; Daily wage of a woman = Rs. 750

EXERCISE 4.1

1. **(i)** $5x + 3$ **(ii)** $-x + 7$ **2.** **(i)** $2(x^2 + 4x + 4)$ **(ii)** $2(x^2 + 4x + 4)$ **(iii)** $\frac{1}{2}$ **(iv)** 2

3. **(i)** **(a)** $2x^3 + 3$ **(b)** $(2x + 3)^3$ **(c)** $4x + 9$ **(d)** x^9

(ii) **(a)** $\frac{2}{2x^2 - 1}$ **(b)** $\frac{8 - x^2}{x^2}$ **(c)** x **(d)** $8x^4 - 8x^2 + 1$

(iii) **(a)** x **(b)** x **(c)** $4x - 3$ **(d)** $\frac{x+3}{4}$

4. $k = -5$ **5.** **(i)** 31 **(ii)** 35 **(iii)** -1

6. **(i)** $\frac{x+3}{2}$ **(ii)** $\sqrt[3]{\frac{x+1}{4}}$ **(iii)** $x^2 + 1$ **(iv)** $\frac{2x+1}{3x-1}$

7. **(i)** $f^{-1}(x) = \frac{x-2}{4},\ g^{-1}(x) = \frac{x+18}{6}$ **(ii)** $x = 42$

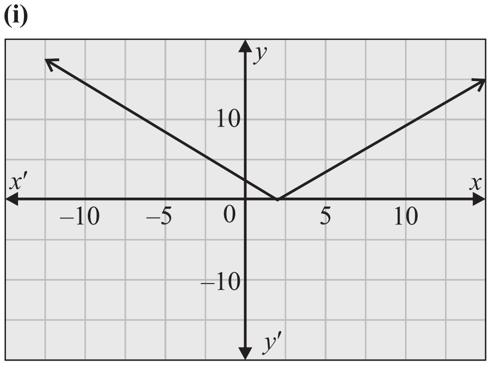
9. **(i)** Domain = $(-\infty, \infty)$; Range = $(-\infty, \infty)$ **(ii)** Domain = $(-\infty, \infty)$; Range = $(-\infty, \infty)$

(iii) Domain = $(-\infty, 1) \cup (1, \infty)$; Range = $(-\infty, -1) \cup (-1, \infty)$

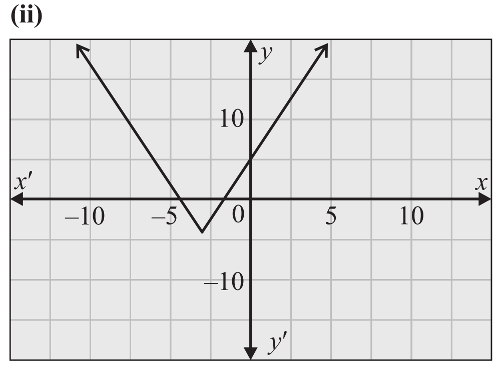
(iv) Domain = $[0, \infty)$; Range = $[2, \infty)$ **10.** $-3, 4$

EXERCISE 4.2

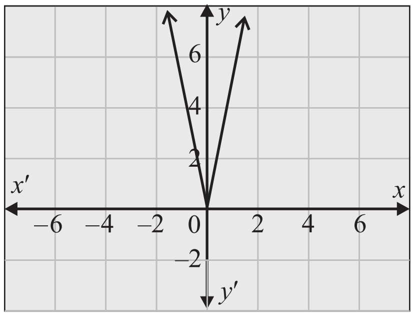
1. **(i)**



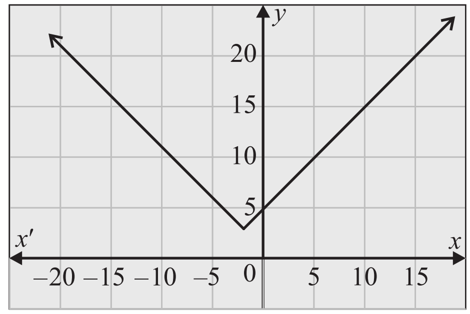
(ii)



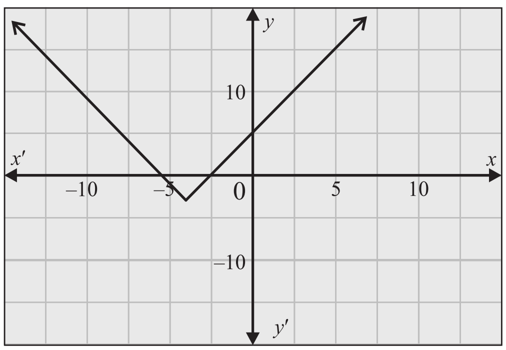
(iii)



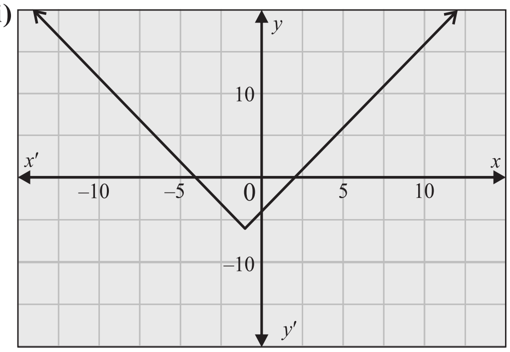
(iv)



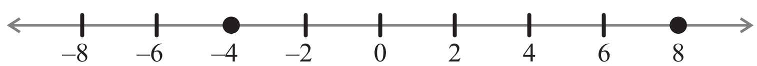
(v)



(vi)

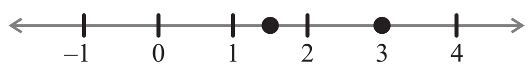


2. **(i)** $\{-4, 8\}$

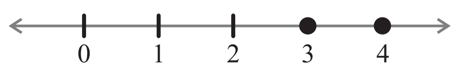


(ii) $\{-4, 3\}$

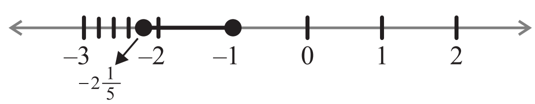
(iii) $\left\{\frac{3}{2}, 3\right\}$



(iv) $\{3, 4\}$

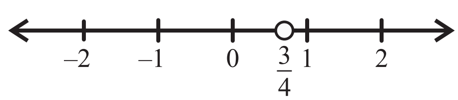


3. **(i)** $\left[-2\frac{1}{5}, -1\right]$

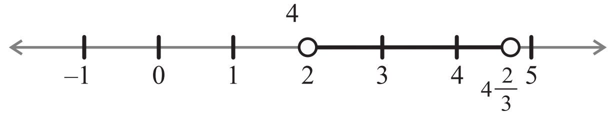


(ii) $x = 3$

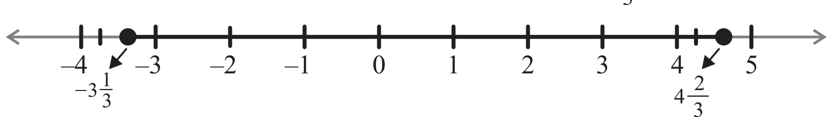
(iii) $\left(-\infty, \frac{3}{4}\right) \cup \left(\frac{3}{4}, \infty\right)$



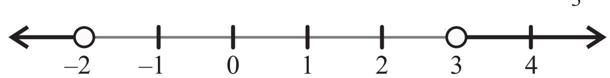
(iv) $\left(2, 4\frac{2}{3}\right)$



(v) $\left[-3\frac{1}{3}, 4\frac{2}{3}\right]$



(vi) $(-\infty, -2) \cup (3, \infty)$



EXERCISE 4.3

1. Rs. 6200 **2.** Rs. 390 **3.** Rs. 275,000 **4.** 3 hours **5.** Rs. 375

6. T < 34.5°C or T > 39.5°C **7.** $2.46 \text{ m} \le x \le 2.54 \text{ m}$ **8.** $x < -0.1$mm or $x > 0.1$ mm

REVIEW EXERCISE 4

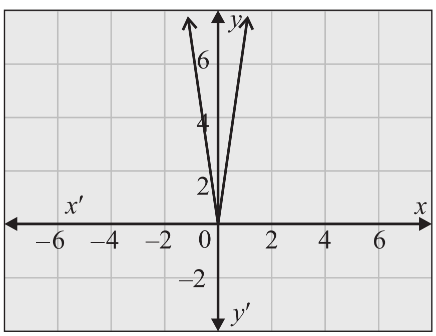
1. **(i)** b **(ii)** c **(iii)** c **(iv)** b **(v)** c
(vi) d **(vii)** d **(viii)** b **(ix)** d **(x)** a

2. **(i)** $-x^2 + x + 30$ **(ii)** $-x^2 - x + 20$ **(iii)** $-x^3 - 5x^2 + 25x + 125$ **(iv)** $5 - x$ **(v)** -24 **(vi)** -3

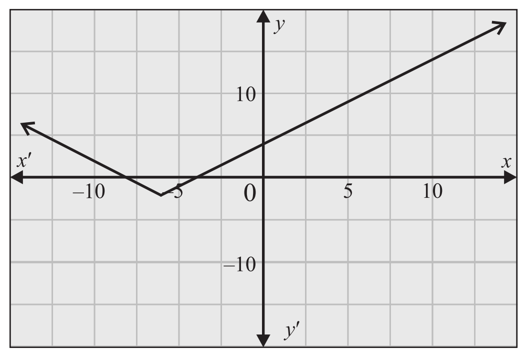
3. **(i)** $(14 + 2x)^3$ **(ii)** $2(7 + x^3)$ **(iii)** x^9 **(iv)** $2(2x + 21)$

4. **(i)** $\frac{x+1}{9}$ **(ii)** $\frac{5+x}{x}$ **(iii)** $x^2 + 5$ **(iv)** $3 - 2x$

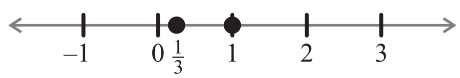
5. **(i)**



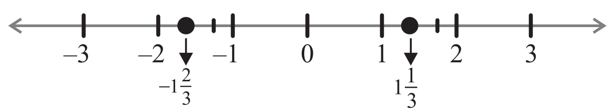
(ii)



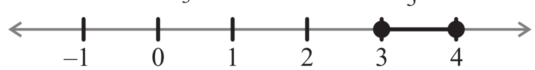
6. **(i)** $\left\{\frac{1}{3}, 1\right\}$



(ii) $\left\{-1\frac{2}{3}, 1\frac{1}{3}\right\}$



7. **(i)** [3, 4]



(ii) [−7, 1]

8. (i) $\frac{x-7}{3}$ (ii) $x = \frac{-7}{2}$ **9.** 100 **10.** Rs. 1700 **11.** $94 \le r \le 106$

EXERCISE 5.1

1. **(i)** $\frac{3}{2b}$ **(ii)** $\frac{b}{c}$ **(iii)** $\frac{c}{x(ax-1)}$ **(iv)** $\frac{3b^2c}{20(1-b)}$

(v) $\frac{2x-3y}{2x}$ **(vi)** $4(x-y)$ **(vii)** $\frac{1}{2a+3x}$ **(viii)** $\frac{x}{x+1}$

(ix) $\frac{3x}{x+2}$ **(x)** $\frac{x+2y}{x^2+xy+y^2}$ **(xi)** $\frac{2x+3}{3x+5}$

EXERCISE 5.2

1. **(i)** $\frac{2}{x-y}$ **(ii)** $\frac{3}{x-2}$ **(iii)** $\frac{5x+y}{18(2x-y)}$ **(iv)** $\frac{1}{x-2y}$ **(v)** $\frac{-3}{x+2}$

(vi) $\frac{9}{(x+3)(2x-3)}$ **(vii)** $\frac{2x^2-3xy-3y^2}{(x-y)(x+y)}$ **(viii)** $\frac{5}{(2x+3)(3x+2)}$ **(ix)** $\frac{3(3x+2)}{(x+3)^2(x-4)}$

(x) $\frac{x-4}{(x-1)(x-2)(x-3)}$ **(xi)** x^2+xy+y^2 **2.** $\frac{2x^3+1}{(x^2+2)^2}$

EXERCISE 5.3

1. **(i)** $\frac{3\ell(\ell-2m)}{2n}$ **(ii)** $\frac{2x(x-2)}{3(x+2)}$ **(iii)** 2 **(iv)** $\frac{(a+b-c)^2}{a(a+2c)}$

(v) $x+2$ **(vi)** 1 **(vii)** $\frac{1}{x^2}$ **(viii)** $\frac{xy}{2(x+y)}$

(ix) $\frac{2(x+2)}{3(x-2)}$ **(x)** $\frac{3(x-2)}{x}$

EXERCISE 5.4

1. **(i)** $\left\{\pm\sqrt{3}, \pm\frac{2\sqrt{5}}{5}\right\}$ **(ii)** $\left\{\pm\sqrt{6}, \pm\frac{\sqrt{3}}{2}\right\}$ **(iii)** $\left\{\pm 2, \pm\frac{\sqrt{10}}{5}\right\}$

(iv) $\{-3, 0\}$ **(v)** $\left\{-\frac{1}{2}, \frac{1}{2}\right\}$ **(vi)** $\{1, 2\}$

(vii) $\left\{-\frac{1}{3}, \frac{2}{3}\right\}$ **(viii)** $\left\{\frac{-7\pm 3\sqrt{5}}{2}, \frac{5\pm\sqrt{21}}{2}\right\}$ **(ix)** $\left\{\frac{-1\pm\sqrt{17}}{4}, \frac{13\pm\sqrt{173}}{2}\right\}$

(x) $\left\{-1, \frac{9\pm\sqrt{65}}{4}\right\}$ **(xi)** $\left\{\frac{3\pm\sqrt{33}}{2}, \frac{3\pm\sqrt{41}}{2}\right\}$ **(xii)** $\left\{\frac{-9\pm\sqrt{5}}{2}\right\}$

EXERCISE 5.5

1. 24 km/h **2.** 12 hours **3.** 3 hours 45 minutes

4 8 hours 34 minutes **5.** 55 km/h, 65 km/h **6.** 12 days

7. 6.67 **8.** 6 m/s **9.** 10 km/h

REVIEW EXERCISE 5

1. **(i)** b **(ii)** b **(iii)** a **(iv)** b **(v)** d

(vi) b **(vii)** c **(viii)** b **(ix)** b **(x)** a

2. **(i)** $\frac{x-4}{x-3}$ **(ii)** $\frac{x+6}{7}$ **3.** **(i)** $\frac{9}{(x+3)(2x-3)}$

(ii) $x+3$ **(iii)** 1 **4.** **(i)** $\{\pm 3, \pm\sqrt{7}\}$

(ii) $\left\{\pm\frac{\sqrt{10}}{2}, \pm\frac{\sqrt{6}}{2}\right\}$ **(iii)** $\{1, 2\}$ **(iv)** $\left\{2\pm\sqrt{3}, \frac{-1\pm\sqrt{-3}}{2}\right\}$

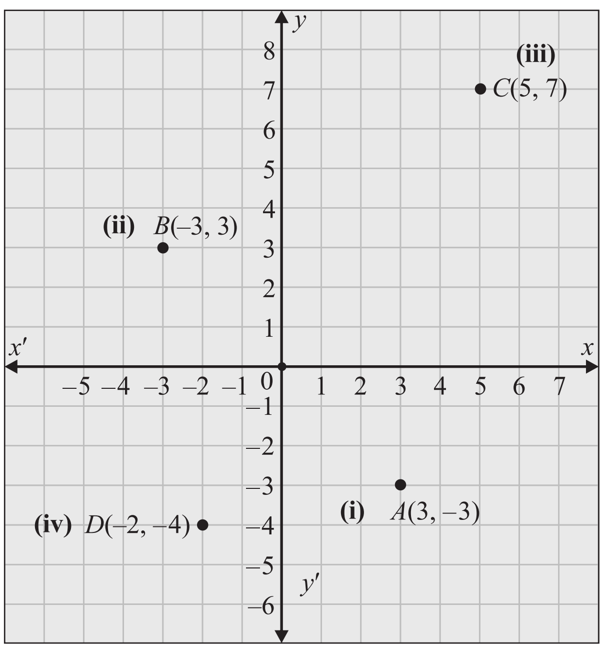
(v) $\left\{1, \frac{-3\pm\sqrt{5}}{2}\right\}$ **(vi)** $\{-4, 2, -1\pm\sqrt{71}\}$ **(vii)** $\{-4, -2, 5, 7\}$

5. 3.16 hours **6.** 18 hours **7.** 10 toys

EXERCISE 6.1

1. **(i)** I **(ii)** IV **(iii)** II **(iv)** III

2.



3. **(i)** Tail = B , Tip = A

(ii) Tail = P , Tip = Q

(iii) Tail = S , Tip = R

(iv) Tail = M, Tip = N

(v) Tail = A , Tip = B

(vi) Tail = D, Tip = C

4. **(i)** $\overrightarrow{AB} = -3\underline{i} + 11\underline{j}$

(ii) $\overrightarrow{AB} = 4\underline{i} - 6\underline{j}$

5. **(i)** $\sqrt{13}$ **(ii)** 5

(iii) $\frac{\sqrt{10}}{2}$

6. **(i)** $\frac{-4}{\sqrt{41}}\underline{i} + \frac{5}{\sqrt{41}}\underline{j}$

(ii) $\frac{3}{5}\underline{i} + \frac{4}{5}\underline{j}$

(iii) $\frac{1}{\sqrt{2}}\underline{i} + \frac{1}{\sqrt{2}}\underline{j}$

(iv) $\frac{2}{\sqrt{13}}\underline{i} - \frac{3}{\sqrt{13}}\underline{j}$ **7.** $\frac{-2\underline{i}}{\sqrt{293}} - \frac{17\underline{j}}{\sqrt{293}}$ **8.** $\frac{-1}{\sqrt{82}}\underline{i} + \frac{9}{\sqrt{82}}\underline{j}$

9. **(i)** Parallel **(ii)** Parallel **(iii)** Non-parallel

(iv) Parallel

10. $-9\underline{i} + 6\underline{j}$ **11.** $6\underline{i} - 10\underline{j}$, $-6\underline{i} + 10\underline{j}$

EXERCISE 6.2

1. (i) $8\underline{i} + 2\underline{j}$ (ii) $10\underline{i} + 12\underline{j}$ (iii) $\frac{43}{2}\underline{i} - \frac{13}{2}\underline{j}$ (iv) $-6\underline{i} + 8\underline{j}$

   (v) $-31\underline{i} + 35\underline{j}$ (vi) $\frac{19}{2}\underline{i} - \frac{19}{2}\underline{j}$

2. (i) 2 (ii) $\sqrt{85}$ (iii) $\sqrt{17}$

3. (i) $x = 5, y = 3$ (ii) $x = -11, y = 0$ (iii) $x = 2, y = 14$

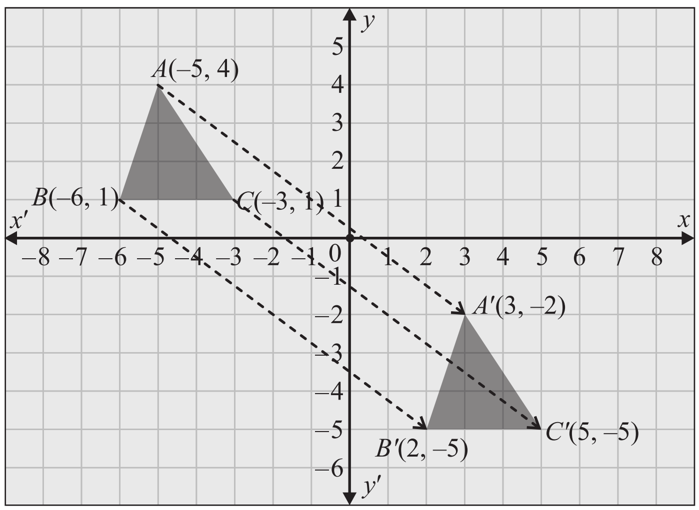
4. $\frac{\sqrt{5}}{2}$

5. $13\underline{i} - 18\underline{j}$

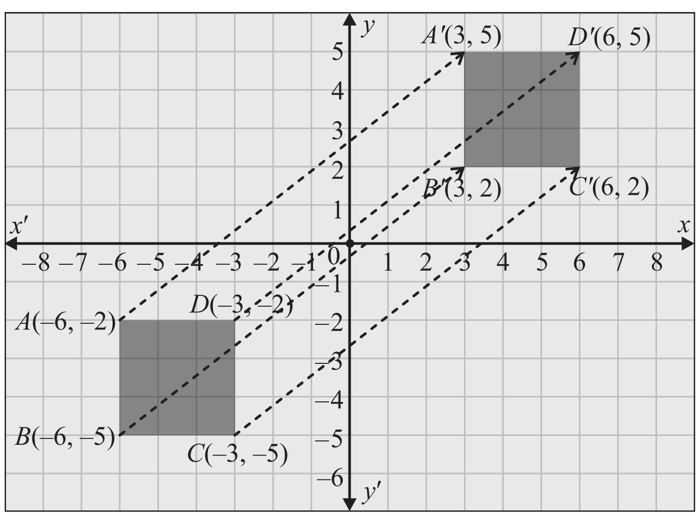
6. $m = \frac{-3}{37},\ n = \frac{47}{37}$

EXERCISE 6.3

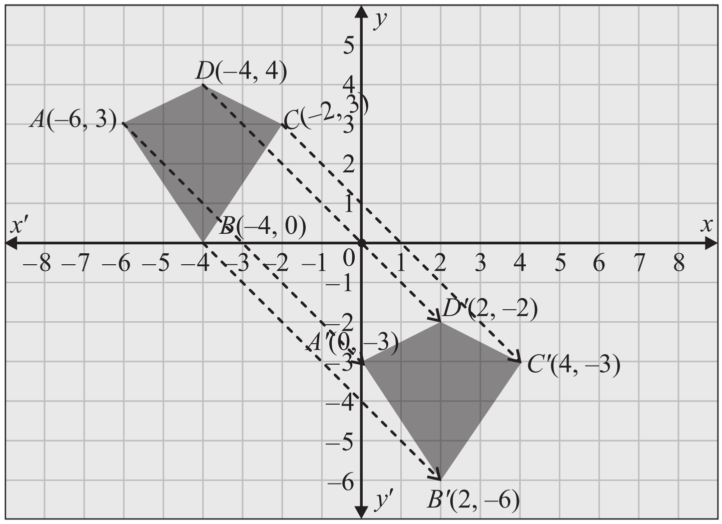
1.



2.



3.



4.	$C(7, 9)$	**6.**	Yes, ABC is an isosceles triangle	**9.**	20 m, 40 m
10.	$30\underline{i} - 30\underline{j}$	**11.**	$20\underline{i} + 50\underline{j}$, 53.85 units		

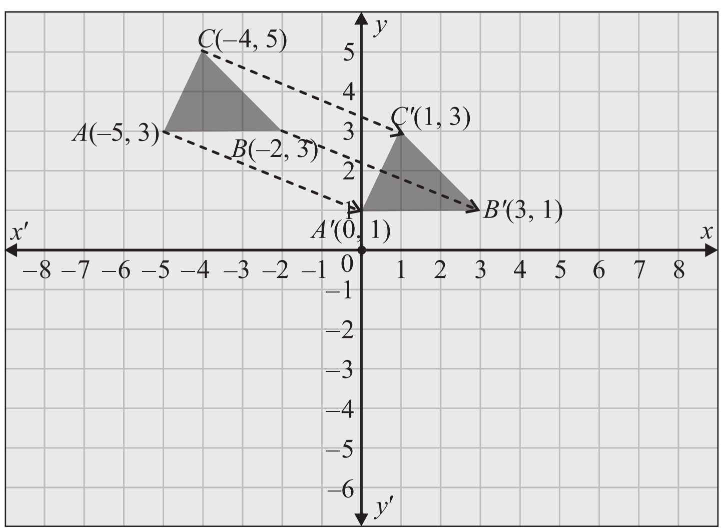
REVIEW EXERCISE 6

1.	**(i)** d	**(ii)** d	**(iii)** c	**(iv)** c	**(v)** b
	(vi) a	**(vii)** c	**(viii)** b	**(ix)** b	**(x)** c

2. **(i)** $\sqrt{410}$ **(ii)** $\sqrt{113}$ **3.** $\frac{5}{\sqrt{26}}\underline{i} + \frac{1}{\sqrt{26}}\underline{j}$

4. **(i)** $17\underline{i} + 7\underline{j}$ **(ii)** $39\underline{i} - 2\underline{j}$ **(iii)** $-\underline{i} + 6\underline{j}$ **5.** $x = \frac{5}{8},\ y = -13$

6.



9. 10 **10.** $200\underline{i} + 60\underline{j}$, 208.8 km/h

EXERCISE 7.1

1. (i) Positive (ii) Negative (iii) Negative (iv) Negative
(v) Positive (vi) Positive
2. (i) – (ii) + (iii) – (iv) – (v) + (vi) +
3. (i) $\frac{1}{2}$ (ii) $\frac{-1}{\sqrt{3}}$ (iii) $-\frac{2}{\sqrt{3}}$ (iv) $\frac{2}{\sqrt{3}}$
(v) $-\frac{1}{2}$ (vi) $-\frac{1}{\sqrt{3}}$ (vii) $\frac{1}{\sqrt{2}}$ (viii) $-\sqrt{2}$ (ix) -1

EXERCISE 7.2

1. (i) $c = 8.6$ cm, $\beta = 67.1°$, $\gamma = 70.9°$ (ii) $b = 20.6$ cm, $\alpha = 36.3°$, $\beta = 93.7°$
(iii) $a = 6.7$ cm, $\alpha = 81.4°$, $\beta = 50.6°$ (iv) $c = 4.3$ cm, $\alpha = 52°$, $\gamma = 57°$
(v) $a = 8.4$ cm, $c = 1.5$ cm, $\gamma = 10°$ (vi) $b = 6.6$ cm, $c = 6.3$ cm, $\beta = 65°$
(vii) $a = 7.7$ cm, $b = 4.4$ cm, $\alpha = 82°$
(viii) $a = 12.3$ cm, $c = 2.4$ cm, $\gamma = 11°$
2. (i) 17.24 cm^2 (ii) 59.82 cm^2 (iii) 13.21 cm^2 (iv) 39.89 cm^2
(v) 4.29 cm^2 (vi) 7.89 cm^2 (vii) 11.71 cm^2 (viii) 21.29 cm^2

EXERCISE 7.3

1. (i) 6.4 cm (ii) 7.1 cm (iii) 25.1°
2. 57.8° 3. 59.5° 4. 124° 5. 29.5° 6. 14.14 units
7. 31.9 m 8. 160 m 9. 3.1 m 10. 9.66 m 11. 376.4 m

EXERCISE 7.4

1. (i) 140° (ii) 290° (iii) 220° (iv) 015°
2. (i) 300° (ii) 120°
3. (i) 310° (ii) 130° (iii) 220° (iv) 040°
4. (i) 316.2 m (ii) 252° 5. 270° 6. 10 km
7. 193.3° 8. 231.8 km 9. 156.2 km

REVIEW EXERCISE 7

1. (i) *d* (ii) *a* (iii) *c* (iv) *c* (v) *d*
(vi) *c* (vii) *c* (viii) *d* (ix) *c* (x) *c*
2. (i) 11.62 cm^2 (ii) 3.71 cm^2 (iii) 4.01 cm^2
3. (i) $c = 6.98$ cm, $\beta = 28.37°$, $\gamma = 102.63°$ (ii) $\beta = 100^o$, $a = 57.8$ cm, $b = 107.3$ cm
4. (i) 9.11 cm (ii) 19.2°
5. (i) 16.1 cm (ii) 29.7°
6. 036.9° 7. 250 km

EXERCISE 8.1

1. 24 cm 2. 1 hoop 3. 7.14 m 4. 6 cm 5. 8 cm
6. 5 cm 7. 20.17 units 8. 18 cm 9. 3.32 cm 10. 6 cm

EXERCISE 8.2

1. $x = 145°$ 2. $x = 77°, y = 6$ cm 3. $x = 28°, y = 4$ cm
4. 10 metres 5. 12 metres
6. The minor arcs subtended by chords *AB* and *CD* are congruent.
7. Yes, the arcs are congruent.

REVIEW EXERCISE 8

1. (i) a (ii) b (iii) c (iv) d (v) b
(vi) d (vii) c (viii) a (ix) c (x) d
2. 3.32 cm 3. 5 cm
4. The angles subtended at the centre by each bench are equal.
5. Yes 6. 7 cm 7. 14.14 m 8. 5 m

EXERCISE 9.1

1. 52^o 2. 57.5^o 3. (i) $28°$ (ii) 62^o
4. 65^o 5. 170^o 6. 6.32m 7. 12cm
8. 12.5cm 9. 2cm 10. 24m

EXERCISE 9.2

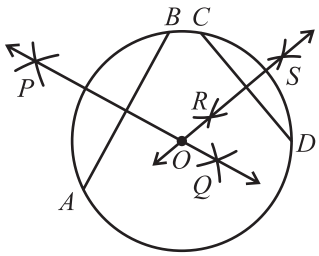
1. 228^o 2. $m\angle BAC = 74^o, m\angle BOC = 148^o, m\angle OBC = 16^o$
3. (i) $x = 21^o, y = 106^o$ (ii) $x = 52^o, y = 90^o$ (iii) $x = 90^o, y = 28^o$
4. $x = 20^o$ 5. 55^o 6. 93^o 7. 1.5 rad
8. (i) 1.36 rad (ii) 2.8 rad 9. (i) 7.8 cm (ii) 10.64cm
10. (i) $\ell = 0.425\text{cm}, A = 0.36125\text{cm}^2$ (ii) $\ell = 2.36\text{cm}, A = 3.53\text{cm}^2$
11. 76.97 cm^2 12. $r = 5$ cm, $\theta = 0.8$ rad or $r = 2$ cm, $\theta = 5$ rad

REVIEW EXERCISE 9

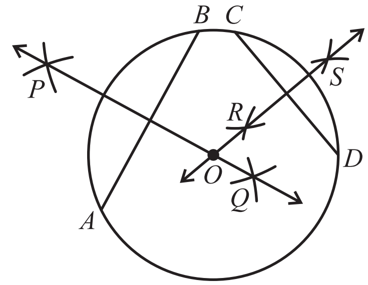
1. (i) *d* (ii) *b* (iii) *a* (iv) *c* (v) *a*
(vi) *b* (vii) *c* (viii) *c* (ix) *c* (x) *a*
2. 7cm 3. $a = 90^o$, $b = 90^o$, $c = 48^o$, $d = 18^o$
4. $a = 68^o$, $b = 56^o$, $c = 56^o$ 5. $a = 40^o$, $b = 40^o$ 6. 22cm 7. 65^o
8. 90^o 9. 75cm^2
10. 15cm^2

EXERCISE 10.1

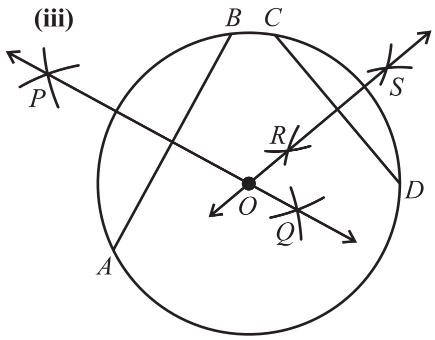
1. **(i)**



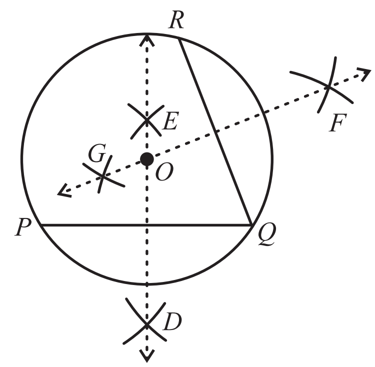
(ii)



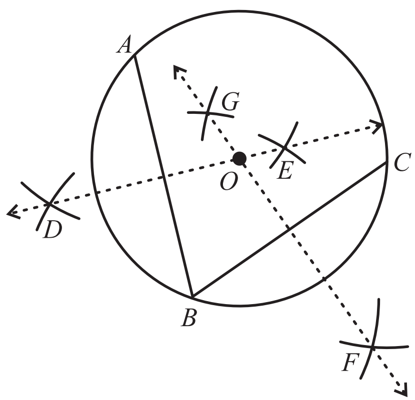
(iii)



2.



3.



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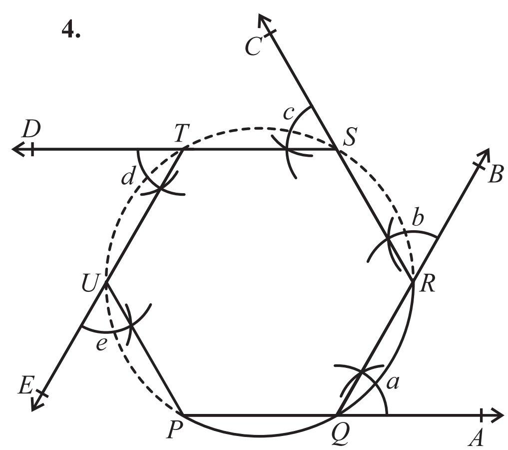
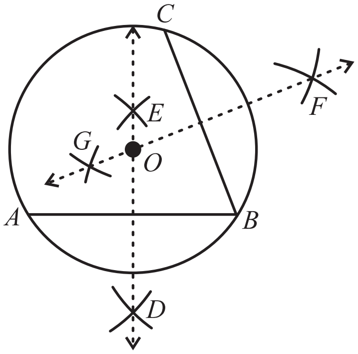


5.



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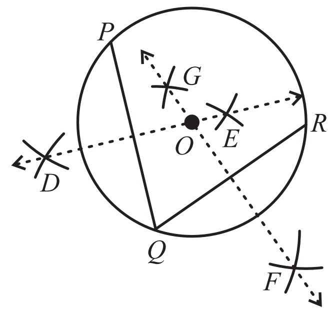

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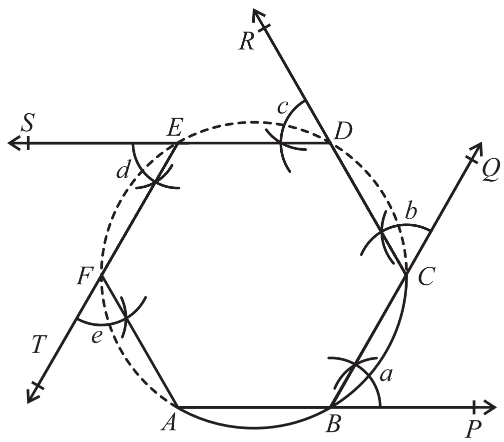
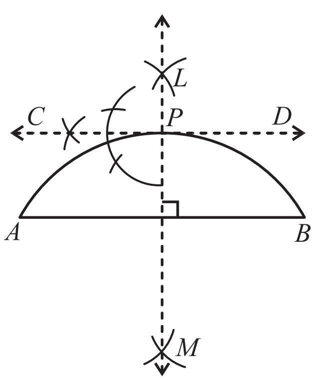
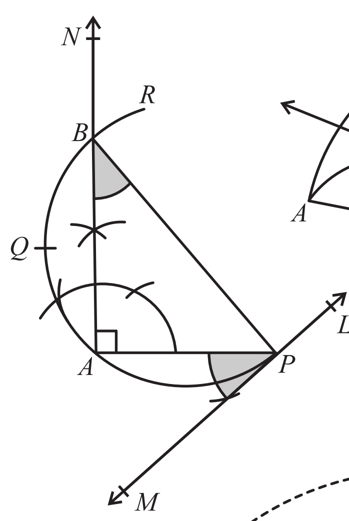


EXERCISE 10.2

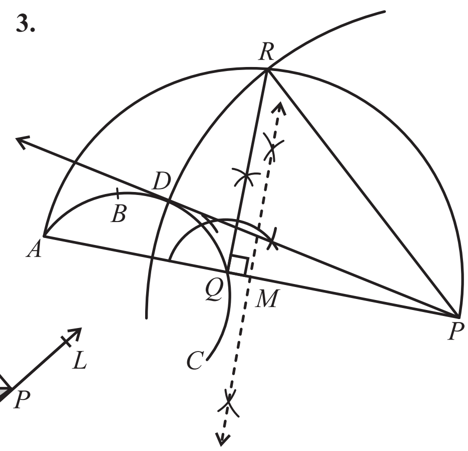
1.



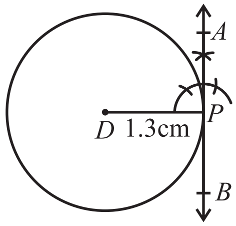
2.



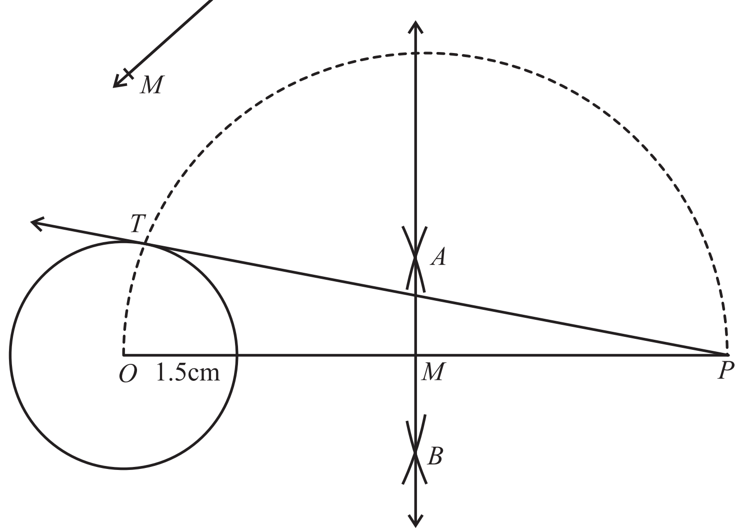
3.



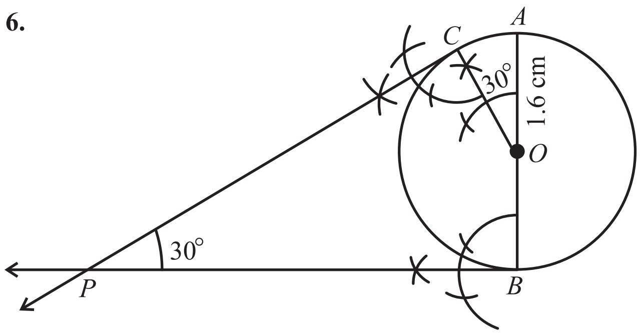
4.



5.



6.



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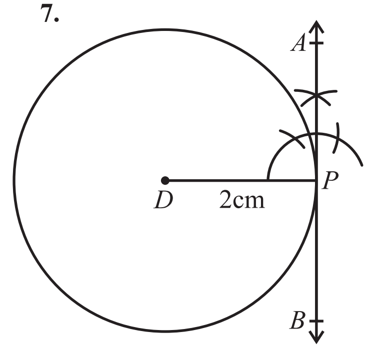

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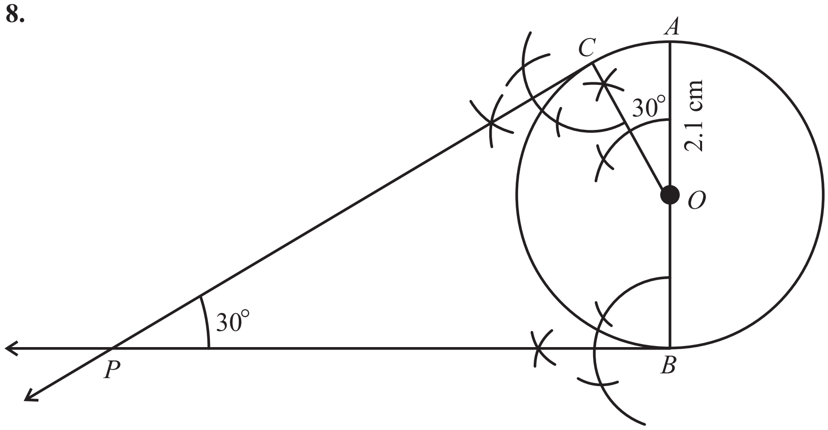
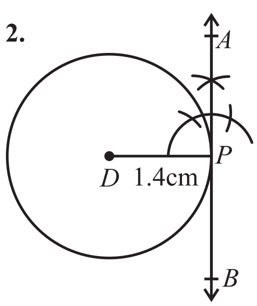


REVIEW EXERCISE 10

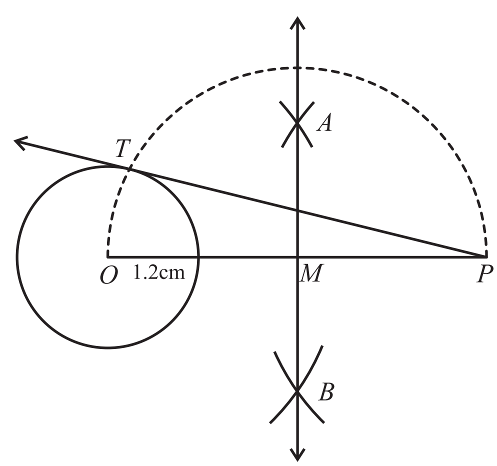
1.

(i)	c	**(ii)**	a	**(iii)**	d	**(iv)**	b	**(v)**	d
(vi)	c	**(vii)**	c	**(viii)**	b	**(ix)**	a	**(x)**	c

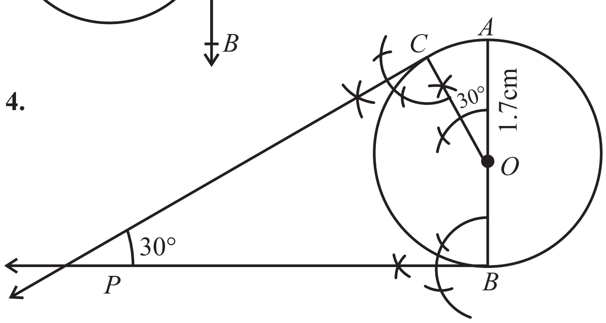
2.



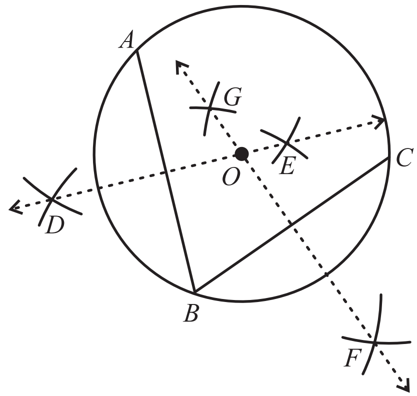
3.



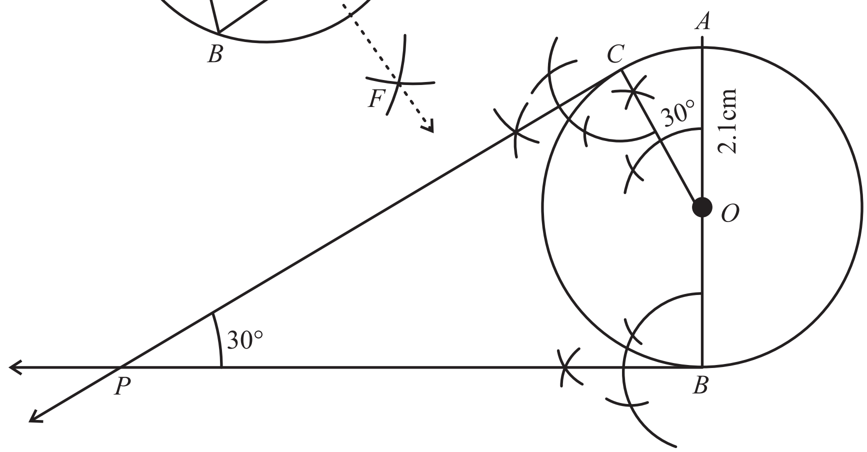
4.



5.

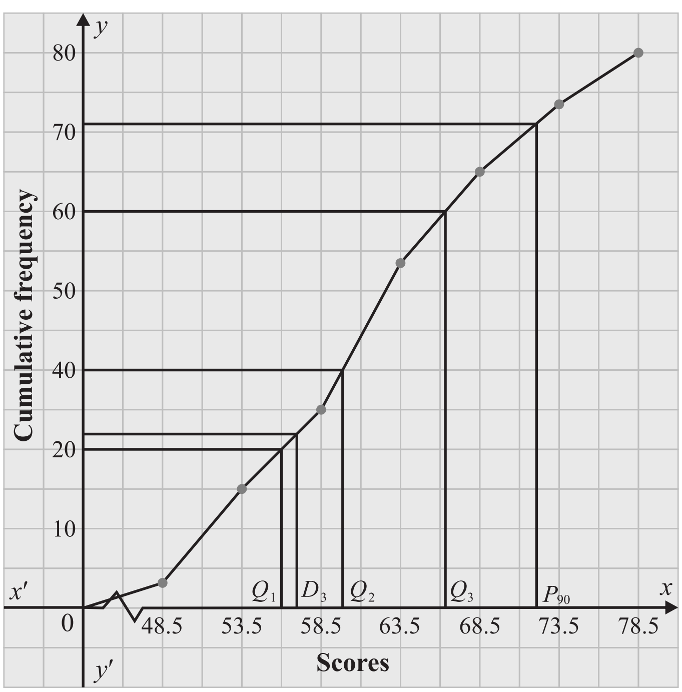


6.



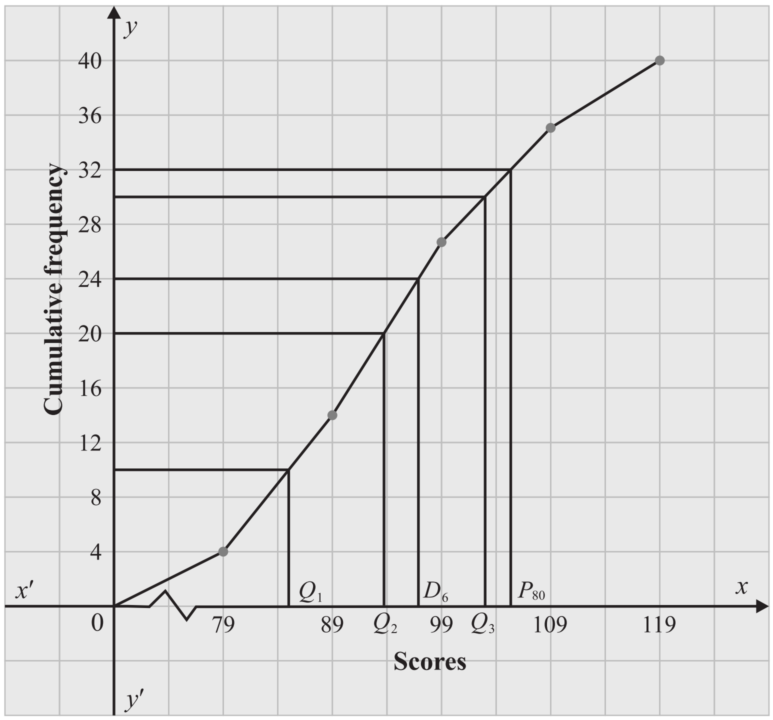
EXERCISE 11.1

1. (i)



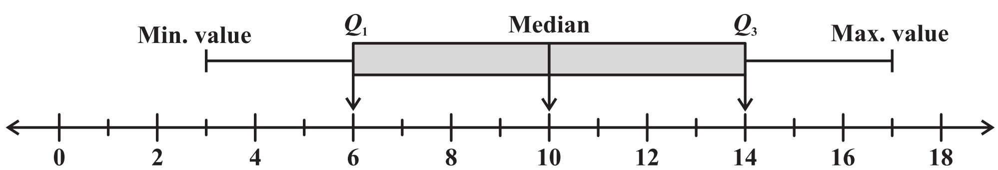
(ii) Median = 61,
$Q_1 = 55$,
$Q_3 = 66$,
$D_3 = 57$,
$P_{90} = 73$,
$IQR = 11$

2.

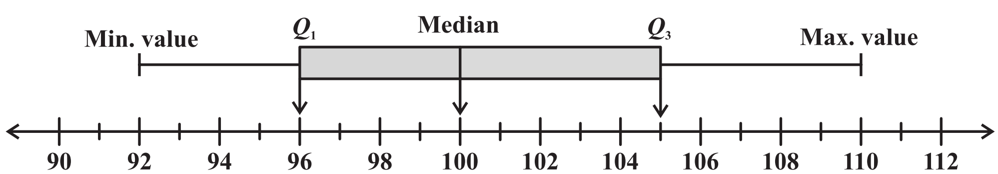


Median = 94,
Q_1 = 85,
Q_3 = 103,
D_6 = 97,
P_{80} = 105
IQR = 18

3. Q_1 = 6, Q_3 = 14, Median = 10, Range = 14, IQR = 8

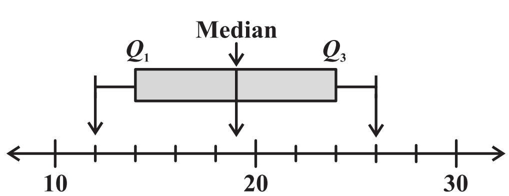


4. Q_1 = 96, Q_3 = 105, Median = 100, Range = 18, IQR = 9
Min. Value = 92, Max. Value = 110

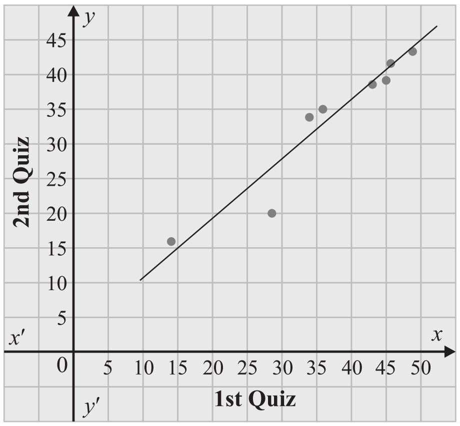


5. Q_1 = 22, Q_3 = 35, Median = 25, Min. Value = 10, Max. Value = 46

6.

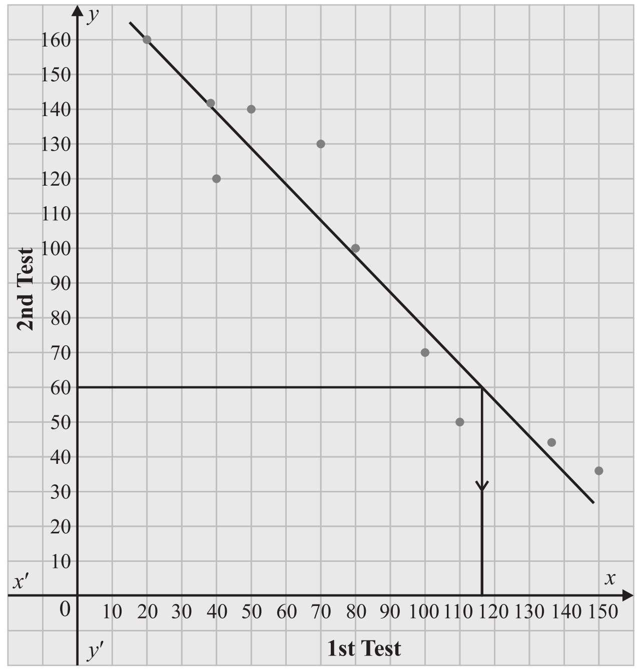


7.



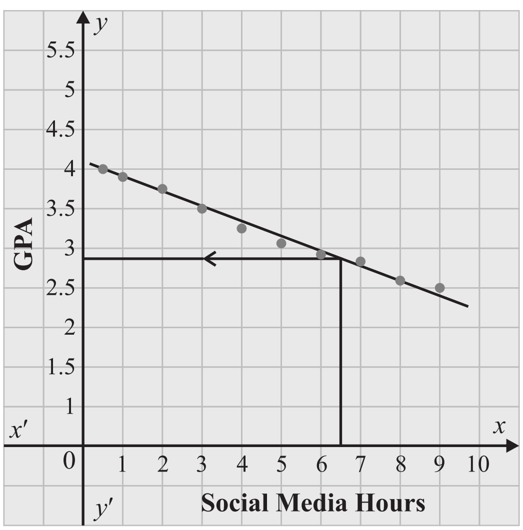
Strong positive correlation between 1st and 2nd quiz.

8. (i)



(ii) Strong negative correlation. **(iii)** 115.5

9. (i)



(ii) Strong negative correlation. **(iii)** 2.9

EXERCISE 11.2

1. (i)	65	(ii)	47.8	**2.**	60.1	**3.**	2500
4.	$\sigma = 1.88, \sigma^2 = 3.55$						

5. $\sigma^2 = 50, \sigma = 7.07$ **6.** Range = 44, $\sigma^2 = 176, \sigma = 13.27$

7. $\sigma^2 = 67.27, \sigma = 8.2$ **8.** (i) Both machines have equal performance.

(ii) Machine A is more consistant. **9.** Person A is better in performance and consistency

10. Mean = 1360.87, variance = 60642.72, standard deviation = 246.26.

11. Variability is 2.61million.

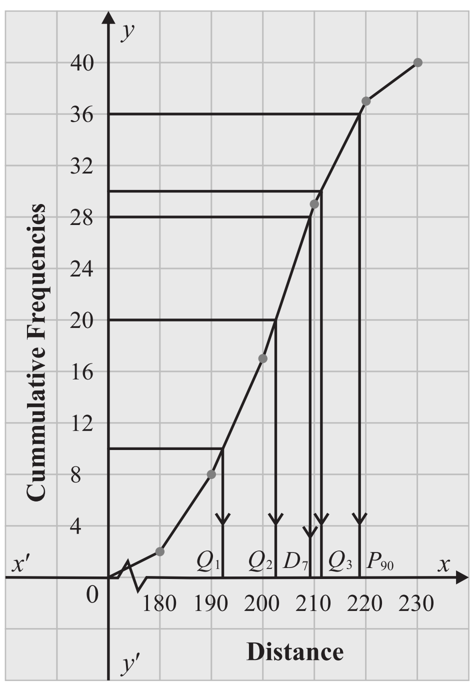
12. $\sigma = 0.51$, unemployement rate are fairly balanced.

13. (i) $\sigma^2 = 3.2, \sigma = 1.79$, (ii) $\sigma^2 = 35, \sigma = 5.92$ (iii) $\sigma^2 = 10, \sigma = 3.16$

REVIEW EXERCISE 11

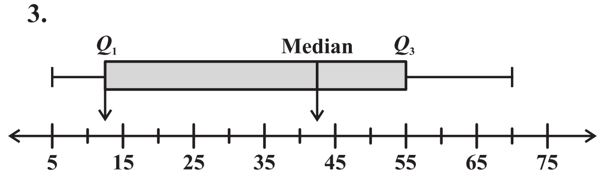
1. (i)	*a*	(ii)	*c*	(iii)	*c*	(iv)	*c*	(v)	*b*	(vi)	*d*
(vii)	*c*	(viii)	*b*	(ix)	*a*	(x)	*c*				

2.

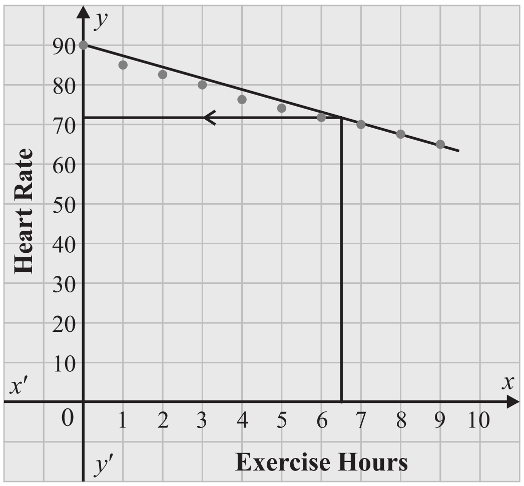


Median = 202.5, $Q_1 = 192.2$, $Q_3 = 211.3$, $D_7 = 209.2$, $P_{90} = 218.8$, $IQR = 19.1$

3.



4. (i)



(ii) Strong negative correlation.

(iii) 71

5. 50 **6.** Range = 30, variance = 38.5, standard deviation = 6.2 **7.** Mean = 9, standard deviation = 1.7889

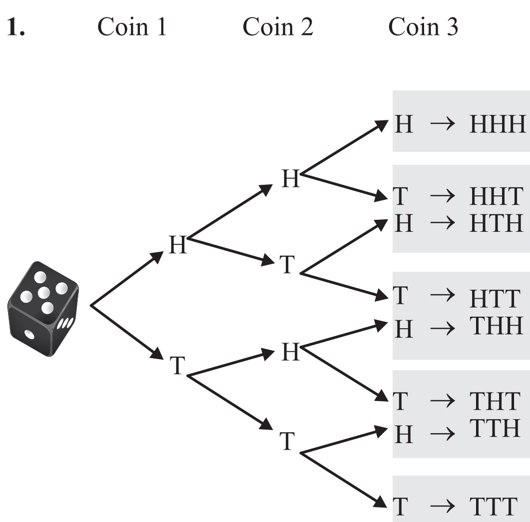
8. Mean (A) = 46.5, standard deviation (A) = 0.96 Mean (B) = 45.17, standard deviation (B) = 5.37

Family A spends more on average and family A's expenses are more stable.

9. Mean = 1470, variance = 57100, standard deviation = 238.96

EXERCISE 12.1

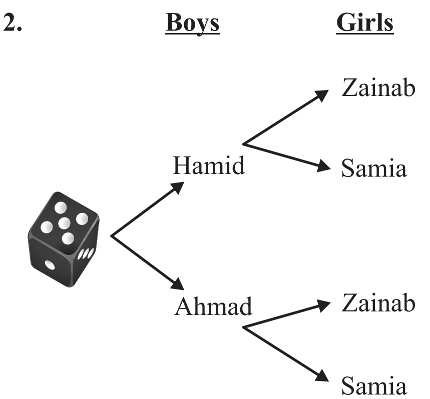
1.



{

Sample space = {(HHH), (HHT), (HTH), (HTT), (THH), (THT), (TTH), (TTT)}

2.



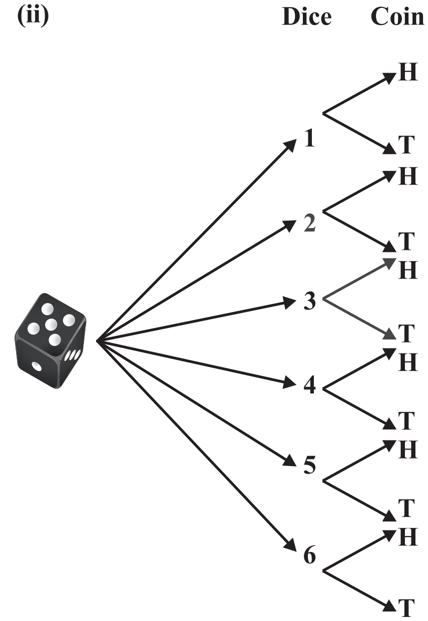
Sample Space = {(Hamid, Zainab), (Hamid, Samia), (Ahmad, Zainab), (Ahmad, Samia)}

3. **(i)** $\frac{15}{32}$ **(ii)** 340 **4.** $\frac{1}{4}$

5. **(i)**

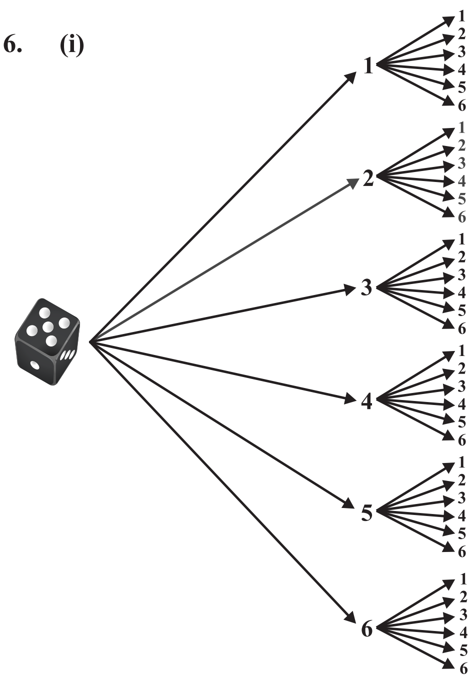
Dice \ Coin	*H*	*T*
1	(1, H)	(1, T)
2	(2, H)	(2, T)
3	(3, H)	(3, T)
4	(4, H)	(4, T)
5	(5, H)	(5, T)
6	(6, H)	(6, T)

(ii)



(iii) $\frac{1}{4}$

6. **(i)**



(ii)

Dice 1 \ Dice 2	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

(iii) **(a)** $\frac{1}{6}$ **(b)** $\frac{1}{6}$

(c) $\frac{1}{2}$ **(d)** 0

7. **(i)** $\frac{1}{8}$ **(ii)** $\frac{7}{8}$ **(iii)** $\frac{7}{8}$ **(iv)** $\frac{3}{8}$ **(v)** $\frac{7}{8}$ **(vi)** $\frac{1}{8}$

8. **(i)** $\frac{5}{18}$ **(ii)** $\frac{2}{9}$ **(iii)** $\frac{13}{18}$ **(iv)** $\frac{5}{6}$

9. **(i)** $\frac{8}{15}$ **(ii)** $\frac{1}{5}$ **(iii)** $\frac{1}{15}$ **(iv)** $\frac{2}{5}$ **(v)** 0 **10.** $\frac{3}{10}$ **11.** $\frac{1}{9}$

EXERCISE 12.2

1. $\frac{5}{8}$ **2.** $\frac{29}{50}$ **3. (i)** $\frac{1}{2}$ **(ii)** $\frac{1}{2}$ **4.** $\frac{2}{9}$ **5.** $\frac{5}{18}$

6. (i) $\frac{2}{13}$ **(ii)** $\frac{11}{13}$ **7.** $\frac{11}{36}$ **8.** $\frac{1}{3}$ **9.** $\frac{3}{4}$ **10.** 0.80

11. $\frac{1}{2}$ **12.** 0.10 **13.** $\frac{11}{15}$ **14. (i)** 0.58 **(ii)** 0.52 **(iii)** 0.74

EXERCISE 12.3

1. (i) $\frac{4}{25}$ **(ii)** $\frac{3}{4}$ **2. (i)** $\frac{1}{6}$ **(ii)** $\frac{1}{2}$

3. $\frac{1}{9}$ **4.** $\frac{1}{2}$ **5.** $\frac{1}{52}$ **6. (i)** $\frac{25}{102}$ **(ii)** $\frac{1}{221}$

(iii) $\frac{1}{17}$ **(iv)** $\frac{1}{17}$ **7.** $\frac{4}{663}$

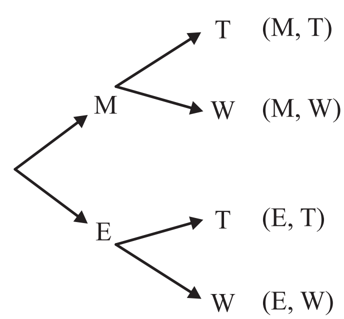
REVIEW EXERCISE 12

1. (i) *a* **(ii)** *c* **(iii)** *c* **(iv)** *a* **(v)** *c*

(vi) *d* **(vii)** *a* **(viii)** *c* **(ix)** *c* **(x)** *c*

2.

	Tuesday	Wednesday
Morning	(M, T)	(M, W)
Evening	(E, T)	(E, W)



3. $\frac{4}{13}$ **4.** $\frac{25}{102}$ **5.** $\frac{1}{9}$ **6.** $\frac{7}{22}$

Glossary

Absolute Value Linear Inequality: An absolute value linear inequality in one variable is an inequality that involves the absolute value of a linear expression with only one variable.

Absolute Valued Function: An absolute valued function is a type of function that includes an algebraic expression enclosed within absolute value bars. Its general form is $f(x) = |x|$.

Algebraic Expression: An algebraic expression is a combination of constants and variables joined with the signs of arithmetic operations.

Algebraic Fraction: An algebraic fraction is a fraction where the numerator, denominator or both contains algebraic expressions.

Arc: Any part of the circumference of a circle is called an arc.

Argand Plane: If we consider x-axis as real axis and y axis as imaginary axis to represent a complex number, then the xy-plane is called complex plane or Argand plane.

Bearing: Bearing is an angle measured in degrees from North direction in a clockwise manner. It is always written in 3 figures.

Box-and-Whisker Plot: A box-and-whisker plot is a graphical representation of a dataset.

Central Angle: If the extremities of an arc or of a chord are joined with the centre of a circle, angle so formed is called a central angle.

Chord: A line segment joining any two distinct points of a circle is called chord of the circle.

Circle: A circle is a locus of a point in a plane, which moves in such a way that its distance from a fixed point is always constant.

Circumference: The length of the boundary of a circle is called circumference.

Column Matrix: A matrix having only one column is called a column matrix.

Complex Number: A complex number is expressed as $x + iy$, where x and y are real numbers and $i = \sqrt{-1}$ is the imaginary number.

Composite Function: If $f(x)$ and $g(x)$ are two functions, then their composite function is written as $(fog)(x) = f(g(x))$. Composition of function is a process of combining two or more functions to produce a new single function.

Compound Event: Compound event is an event that consists of two or more simple events happening together.

Conjugate of a Complex Number: The conjugate of the complex number $x + iy$ is defined as the complex number $x - iy$. The complex conjugate of z is denoted by $\overline{z}$.

Coplanar Vectors: Two or more vectors are said to be coplanar if they lie on the same plane or parallel to the same plane or in the parallel planes.

Cumulative Frequency Polygon: A cumulative frequency polygon, also known as an ogive, is a graphical representation that shows the cumulative total of frequencies up to each class boundary in a frequency distribution.

Deciles: Deciles divide an arranged data set into ten equal parts, each representing 10% of the data.

Dependent Events: Two or more events are said to be dependent, if the outcome of one does affect the outcome of the other.

Diagonal Matrix: A square matrix is called a diagonal matrix if at least one of the elements of its main diagonal is non-zero and non-diagonal elements are zero.

Diameter: A chord passing through the centre of circle is called diameter.

Equal Complex Numbers: Two complex number $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are said to be equal if and only if $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$, that is $x_1 = x_2$ and $y_1 = y_2$.

Equal Matrices: Two matrices A and B are said to be equal if and only if they are of the same order and their corresponding elements are same or equal and in such a case we say that $A = B$.

Equal Vectors: Two vectors are said to be equal if they have equal length and the same direction.

Exponential Equations: Equations in which variable occurs in exponents are called exponential equations.

Event: In a random experiment, each possible outcome is called an event.

Function: A function from a set X to a set Y is a rule that assigns each element of set X exactly one element in set Y. It is written as $f: X \to Y$.

Identity Matrix: A scalar matrix is called an identity or unit matrix if all of its main diagonal entries are 1.

Independent Events: Two or more events are said to be independent, if the outcome of one does not affect the outcome of the other.

Inverse Function: An inverse function is a function that 'reverses' the effect of the original function, meaning if a function f maps X to Y, then its inverse denoted by f^{-1}, maps Y back to X.

Major Arc: A major arc is an arc greater than the semi-circle.

Matrix: A matrix is a rectangular array in shape whose elements are written within square brackets in a definite order, in rows and columns. The arrays obey certain algebraic operations.

Measures of Dispersion: Measure of dispersion are statistical tools used to describe how spread out or scattered the values in a data set are around a central value.

Minor Arc: A minor arc is an arc smaller than a semi-circle.

Modulus of a Complex Number: If $z = x + iy$, then the modulus of z is denoted by $|z|$ and defined as $|z| = \sqrt{x^2 + y^2}$.

Multiplication of Matrices: A matrix A is said to be conformable for multiplication with a matrix B if the number of columns of A is equal to the number of rows of B.

Mutually Exclusive Events: Two or more events are called mutually exclusive events if they cannot happen at the same time.

Negative of a matrix: The negative of a matrix A is obtained by multiplying each of its elements by -1. It is denoted by $-A$.

Negative of a vector: A vector that has the same magnitude as the given vector but opposite direction is called negative of a vector.

Non-Mutually Exclusive Events: Two or more events are called non-mutually exclusive events if they can happen at the same time.

Order of Matrix: If a matrix A has 'm' number of rows and 'n' number of columns then the order of the matrix A is "$m \times n$" (read as m by n).

Outcome: The possible result of a random experiment is known as outcome.

Percentiles: Percentiles divide an arranged data set into 100 equal parts, each containing 1% of the data.

Polynomial: A polynomial in one variable x is an expression of the form:

$P(x) = a_n x^n + a_{n-1}\ x^{n-1} + ... + a_1 x + a_0$, where $a_0, a_1, ..., a_n$ are real numbers, n is a non-negative integer, x is a variable and $a_n \neq 0$.

Quadratic Equation: A quadratic equation is a second degree polynomial equation in one variable.

Quartiles: Quartiles (Q_1, Q_2, Q_3) divide an arranged data set into four parts.

Radial Segment: The line segment joining any point of the circle to its centre is called radial segment.

Random Experiment: A random experiment is an experiment in which the set of all possible outcomes are known and exact outcome in not known.

Range: The difference between highest value and the lowest value is called range.

Rational Expression: A rational expression is the quotient $\dfrac{P(x)}{Q(x)}$, where $Q(x) \neq 0$, of two polynomials $P(x)$ and $Q(x)$.

Reciprocal Equation: A reciprocal equation is an equation when variable is changed to its reciprocal, the equation remains the same.

Rectangular Matrix: If the number of rows and number of columns in a matrix are not equal, it is called a rectangular matrix.

Row Matrix: A matrix having only one row is called a row matrix.

Sample Space: The set of all possible outcomes in a random experiment is called a sample space.

Scalar Matrix: A diagonal matrix is called a scalar matrix if all the entries in the main diagonal are same and non-zero.

Scalar: A scalar is a quantity having magnitude but no direction.

Scatter Diagram: Scatter diagram are used to visually represent the relationship between two variables.

Secant: A straight line cutting the circumference of a circle at two distinct point is called a secant.

Sector of the Circle: The area (region) enclosed between two radial segments (Radii) and arc of circle is called sector of the circle.

Segment of the Circle: The area (region) enclosed between a chord and the arc which it cuts off is called a segment of the circle.

Semi-Circle: A semi-circle is half the circle, that is a figure bounded by a diameter and circumference cut by the diameter.

Simultaneous Linear Equations: A set of two or more linear equations with two or more variables that are solved together (simultaneously) to find a common solution.

Skew-Symmetric Matrix: A square matrix A is called skew-symmetric if $A^t = -A$.

Square Matrix: A matrix in which number of rows is equal to the number of columns is called a square matrix.

Standard Deviation: Standard deviation is the positive square root of the average of the squares of deviations of the given values from their mean.

Symmetric Matrix: A square matrix A is called symmetric if it is equal to its transpose. i.e., $A^t = A$.

Tangent to a Circle: In a plane, a line is a tangent to a circle that touches only one point on the circumference of a circle.

Transpose of a Matrix: A matrix obtained from a given matrix by interchanging the rows and columns is called the transpose of the given matrix.

Tree Diagram: A tree diagram is a visual representation used in probability to show all possible outcomes of one or more events.

Unit Vector: A unit vector is a vector whose magnitude is 1. If $\underline{a}$ is any vector, then its unit vector is represented as $\hat{\underline{a}}$ (read as $\underline{a}$ hat). It is defined as $\hat{\underline{a}} = \frac{\underline{a}}{|\underline{a}|}$.

Variance: The average of the squares of the deviations of the values from their mean is called variance.

Vector: A vector is a quantity having both magnitude and direction.

Vertical Line Test: A curve drawn in a graph represents a function, if every vertical line intersects the curve at most one point.

Zero Correlation: If there is not consistent pattern between the two variables, then it is called zero correlation.

Zero Matrix: If all the entries in a matrix are zero, it is called zero or null matrix.

Zero Vector: A vector whose initial and terminal points are coincident is called a zero vector.

Symbols

Symbols	Stands for
$=$	is equal to
$\neq$	is not equal to
$\in$	belongs to/element of
$\notin$	not belongs to/not element of
$\cup$	union
$\cap$	intersection
$>$	is greater than
$<$	is less than
i	iota
$\leq$	is less than or equal to
$\geq$	is greater than or equal to
$\lvert z \rvert$	modulus or absolute value
$\therefore$	therefore/so
$\because$	since
α	alpha
β	beta
γ	gamma
$\Rightarrow$	implies that
$\rightarrow$	if . . . then or implies
θ	theta
ϕ	phi

Symbols	Stands for
π	pi
(a, b)	ordered pair
$\underline{\hat{a}}$	unit vector a
$n(A)$	number of elements in set A
$P(A)$	probability of event A
$\overline{AB}$	line segment AB
$m\,\overline{AB}$	measure of line segment AB
$\overrightarrow{AB}$	ray AB
$\bar{z}$	conjugate of z
$\overleftrightarrow{AB}$	line AB
$m\angle ABC$	measure of angle ABC
$\angle ABC$	angle ABC
ΔABC	triangle ABC
$\lvert\overline{AB}\rvert$	length of $\overline{AB}$
$\overset{\frown}{AB}$	arc AB
$\parallel$	is parallel to
$\underline{a}$	vector a
$\perp$	is perpendicular to
$\lvert x \rvert$	absolute value of x
$\circ$	degree
λ	lambda

Instructions for Preparation of Exam Paper of Mathematics for Grade-10

- **The paper of Mathematics for Grade-10 will consist of 75 marks.**
- **Objective Type = 15 + Subjective Type = 60 marks.**
- **Timing of the paper will be 2:30 hours.**
- **Objective Type = 20 minutes + Subjective Type = 2:10 hours.**
- **The paper will be made as per following details:**

Objective (MCQs)	**Q-1:** There will be 15 Multiple Choice Questions (MCQs) from the entire content of the textbook. The distribution is as follows: One MCQ each from chapters 1, 2, 3, 4, 5, 6, 7, 8 and 10 and two MCQs each from chapters 9, 11, and 12.	1 × 15 = 15
Part-I: Subjective	This section contains three short questions. The details are as follows: **Q-2:** Students are required to attempt any 6 out of 9 short questions. The details are as follows: • At least two short questions will be drawn each from chapters 1, 2, 3, and 4.	2 × 6 = 12
	Q-3: Students are required to attempt any 6 out of 9 short questions. The details are as follows: • At least two short questions will be drawn each from Chapters 5, 6, 7, and 8.	2 × 6 = 12
	Q-4: Students are required to attempt any 6 of the 9 short questions. The details are as follows: • At least two short questions will be drawn each from Chapters 9 10 11 and 12	2 × 6 = 12
Part-II: Subjective:	This section will contain three long questions bifurcated in two-parts **a & b** (carrying 4 marks each) and students have to attempt two questions. The detail is as follows: **Q-5: (a)** One long-answer question will be selected from chapter 1. **(b)** One long-answer question will be selected from chapter 2. **Q-6: (a)** One long-answer question will be selected from chapter 3. **(b)** One long-answer question will be selected from chapter 4. **Q-7: (a)** One long-answer question will be selected from chapter 5. **(b)** One long-answer question will be selected from chapter 7.	2 × 8 = 16

Part-III: Subjective:	This section will contain two long questions bifurcated in two-parts **a & b** (carrying 4 marks each) and students have to attempt one question. The detail is as follows: **Q-8:** **(a)** One long-answer question will be selected from chapter 6. **(b)** One long-answer question will be selected either from chapter 8 or chapter 11. **Q-9:** **(a)** One long-answer question will be selected from chapter 10. **(b)** One long-answer question will be selected either from chapter 9 or chapter 12.	1× 8 = 8

MODEL PAPER OF MATHEMATICS FOR GRADE-10

Objective Type (معروضی طرز)

Time Allowed: 20 Min. — **وقت: 20 منٹ**

Max. Marks: 15 — **کل نمبر: 15**

1- **نوٹ:** ہر سوال کے چار ممکنہ جوابات A، B، C اور D دیئے گئے ہیں۔ جوابی کاپی پر ہر سوال کے سامنے دیئے گئے دائروں میں سے درست جواب کے مطابق متعلقہ دائرہ کو مارکر یا پین سے بھر دیجیے۔ ایک سے زیادہ دائروں کو پر کرنے یا کاٹ کر پر کرنے کی صورت میں مذکورہ جواب غلط تصور ہوگا۔

Note: Four possible answers A, B, C and D to each question are given. The choice which you think is correct, fill that circle in front of that question with Marker or ink pen in the answer book. Cutting or filling two or more circles will result in zero mark in that question.

D	C	B	A	Questions/سوالات	نمبر شمار
2	1	0	-1	$i^2 + i^4 =$	-i
45	25	20	-20	$x^2+5x-5=0$ کا فرق کنندہ کیا ہے؟ What is the discriminant of $x^2+5x-5=0$?	-ii
سکیو سمیٹرک skew-symmetric	مستطیلی rectangular	قطاری row	سمیٹرک symmetric	اگر $A^t=-A$ ہو تو A ________ قالب ہے: If $A^t=-A$, then A is ________ matrix.	-iii
$f:\frac{X}{Y}$	$f:X\to Y$	$f:Y\to X$	$f:XY$	X سے Y تفاعل f کو ________ سے ظاہر کیا جاتا ہے: A function f from X to Y is represented by:	-iv
$\frac{xyz}{2}$	xyz	$\frac{3}{2}$	$\frac{3xyz}{2}$	$\frac{15xyz}{10}$ کی مختصر ترین شکل ہے: Lowest form of $\frac{15xyz}{10}$ is:	-v
چوتھا fourth	تیسرا third	دوسرا second	پہلا first	$P(4,-4)$ ________ ربع میں واقع ہے: $P(4,-4)$ lies in ________ quadrant.	-vi
$-\cos\theta$	$\cos\theta$	$-\sin\theta$	$\sin\theta$	$\sin(90^\circ+\theta)=$	-vii

D	C	B	A	Questions/سوالات	نمبرشمار
360^0	270^0	180^0	90^0	ایک دائرہ کے مرکزی زاویوں کا مجموعہ ہوتا ہے: The sum of the measures of central angles of a circle is:	-viii
بہت سے many	تین three	دو two	ایک one	دائرہ کے بیرونی نقطہ سے دائرہ پر ______ مماس کھینچے جاسکتے ہیں: ________ tangents can be drawn to a circle from a point outside the circle.	-ix
3 cm	6 cm	8 cm	12 cm	اگر $r = 6$cm ، ریڈین $\theta = 2$ ہو تو قوس کی لمبائی ہے: If $r = 6$cm, $\theta = 2$ radians, then arc length is:	-x
4	3	2	1	دائرہ کا مرکز معلوم کرنے کے لیے کم از کم کتنے وتروں کی ضرورت ہوتی ہے؟ At least how many chords are needed to locate the centre of the circle?	-xi
تغیریت variance	عادہ mode	وسطانیہ median	اوسط mean	دوسرا چہارمی حصہ ظاہر کرتا ہے: Second quartile represents:	-xii
سعت range	معیاری انحراف standard daviation	وسطانیہ median	اوسط mean	تغیریت کا مثبت جذرالمربع کہلاتا ہے: Positive square root of variance is called:	-xiii
1.6	1	0.6	0.4	اگر $P(A) = 0.6$ ہو تو واقعہ A وقوع پذیر نا ہونے کا احتمال ہے: If $P(A) = 0.6$, then the probability of event A not happening is:	-xiv
26	52	1	13	تاش کی گڈی میں سے ایک پتا نکالنے کے ______ ممکنہ نتائج ہوتے ہیں: ________ outcomes are possible when we draw a card from deck of cards.	-xv

Subjective Type (انشائیہ طرز)

Time Allowed: 2:10 hrs. وقت: 2:10 گھنٹے

Max. Marks: 60 کل نمبر: 60

(Part-I حصہ اول)

2.	**Write short answers to any six (06) questions:**		**کوئی سے چھ (6) سوالات کے مختصر جوابات لکھیے۔**	**-2**
(i)	Simplify:	i^{17}	مختصر کیجیے۔	(i)
(ii)	Find the modulus of	$-5-4i$	ماڈیولس معلوم کیجیے۔	(ii)
(iii)	Define quadratic equation with example.		دودرجی مساوات کی مثال کے ساتھ تعریف کیجیے۔	(iii)
(iv)	Write the quadratic equation in standard form.	$2x\,(x + 1) = 4\,(2x + 3)$	دودرجی مساوات کو معیاری شکل میں لکھیے۔	(iv)
(v)	Examine the nature of roots of equation	$3x^2 - 9x - 2 = 0$	مساوات کے روٹس کی نوعیت کا جائزہ لیجیے۔	(v)
(vi)	Find $A-B$ and $B-A$.	$A = \begin{bmatrix} 5 & 9 \\ 4 & 8 \end{bmatrix}, B = \begin{bmatrix} 3 & 6 \\ 2 & 1 \end{bmatrix}$	$A-B$ اور $B-A$ معلوم کیجیے۔	(vi)
(vii)	Find the multiplicative inverse of	$\begin{bmatrix} -4 & 8 \\ 7 & 2 \end{bmatrix}$	ضربی معکوس معلوم کیجیے۔	(vii)

(viii) Find $f(x)\cdot g(x)$, if $f(x) = x + 1,\ g(x) = 2x - 3$
(viii) $f(x)\cdot g(x)$ معلوم کیجیے، اگر

(ix) Solve and express the solution on number line: $|4x-9|=3$
(ix) حل کیجیے اور حل کو عددی خط پر ظاہر کیجیے۔

3: Write short answers to any six (06) questions:
-3 کوئی سے چھ (6) سوالات کے مختصر جوابات لکھیے۔

(i) Simply: $\dfrac{3}{x-y}+\dfrac{1}{y-x}$
(i) مختصر کیجیے۔

(ii) Define reciprocal equation with example.
(ii) معکوس مساوات کی مثال کے ساتھ تعریف کیجیے۔

(iii) Reduce to the lowest form: $\dfrac{x^2-5x}{x^2-4x-5}$
(iii) مختصر ترین شکل میں تبدیل کیجیے۔

(iv) Show that $\underline{a}=5\underline{i}-2\underline{j}$ and $\underline{b}=-10\underline{i}+4\underline{j}$ are parallel.
(iv) ثابت کیجیے کہ $\underline{a}=5\underline{i}-2\underline{j}$ اور $\underline{b}=-10\underline{i}+4\underline{j}$ متوازی ہیں۔

(v) Find a unit vector in the direction of $\underline{a}=\dfrac{5}{3}\underline{i}+\dfrac{1}{3}\underline{j}$
(v) $\underline{a}=\dfrac{5}{3}\underline{i}+\dfrac{1}{3}\underline{j}$ کی سمت میں اکائی ویکٹر معلوم کیجیے۔

(vi) State law of sines.
(vi) سائن کا قانون بیان کیجیے۔

(vii) Calculate area of triangle ABC in which $a = 6.12$ cm, $b = 8.34$ cm, $c = 7.12$ cm:
(vii) ΔABC کا رقبہ معلوم کیجیے جس میں:

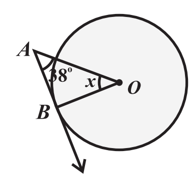
(viii) Define "Secant" and draw its figure.
(viii) قاطع خط کی تعریف کیجیے اور اس کی شکل بھی بنائیے۔

(ix) Prove that a unique circle can pass through the points $A(1, 1)$, $B(4, 2)$ and $C(3, 5)$.
(ix) ثابت کیجیے کہ نقاط $A(1, 1)$، $B(4, 2)$ اور $C(3, 5)$ میں سے ایک منفرد دائرہ گزر سکتا ہے۔

4: Write short answers to any six (06) questions:
-4 کوئی سے چھ (6) سوالات کے مختصر جوابات لکھیے۔

(i) Define tangent to a circle with Figure.
(i) شکل کے ساتھ دائرہ کے مماس کی تعریف کیجیے۔

(ii) Find the value of x.
(ii) x کی قیمت معلوم کیجیے۔



(iii) Find the area of sector of a circle having central angle 60° and radius 7cm. Also find the length of the arc.
(iii) دائرہ کے سیکٹر کا رقبہ معلوم کیجیے۔ جس کا مرکزی زاویہ 60° اور رداس 7 cm ہے۔ قوس کی لمبائی بھی معلوم کیجیے۔

(iv) Construct a circle with the help of radius = 2 cm and locate its centre by construction.
(iv) 2 cm رداس کی مدد سے دائرہ بنائیے اور عمل کے ذریعے اور بناوٹ سے مرکز کی تصدیق کیجیے۔

(v) Draw a circle of radius 1.3 cm draw a tangent at point P, when P lies on its circumference.
(v) 1.3 cm رداس کا ایک دائرہ بنائیے اور نقطہ P پر ایک مماس کھینچیں، جبکہ P اس کے محیط پر واقع ہو۔

(vi) Find median for the given data:
(vi) دیئے گئے مواد کا وسطانیہ معلوم کیجیے۔

102, 98, 95, 100, 93, 110, 108, 104, 97, 96, 92, 101, 99, 105, 107

(vii) Find the range of the given data: 15, 22, 18, 30, 17, 25
(vii) دیئے گئے مواد کی سعت معلوم کیجیے۔

(viii) Two coins are tossed together, what is probability of getting different faces on the coins.
(viii) دو سکے ایک ساتھ اُچھالے گئے۔ دونوں سکوں پر مختلف رُخ آنے کا احتمال کیا ہے؟

(ix) Two numbers are randomly chosen from 1 to 10, with replacement, find the probability that both numbers are prime.

(ix) 1 سے 10 تک کے اعداد میں سے دو اعداد کو واپس رکھنے کے ساتھ بلا ترتیب منتخب کیا گیا، احتمال معلوم کیجیے کہ دونوں اعداد مفرد ہوں۔

(Part-II حصہ دوئم)

Note: Attempt any two (02) questions.

نوٹ: کوئی سے دو سوالات کے جوابات لکھیے۔

5۔ (ا) کمپلیکس عددی سروں z اور w کے لیے $3z+(2+i)w=11-i,\ (2-i)z-w=-1+i$ کو حل کیجیے۔

5: (a) Solve $3z+(2+i)w=11-i,\ (2-i)z-w=-1+i$ with complex coefficient for w and z.

(ب) اگر مساوات $x^2+2(1+k)x+k^2=0$ کے روٹس مساوی ہوں تو k کی قیمت معلوم کیجیے۔

(b) If the equation $x^2+2(1+k)x+k^2=0$ has equal roots, then find the value of k.

6: (a) Solve by matrix inverse method: $\begin{aligned}3x+2y&=2\\ x-2y&=-2\end{aligned}$

6۔ (ا) معکوس قالب کے طریقہ سے حل کیجیے:

(ب) تصدیق کیجیے $f\left(f^{1}(x)\right)=f^{-1}\left(f(x)\right)=x$ جہاں $f(x)=\dfrac{x-4}{x+2}$, $x\neq -2$ ہے۔

(b) Verify $f\left(f^{1}(x)\right)=f^{-1}\left(f(x)\right)=x$, where $f(x)=\dfrac{x-4}{x+2},\ x\neq -2$.

7: (a) Solve: $4\cdot 2^{2x+1}-9\cdot 2^{x}+1=0$

7۔ (ا) حل کیجیے۔

(b) Solve the triangle ABC: $c = 7cm,\ \beta=34^{\circ},\ \gamma=64^{\circ}$

(ب) مثلث ABC کو حل کیجیے۔

(Part-III حصہ سوئم)

Note: Attempt any one (01) question.

نوٹ: کوئی سے ایک سوال کا جواب لکھیے۔

8۔ (ا) A، B اور D کے محددات بالترتیب (1, 2)، (6, 3) اور (2, 8) ہیں۔ اگر $ABCD$ متوازی الاضلاع ہو تو ویکٹر کے طریقہ سے C کے محددات معلوم کیجیے۔

8: (a) The coordinates of A, B and D are (1, 2), (6, 3) and (2, 8) respectively. Find the coordinates of C by using vector method if $ABCD$ is a parallelogram.

(b) A factory produces rods with lengths (in cm).

(ب) ایک فیکٹری درج ذیل لمبائیوں (سینٹی میٹر میں) کی سلاخیں تیار کرتی ہے۔

50.1, 49.9, 50.0, 50.2, 49.8

check consistency using standard deviation.

معیاری انحراف کا استعمال کرتے ہوئے یکسانیت کی جانچ کیجیے۔

9۔ (ا) 1.5 cm رداس کا دائرہ بنائیے اور دو مماس کھینچیں جو 30° کے زاویہ پر ملتے ہوں۔

9: (a) Draw a circle of radius 1.5 cm. Draw two tangents that meet an angle of 30°.

(ب) دو ڈائس ایک ساتھ پھینکے گئے۔ ایک ہی عدد یا اعداد کا مجموعہ 10 حاصل کرنے کا احتمال معلوم کیجیے۔

(b) Two dice are rolled together. Find the probability of getting a same number or sum of faces as 10.