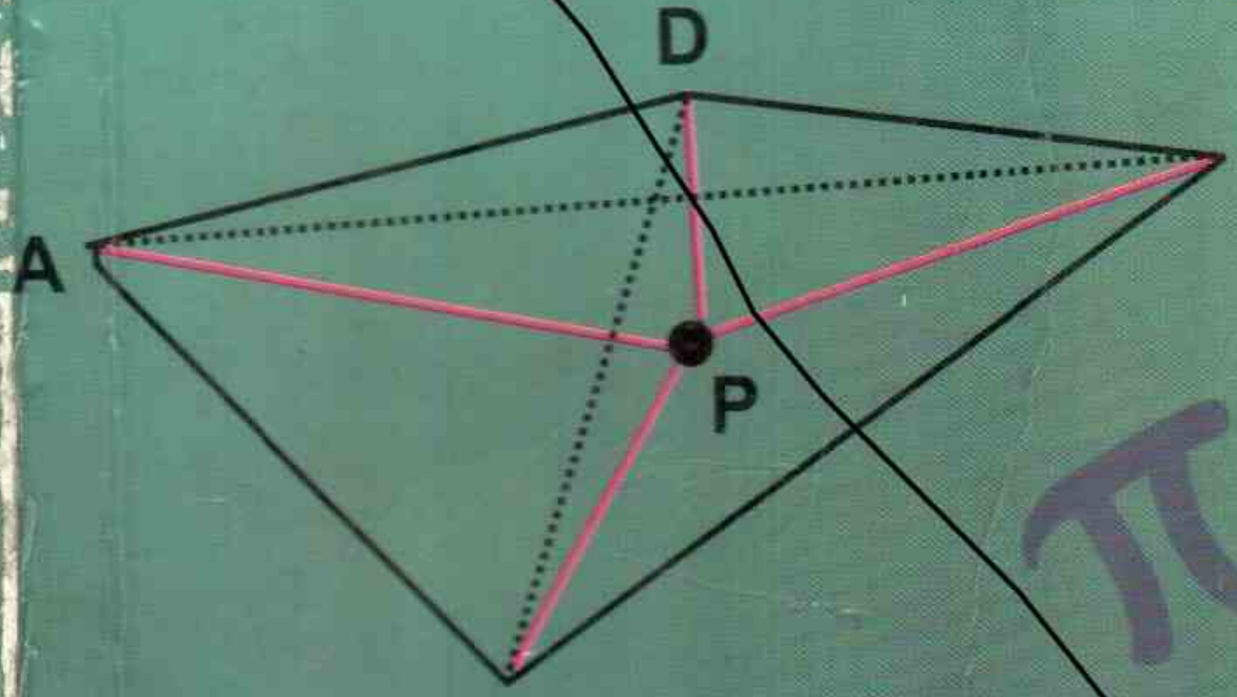
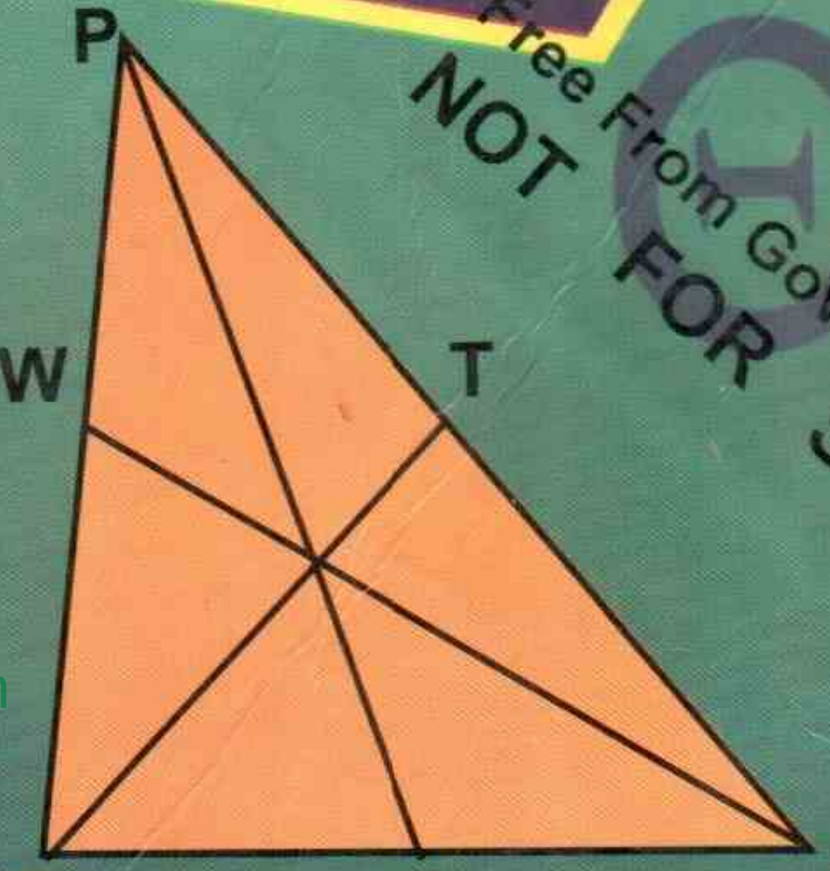
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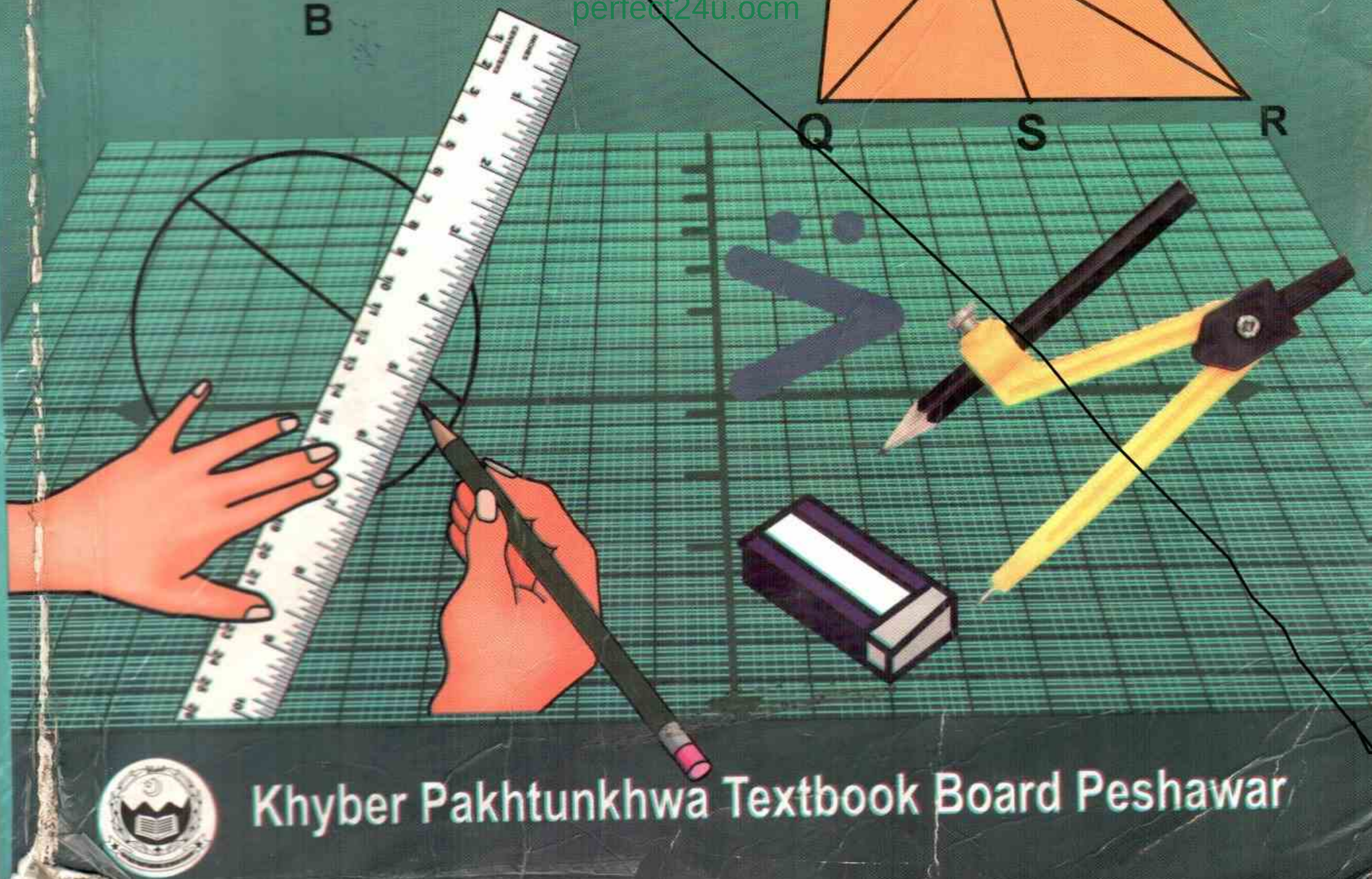
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UNIT 01

MATRICES

After reading this unit, the students will be able to:

- Define a matrix with real entries and relate its rectangular layout (formation) with real life,
- Define rows and columns of a matrix,
- Define the order of a matrix,
- Define equality of two matrices.
- Define and identify row matrix, column matrix, rectangular matrix, square matrix, zero/null matrix, identity matrix, scalar matrix, diagonal matrix, transpose of a matrix, symmetric and skew-symmetric matrices.
- Know whether the given matrices are conformable for addition/subtraction.
- Add and subtract matrices.
- Multiply a matrix by a real number.
- Verify commutative and associative laws under addition.
- Define additive identity of a matrix.
- Find additive inverse of a matrix.
- Know whether the given matrices are conformable for multiplication.
- Multiply two (or three) matrices.
- Verify associative law under multiplication.
- Verify distributive laws.
- Show with the help of an example that commutative law under multiplication does not hold in general (i.e. $AB \neq BA$).
- Define multiplicative identity of a matrix.
- Verify the result $(AB)^t = B^t A^t$
- Define the determinant of a square matrix.
- Evaluate determinant of a matrix.
- Define singular and non-singular matrices.
- Define adjoint of a matrix.
- Find multiplicative inverse of a non-singular matrix A and verify that $AA^{-1} = I = A^{-1}A$ where I is the identity matrix.
- Use adjoint method to calculate inverse of a non-singular matrix.
- Verify the result $(AB)^{-1} = B^{-1}A^{-1}$.
- Solve a system of two linear equations and related real life problems in two unknowns using
  - Matrix inversion method,
  - Cramer's rule.

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1.1 INTRODUCTION TO MATRICES

The word 'matrices' is plural of the word 'matrix'. The term matrix was first introduced by the mathematician Arther Cayley in 1860. The knowledge of matrices is necessary in various areas of Mathematics. It has widely been used in the fields of pure mathematics, statistics, engineering and physical and social sciences. Thus, matrix theory finds an important place in modern age and has become an integral part of mathematics.

Matrices make presentation of numbers clearer and make calculations easier. The following table presents the information about a series of hockey matches played between Pakistan and India.

	Played	Won	Drawn	Lost
Pakistan	8	4	1	3
India	8	3	1	4

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The information is readily available when presented in this way. For example, if we want to know how many matches India lost against Pakistan, we go along the row 'India' and column 'Lost' and find that it is 4. Similarly, if we want to know how many matches Pakistan drew with India, we go along the row 'Pakistan' and column 'D' and find that it is 1. As long as we remember what each number represents, we could remove the row and column headings and write just the numbers, enclosing them in square brackets or parenthesis such as

$$\begin{bmatrix} 8 & 4 & 1 & 3 \\ 8 & 3 & 1 & 4 \end{bmatrix}$$

Let us denote this by A.

Thus $A = \begin{bmatrix} 8 & 4 & 1 & 3 \\ 8 & 3 & 1 & 4 \end{bmatrix}$ is a matrix.

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1.1.1 Matrix

A matrix is a rectangular array (arrangements) of real numbers enclosed in square brackets. Each number in a matrix is called an element or entry of the matrix. For example $\begin{bmatrix} 2 & 3 \\ 6 & 5 \end{bmatrix}$, $\begin{bmatrix} -1 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix}$ and $\begin{bmatrix} 3 & 4 & 2 \\ -1 & 1 & -2 \\ -4 & -3 & -3 \end{bmatrix}$ are all matrices. In the matrix $\begin{bmatrix} 2 & 3 \\ 6 & 5 \end{bmatrix}$ the number 2, 3, 6, 5 are the elements or entries of the matrix. Similarly in the matrix $\begin{bmatrix} 3 & 4 & 2 \\ -1 & 1 & -2 \\ -4 & -3 & -3 \end{bmatrix}$, the numbers 3, 4, 2, −1, 1, −2, −4, −3, −5 are the elements or entries of the given matrix. Matrices are frequently denoted by capital letters such as A, B, C and so on. Thus we can write

$$A=\begin{bmatrix} 2 & 3 \\ 6 & 5 \end{bmatrix}, \quad B=\begin{bmatrix} -1 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix}, \quad C=\begin{bmatrix} 3 & 4 & 2 \\ -1 & 1 & -2 \\ -4 & -3 & -3 \end{bmatrix}$$

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1.1.2 Rows and Columns of Matrix

The rows of a matrix run horizontally, and the columns of a matrix run vertically. To make the idea more clear, let us consider the following matrix.

$$A=\begin{bmatrix} 4 & 3 & -2 \\ 1 & 5 & 2 \\ 3 & 1 & 1 \end{bmatrix}$$

The numbers 4, 3, −2 are arranged horizontally, so they continue the first row.

The numbers 1, 5, 2 are arranged horizontally, so they continue second row.

The numbers 3, 1, 1 are arranged horizontally, so they continue third row.

Again consider the same matrix, $A=\begin{bmatrix} 4 & 3 & -2 \\ 1 & 5 & 2 \\ 3 & 1 & 1 \end{bmatrix}$

...rs 4, 1, 3 run vertically, ...stitute the first column.

The numbers 3, 5, 1 run vertically, so they constitute the second column.

The numbers −2, 2, 1 run vertically, so they constitute the third column.

...rices)

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1.1.3 Order (or dimension) of a Matrix

A matrix with m rows and n columns has order $m \times n$ (read "m by n").

If a matrix has order $m \times n$ then m represents the number of rows and n represents the number of columns.

$m \times n$

number of rows ← → number of columns

For example, consider $A = \begin{bmatrix} -3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ -4 & 3 \end{bmatrix}$

The order of the matrix A is 2-by-3 or A is a 2-by-3 matrix. Similarly the order of the matrix B is 2-by-2 or B is a 2-by-2 matrix.

Do You Know?

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Matrix

$\begin{bmatrix} 6 & 10 \\ 5 & 3 \\ 5 & 2 \end{bmatrix}$

Row 1 → Row 2 → Row 3 →

An element of the matrix.

Column 1 Column 2

Dimension of this matrix is 3 x 2.

Remember

Sometimes we will prefer to write the order of a matrix as m-by-n or sometimes ($m \times n$).

Note

Order of a matrix $m \times n$ does not mean to multiply m & n.

Tid-Bit

Order of a matrix is also called dimension or size.

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Example 1

Write the number of rows and column of the following matrices and hence mention their orders.

i) $A=\begin{bmatrix} p & q \\ r & s \end{bmatrix}$ ii) $B=\begin{bmatrix} 3 & 4 & 7 \\ 5 & 6 & 8 \end{bmatrix}$

$\begin{bmatrix} 2 & -5 & 10 \\ -4 & 19 & 4 \end{bmatrix}$

Solution:

i) Given $A=\begin{bmatrix} p & q \\ r & s \end{bmatrix}$, we see that:

A has two rows and two columns, so order of A is 2-by-2, sometimes we will prefer to write the order of A is 2 by 2. Thus A is 2 by 2 matrix.

ii) Given $B=\begin{bmatrix} 3 & 4 & 7 \\ 5 & 6 & 8 \end{bmatrix}$

B has two rows and three columns. The order of B is 2-by-3, or 2 by 3 i.e. B is a 2 by 3 matrix.

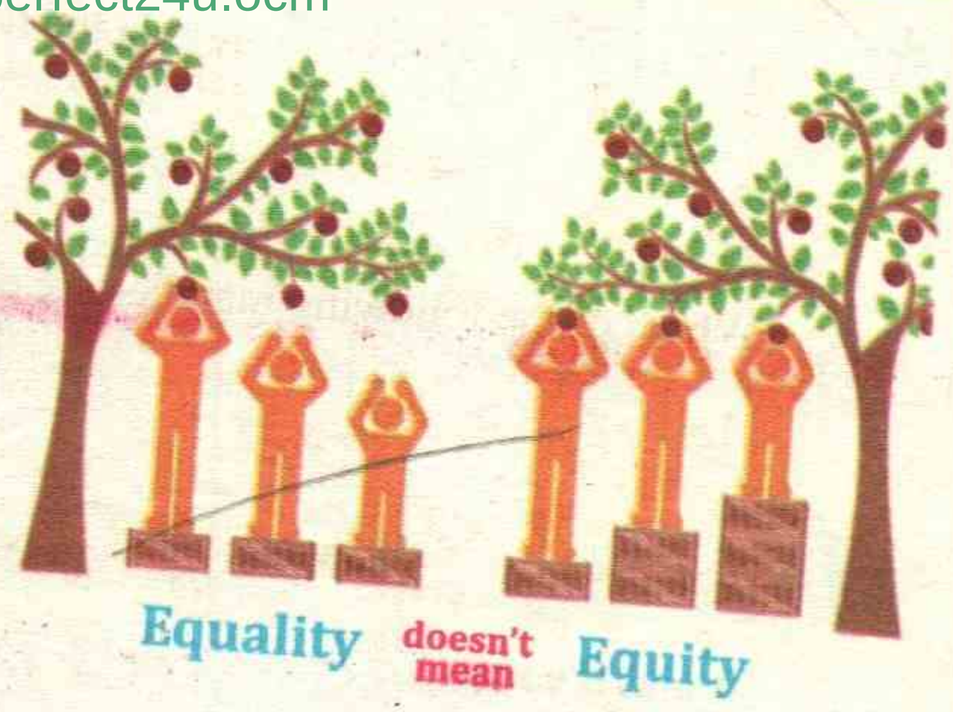
1.1.4 Equality of two Matrices

Two given A and B are said to be equal if:

(i) Both the matrices are of the same order i.e. they respectively have the same number of rows and columns.

(ii) The elements in the corresponding positions in A and B are equal.

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For example, the matrices

$\begin{bmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1+1 & \frac{12}{4} \\ 4+2 & \frac{10}{2} & \frac{8}{2} \end{bmatrix}$ are equal whereas the matrices

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \end{bmatrix}$ are not equal.

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Exercise 1.1

1. Which of the following are square and which are rectangular matrices?

i) $A=\begin{bmatrix}2 & 3\\ 0 & 5\end{bmatrix}$ Square

ii) $B=\begin{bmatrix}6 & 3 & -1\\ 1 & 5 & 2\end{bmatrix}$

iii) $C=\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}$ Square

iv) $B=[-5]$ square

v) $E=[-3 \quad 4]$

vi) $E=\begin{bmatrix}-1\\ 7\end{bmatrix}$

2. List the order of the following matrices.

i) $A=\begin{bmatrix}1 & 2 & -1\\ 3 & 4 & 2\end{bmatrix}$ 2by3

ii) $B=[-4]$ 1by1

iii) $C=\begin{bmatrix}2 & 3 & -1\\ 1 & 2 & 5\end{bmatrix}$ 2 by 3

iv) $F=\begin{bmatrix}2 & 1\\ 3 & 2\\ 4 & -1\end{bmatrix}$ 3by2

v) $E=[3 \quad 2]$ 1

vi) $D=\begin{bmatrix}1 & 2 & 3\\ 6 & 5 & 9\\ 0 & 0 & 0\end{bmatrix}$

3. If $A=\begin{bmatrix}3 & 2 & -4\\ -2 & 5 & 0\\ 2 & 1 & 5\\ -3 & 4 & 6\end{bmatrix}$, give the following elements.

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(i) a_{12} (ii) a_{23} (iii) a_{32} (iv) a_{43} (v) a_{13} (vi) a_{33}

4. Which of the following matrices are equal.

$A=\begin{bmatrix}2 & 5\\ 1 & 3\end{bmatrix}$, $B=\begin{bmatrix}2 & 5\\ 4 & 3\end{bmatrix}$, $C=\begin{bmatrix}1+1 & 3+2\\ 4 & 2+1\end{bmatrix}$, $D=\begin{bmatrix}2 & 4+1\\ 1 & 3\end{bmatrix}$

5. Let $A=\begin{bmatrix}2 & -3\\ u & o\end{bmatrix}$ and $B=\begin{bmatrix}v & -3\\ 5 & w\end{bmatrix}$, for what values of u, v and w are A and B equal.

6. If $\begin{bmatrix}x+3 & z+4 & 2y-7\\ -6 & a-1 & 0\\ b-3 & -21 & 0\end{bmatrix}=\begin{bmatrix}0 & 6 & 3y-2\\ -6 & -3 & 2c+2\\ 2b+4 & -21 & 0\end{bmatrix}$, find the values of a, b, c, x, y and z.

7. Solve the following equation for a, b, c, d.

$$\begin{bmatrix}a+b & b+2c\\ 2c+d & 2a-d\end{bmatrix}=\begin{bmatrix}-1 & 4\\ 8 & 0\end{bmatrix}$$

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1.2 TYPES OF MATRICES

a) Row-matrix

A matrix which has just one and only row in it is called a Row-matrix. For example, the matrices $[a,b]$, $[1 \quad 3 \quad 4]$, $[2 \quad 4 \quad 6 \quad 8]$ are all row-matrices.

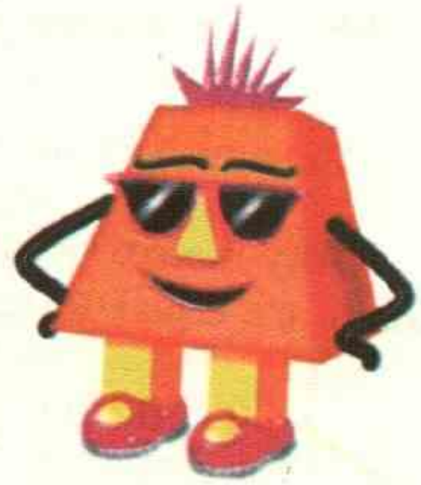
b) Column-matrix

A matrix which has just one and only one column in it is called a column-matrix.

For example,

$$\begin{bmatrix} a \\ b \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \end{bmatrix}$$

are all column matrices.



c) Square-matrix

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A matrix in which number of rows and columns are equal is called a square matrix. For example the matrix $\begin{bmatrix} l & m \\ n & p \end{bmatrix}$ has two rows and two columns, so it is a square matrix. The order of the matrix $\begin{bmatrix} l & m \\ n & p \end{bmatrix}$ is 2-by-2 but for brevity we write the above matrix as a square matrix of order 2 or 2-square matrix. Similarly the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ -1 & -4 & 2 \end{bmatrix}$ is a square matrix of order 3 or 3-square matrix. As a special case the matrix consisting of a single element say $[3]$ is also a square matrix of order 1 or a 1-square matrix.



Remember

$[1 \quad 3 \quad 4]$ **Row matrix**

$\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$ **Column matrix**

$\begin{bmatrix} 1 & 3 & 4 \\ 3 & 1 & 4 \\ 4 & 1 & 3 \end{bmatrix}$ **Square matrix**

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d) Rectangular-matrix

A matrix whose number of rows and number of columns are not equal is called rectangular matrix. Thus the matrix $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$ is a rectangular matrix because its number of rows $=2=m$ and its number of columns $=3=n$ obviously $2 \neq 3$. Similarly the matrix $\begin{bmatrix} 1 & 4 \\ 3 & 2 \\ 5 & -3 \end{bmatrix}$ is also a rectangular matrix with its order 3-by-2 i.e. number of rows and number of columns are not equal.

e) Zero matrix or Null matrix

Any matrix (weather it is rectangular or square) of which all the elements (entries) are equal to zero is said to be a Zero matrix or a Null matrix. Thus $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are some examples of zero matrices or null matrices.

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The following points are very important to remember.

i) A zero or null matrix is not necessarily a square matrix.

ii) A null matrix is generally donated by O.

iii) Sometimes we prefer to put the order of a null matrix as a proper subscript with the general symbol O.

Thus,

$O_{2\times2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a null-matrix of order 2.

$O_{2\times3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is a null-matrix of order 2-by-3.



The role of the Zero matrix in matrix manipulation is very similar to that of Zero in ordinary algebra. Provided that they are compatible for multiplication (compatibility of multiplication in matrices will be discussed later on in this chapter), the product of any matrix and the Zero matrix is a zero matrix. When working with matrices involving zero matrices, we choose a zero matrix of such an order which is suitable for solving the problem in hand.

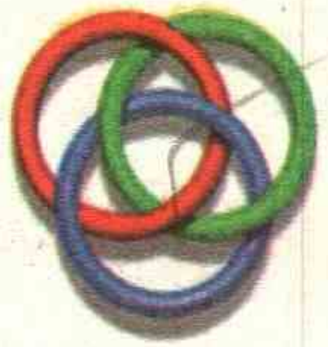
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f) Diagonal matrix

A square matrix in which all elements are zero except the diagonal elements is known as diagonal matrix. For example

$A = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ is a diagonal matrix of order 2.



$B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -5 \end{bmatrix}$ is a diagonal matrix of order 3.



Remember

Main diagonal

Start at the top left and goes to the bottom right in the matrix.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

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g) Scalar matrix

A square matrix in which all the elements lying on the main diagonal of the matrix are equal and the remaining elements off the main diagonal are all zero is called scalar matrix.

Equivalently scalar matrix can also be defined as under:

A diagonal matrix in which all the diagonal elements are equal is said to be a scalar matrix.

For instance

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$ are scalar matrices of order 2 and 3 respectively.

Tid-Bit

Scalar matrix is a special case of a diagonal matrix.

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Note

Every scalar matrix is a diagonal matrix but every diagonal matrix is not necessarily a scalar matrix. For example,

$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ is a diagonal matrix,

because all elements of the matrix are zero except the elements of the main diagonal but the above matrix is not a scalar matrix because elements lying on the main diagonal are not all equal.

h) Identity matrix

The identity matrix (sometimes called the unit matrix) is a square matrix and is denoted by I. It is characterized by the fact that all elements on its main diagonal are 1's whereas all other elements are zero. Sometime it is useful to write the order of the identity matrix as a subscript of I. thus

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$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the identity matrix of order 2.

$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the identity matrix of order 3.



Identity Matrix

When there is no danger of ambiguity the identity matrices I_2 & I_3 are simply denoted by I.

i) Transpose of a matrix

Suppose we have a matrix A of any given order. The matrix obtained by interchanging mutually the rows and columns in A is called the transpose of A and is denoted by A^t. For example

If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ then $A^t = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

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j) Symmetric matrix

A square matrix A is said to be symmetric if the transpose of A denoted by A^t is again equal to A i.e. $A = A^t$. For example

If $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ then $A^t = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$,

We see that $A = A^t$, so $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ is symmetric.

Similarly if $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ then $B^t = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} = B$

i.e. $B = B^t$, so $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ is a symmetric matrix.

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k) Skew-Symmetric matrix

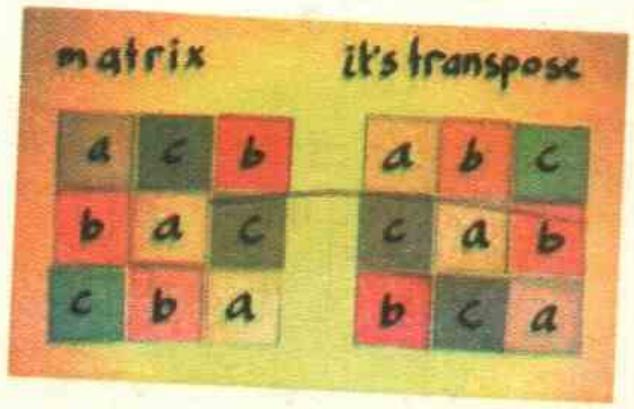
A given square matrix A is said to be Skew-Symmetric if $A^t = -A$.
For example:

If $A = \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix}$ so that $A^t = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}$

then,

$$A^t = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix} = -A$$

i.e. $A^t = -A$



so $A = \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix}$ is skew symmetric matrix.

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Exercise 1.2

1. Write transpose of the following matrices:

i. $P=\begin{bmatrix}1 & 2\\ 3 & 1\end{bmatrix}$

ii. $Q=\begin{bmatrix}l & m\\ n & p\end{bmatrix}$

iii. $R=[6]$

iv. $S=\begin{bmatrix}-5 & 1\\ -2 & 1\\ 4 & 4\end{bmatrix}$

v. $T=\begin{bmatrix}6 & 7 & 8\\ 13 & 1 & 3\\ 2 & 4 & 5\end{bmatrix}$

2. Which of the following matrices are transpose of each other?

i. $A=\begin{bmatrix}a_1 & a_2\\ b_1 & b_2\end{bmatrix}$

ii. $B=\begin{bmatrix}a_1 & b_1\\ a_2 & b_2\end{bmatrix}$

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iii. $C=\begin{bmatrix}-3 & 1 & -1\\ 4 & 2 & 7\end{bmatrix}$

iv. $D=\begin{bmatrix}-3 & 4\\ 1 & 2\\ -1 & 7\end{bmatrix}$

3. Which of the following matrices are symmetric?

i. $A=\begin{bmatrix}5 & -7\\ -1 & 5\end{bmatrix}$

ii. $B=\begin{bmatrix}-1 & 2\\ 2 & 3\end{bmatrix}$

iii. $C=\begin{bmatrix}3 & 4\\ 5 & 6\end{bmatrix}$

iv. $D=\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix}$

4. Which of the following matrices are skew-symmetric?

i. $A=\begin{bmatrix}0 & 4\\ -4 & 0\end{bmatrix}$

ii. $B=\begin{bmatrix}0 & -5\\ 5 & 0\end{bmatrix}$

iii. $C=\begin{bmatrix}0 & 7\\ 7 & 0\end{bmatrix}$

iv. $D=\begin{bmatrix}0 & 3 & 2\\ -3 & 0 & 1\\ -2 & -1 & 0\end{bmatrix}$

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1.3 ADDITION AND SUBTRACTION OF MATRICES

1.3.1 Conformability for Addition/subtraction of matrices

Let we have two given matrices. These two matrices may be added or subtracted if & only if they have the same orders.

Example 2

i) Let $A=\begin{bmatrix}1 & 3\\ -3 & 2\end{bmatrix}$ and $B=\begin{bmatrix}4 & 7\\ 10 & 13\end{bmatrix}$ then these matrices are conformable for addition and subtraction because both are 2-by-2 matrices, that is both matrices are of the same order.

ii) Let $C=\begin{bmatrix}7 & 3 & 1\\ -2 & 0 & 5\end{bmatrix}$ and $D=\begin{bmatrix}9 & 5 & 13\\ 10 & -1 & 1\\ 2 & 0 & 3\end{bmatrix}$ then C & D are not conformable for addition and subtraction because their orders are not same.

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1.3.2 Addition and subtraction of matrices

a) Addition of two matrices

Suppose A & B are conformable for addition. Their sum A + B is obtained by adding corresponding elements of the matrices A & B.

Let $A=\begin{bmatrix}3 & 8\\ 4 & 6\end{bmatrix}$ and $B=\begin{bmatrix}4 & 0\\ 1 & -9\end{bmatrix}$

We see that A & B both are 2-by-2 matrices, so these are conformable for addition.

$3+4=7$

$$\begin{bmatrix}3 & 8\\ 4 & 6\end{bmatrix}+\begin{bmatrix}4 & 0\\ 1 & -9\end{bmatrix}=\begin{bmatrix}7 & 8\\ 5 & -3\end{bmatrix}$$

$$A+B=\begin{bmatrix}3 & 8\\ 4 & 6\end{bmatrix}+\begin{bmatrix}4 & 0\\ 1 & -9\end{bmatrix}=\begin{bmatrix}3+4 & 8+0\\ 4+1 & 6-9\end{bmatrix}=\begin{bmatrix}7 & 8\\ 5 & -3\end{bmatrix}$$

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b) Subtraction of two matrices

Let A & B are two matrices which are of the same order and so these are conformable for subtraction. Their difference A-B is obtained by subtracting each element of B form the corresponding element of A. For example

Let $A=\begin{bmatrix}3 & 8\\ 4 & 6\end{bmatrix}$ and $B=\begin{bmatrix}4 & 0\\ 1 & -9\end{bmatrix}$

Both the matrices are of the same order, so these are conformable for subtraction.

$3-4=-1$

$$\begin{bmatrix}3 & 8\\ 4 & 6\end{bmatrix}-\begin{bmatrix}4 & 0\\ 1 & -9\end{bmatrix}=\begin{bmatrix}-1 & 8\\ 3 & 15\end{bmatrix}$$

$$A-B=\begin{bmatrix}3 & 8\\ 4 & 6\end{bmatrix}-\begin{bmatrix}4 & 0\\ 1 & -9\end{bmatrix}$$

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$$=\begin{bmatrix}3-4 & 8-0\\ 4-1 & 6-(-9)\end{bmatrix}=\begin{bmatrix}3-4 & 8-0\\ 4-1 & 6+9)\end{bmatrix}$$

$$=\begin{bmatrix}-1 & 8\\ 3 & 15\end{bmatrix}$$

1.3.3 Multiplication of a matrix by a real number

Let A be any matrix and k is any real number. The matrix obtained by multiplying each element of A by the real number k is called the scalar multiplication of A by k and it is denoted by kA.

$2\times4=8$

$$2\begin{bmatrix}4 & 0\\ 1 & -9\end{bmatrix}=\begin{bmatrix}8 & 2\\ 2 & -18\end{bmatrix}$$

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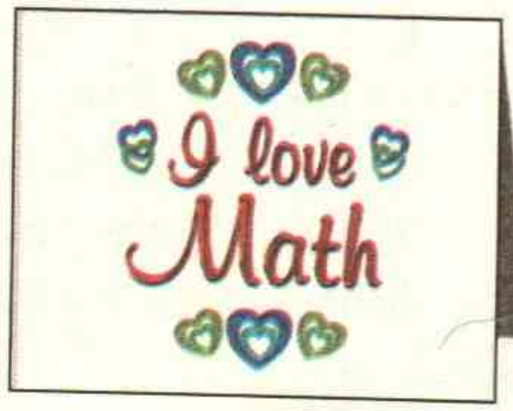
Example 3

i. If $A=\begin{bmatrix} 6 & 2 \\ -3 & 1 \end{bmatrix}$

then $3A=\begin{bmatrix} 6\times3 & 2\times3 \\ -3\times3 & 1\times3 \end{bmatrix}=\begin{bmatrix} 18 & 6 \\ -9 & 3 \end{bmatrix}$

ii. If $B=\begin{bmatrix} 5 & 4 & 7 \\ -3 & a & b \end{bmatrix}$

then $7B=\begin{bmatrix} 7\times5 & 7\times4 & 7\times7 \\ 7\times(-3) & 7a & 7b \end{bmatrix}=\begin{bmatrix} 35 & 28 & 49 \\ -21 & 7a & 7b \end{bmatrix}$



1.3.4 Commutative and associative laws under addition

a) Commutative law under addition

Let A & B are two given matrices of the same order then A+B=B+A. This is called commutative law of matrices under addition.

Now we give some examples in support of the above law.

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Example 4

Let $A=\begin{bmatrix} 2 & 5 \\ 4 & 7 \end{bmatrix}$ and $B=\begin{bmatrix} -2 & 1 \\ -3 & 6 \end{bmatrix}$, then A & B are of the same order and hence these matrices can be added.

$$A+B=\begin{bmatrix} 2 & 5 \\ 4 & 7 \end{bmatrix}+\begin{bmatrix} -2 & 1 \\ -3 & 6 \end{bmatrix}=\begin{bmatrix} 2+(-2) & 5+1 \\ 4+(-3) & 7+6 \end{bmatrix}=\begin{bmatrix} 0 & 6 \\ 1 & 13 \end{bmatrix}$$

i.e. $A+B=\begin{bmatrix} 0 & 6 \\ 1 & 13 \end{bmatrix}$

Again $B+A=\begin{bmatrix} -2 & 1 \\ -3 & 6 \end{bmatrix}+\begin{bmatrix} 2 & 5 \\ 4 & 7 \end{bmatrix}=\begin{bmatrix} -2+2 & 1+5 \\ -3+4 & 6+7 \end{bmatrix}=\begin{bmatrix} 0 & 6 \\ 1 & 13 \end{bmatrix}$

i.e. $B+A=\begin{bmatrix} 0 & 6 \\ 1 & 13 \end{bmatrix}$

This proves that $A+B=B+A$.

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(b) Associative law for matrix addition

Let A, B & C are three matrices of the same order, that is, these are conformable for addition.

Then $A+(B+C)=(A+B)+C$

This is called associative law for matrix addition.

Now we give an example in support of the above law.

Example 5

Let $A=\begin{bmatrix}-1 & 2\\ 4 & -3\end{bmatrix}$, $B=\begin{bmatrix}4 & -5\\ 6 & 7\end{bmatrix}$ and $C=\begin{bmatrix}3 & -2\\ 1 & 0\end{bmatrix}$,

then all these matrices are of the same order, so these are conformable for addition.

Now

$$A+(B+C)=\begin{bmatrix}-1 & 2\\ 4 & -3\end{bmatrix}+\left(\begin{bmatrix}4 & -5\\ 6 & 7\end{bmatrix}+\begin{bmatrix}3 & -2\\ 1 & 0\end{bmatrix}\right)$$

$$=\begin{bmatrix}-1 & 2\\ 4 & -3\end{bmatrix}+\begin{bmatrix}4+3 & -5-2\\ 6+1 & 7+0\end{bmatrix}=\begin{bmatrix}-1 & 2\\ 4 & -3\end{bmatrix}+\begin{bmatrix}7 & -7\\ 7 & 7\end{bmatrix}$$

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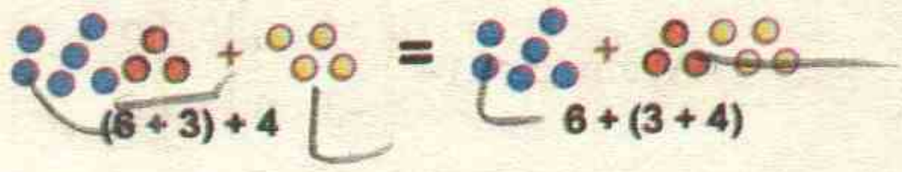
$$=\begin{bmatrix}-1+7 & 2-7\\ 4+7 & -3+7\end{bmatrix} \quad \therefore A+(B+C)=\begin{bmatrix}6 & -5\\ 11 & 4\end{bmatrix}$$

Again $(A+B)+C=\left(\begin{bmatrix}-1 & 2\\ 4 & -3\end{bmatrix}+\begin{bmatrix}4 & -5\\ 6 & 7\end{bmatrix}\right)+\begin{bmatrix}3 & -2\\ 1 & 0\end{bmatrix}$

$$=\begin{bmatrix}-1+4 & 2+(-5)\\ 4+6 & -3+7\end{bmatrix}+\begin{bmatrix}3 & -2\\ 1 & 0\end{bmatrix}$$

$$=\begin{bmatrix}3 & -3\\ 10 & 4\end{bmatrix}+\begin{bmatrix}3 & -2\\ 1 & 0\end{bmatrix}$$

$$=\begin{bmatrix}3+3 & -3-2\\ 10+1 & 4+0\end{bmatrix}$$



$$=\begin{bmatrix}6 & -5\\ 11 & 4\end{bmatrix}$$

$\therefore$ $A+(B+C)=(A+B)+C$

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1.3.5 Additive identity of matrices

In ordinary arithmetic 0 (zero) is called additive identity because when 0 is added to any real number or when any real number is added to 0, the value of the number does not change. For example

$0+5=5+0=5$, $\frac{1}{3}+0=0+\frac{1}{3}=\frac{1}{3}$ and $\sqrt{7}+0=0+\sqrt{7}=\sqrt{7}$ & so on.

In theory of matrices, zero matrix (or Null-matrix) denoted by O performs the same function as 0 in ordinary arithmetic because when O is added to any matrix A or if A is added to the matrix O, the matrix A does not change provided O & A are conformable for addition. In such a situation A+O = O+A = A.

Definition:

In the theory of matrix a Zero or (Null) matrix of some specific order serves as the additive identity for the matrices of the same specific order.

Here we give some examples to make the idea clearer.

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Example 6

If $A=\begin{bmatrix}3 & -1\\ 2 & 5\end{bmatrix}$ then the additive identity for such a case is $O=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$ because A & O are conformable for addition and

$$A+O=\begin{bmatrix}3 & -1\\ 2 & 5\end{bmatrix}+\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=\begin{bmatrix}3+0 & -1+0\\ 2+0 & 5+0\end{bmatrix}=\begin{bmatrix}3 & -1\\ 2 & 5\end{bmatrix}=A$$

$$\text{Also } O+A=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}+\begin{bmatrix}3 & -1\\ 2 & 5\end{bmatrix}=\begin{bmatrix}0+3 & 0-1\\ 0+2 & 0+5\end{bmatrix}=\begin{bmatrix}3 & -1\\ 2 & 5\end{bmatrix}=A$$

i.e. $A+O=O+A=A$

$O=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$ is additive identity of $A=\begin{bmatrix}3 & -1\\ 2 & 5\end{bmatrix}$. But at the same time $O=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$ is additive identity for all 2-square matrices.

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1.3.6 Additive inverse of a matrix

Let A and B be two matrices of the same order. If A & B are so related together that $A+B=O=B+A$ where O is the identity matrix of the same order as that of order of A or B, then B is called the additive inverse of A and likewise A is called the additive inverse of B. In other words $B=-A$ which implies that $A=-B$ i.e. A & B are additive inverses of each other.

Example 7

Prove that $P=\begin{bmatrix}3 & 2 & -1\\ -2 & 4 & 6\end{bmatrix}$, $Q=\begin{bmatrix}-3 & -2 & 1\\ 2 & -4 & -6\end{bmatrix}$ are additive inverse of each other.

Solution:

Since

$$P+Q=\begin{bmatrix}3 & 2 & -1\\ -2 & 4 & 6\end{bmatrix}+\begin{bmatrix}-3 & -2 & 1\\ 2 & -4 & -6\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}=O$$

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$$Q+P=\begin{bmatrix}-3 & -2 & 1\\ 2 & -4 & -6\end{bmatrix}+\begin{bmatrix}3 & 2 & -1\\ -2 & 4 & 6\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}=O$$

Hence P and Q are additive inverses of each other.

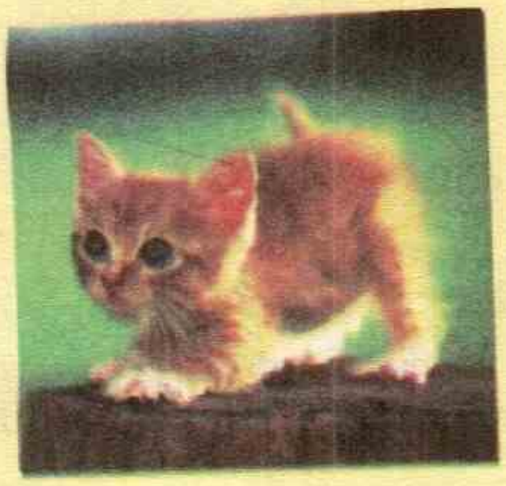
How?

How computers represent matrices.

[[1, 2, 3], [4, 5, 6]]

$$\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix}$$

How people represent matrices.



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Exercise 1.3

1. Let A & B be 2-by-3 matrices and let C & D be 2-square matrices. Which of the following matrix operations are defined. For those which are defined, give the dimension of the resulting matrix.

   i) A+B ii) B+D iii) 3A-2C iv) 7C+2D

2. Multiply the following matrices by the real numbers as indicated.

   i) Multiply $A=\begin{bmatrix}1\\2\\3\end{bmatrix}$ by 2 ii) Multiply $B=\begin{bmatrix}a & b & c\\ d & e & f\end{bmatrix}$ by $p\in R$

3. Find a matrix X such that $4X=\begin{bmatrix}1 & 2 & 1\\ 4 & 2 & 3\\ -1 & 9 & 7\end{bmatrix}$.

4. If $A=\begin{bmatrix}1 & 2\\ 3 & 4\\ 5 & 6\end{bmatrix}$ and $B=\begin{bmatrix}-3 & -2\\ 1 & -5\\ 4 & 3\end{bmatrix}$, then find $3A-B$.

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5. Given $A=\begin{bmatrix}1 & 2 & -3\\ 5 & 0 & 2\\ 1 & -1 & 1\end{bmatrix}$ and $B=\begin{bmatrix}3 & -1 & 2\\ 4 & 2 & 5\\ 2 & 3 & 0\end{bmatrix}$, find the matrix C such that $A+2B=C$.

6. If $A=\begin{bmatrix}2 & -2\\ 4 & 2\\ -5 & 1\end{bmatrix}$ and $B=\begin{bmatrix}8 & 0\\ 4 & -2\\ 3 & 6\end{bmatrix}$, then find the matrix X such that $2A+3X=5B$.

7. Find x,y,z and w if

$$3\begin{bmatrix}x & y\\ z & w\end{bmatrix}=\begin{bmatrix}x & 6\\ -1 & 2w\end{bmatrix}+\begin{bmatrix}4 & x+y\\ 3+w & 3\end{bmatrix}.$$

8. Find X and Y if $X+Y=\begin{bmatrix}5 & 2\\ 0 & 9\end{bmatrix}$ and $X-Y=\begin{bmatrix}3 & 6\\ 0 & -1\end{bmatrix}$.

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9. Let $A=\begin{bmatrix}2 & -3\\ 4 & 5\end{bmatrix}$, $B=\begin{bmatrix}2 & 5\\ -1 & 3\end{bmatrix}$ and $C=\begin{bmatrix}3 & -1\\ 0 & 4\end{bmatrix}$. If $c=2$ and $d=-4$ then verify that:

i) $(c+d)A=cA+dA$ ii) $c(A+B)=cA+cB$ iii) $cd(A)=c(dA)$

10. Let $A=\begin{bmatrix}-1 & 2 & 3\\ 4 & 2 & 0\\ -3 & 2 & 5\end{bmatrix}$, $B=\begin{bmatrix}3 & -1 & 2\\ -5 & 3 & 4\\ -3 & -4 & 0\end{bmatrix}$ and $C=\begin{bmatrix}2 & -3 & 6\\ 0 & 4 & -1\\ -5 & 1 & 3\end{bmatrix}$.

Compute the following if possible.

i) $A+2B$ ii) $3A-4B$ iii) $(A+B)-C$ iv) $A+(B+C)$

11. Prove that in the following matrices commutative law of addition holds.

i) $A=\begin{bmatrix}7 & 1\\ 2 & 4\end{bmatrix}$, $B=\begin{bmatrix}1 & 1\\ 2 & 2\end{bmatrix}$

ii) $C=\begin{bmatrix}-3 & 4 & -5\\ 2 & 3 & 1\end{bmatrix}$, $D=\begin{bmatrix}-3 & -4 & 5\\ 1 & 2 & 3\end{bmatrix}$

12. Verify $A+(B+C)=(A+B)+C$ for the following matrices.

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i) $A=\begin{bmatrix}2 & -3\\ 4 & 1\end{bmatrix}$, $B=\begin{bmatrix}5 & -2\\ 3 & 6\end{bmatrix}$, $C=\begin{bmatrix}1 & 7\\ -6 & -3\end{bmatrix}$

ii) $A=\begin{bmatrix}a & b & c\\ 3 & 4 & 5\end{bmatrix}$, $B=\begin{bmatrix}1 & 2 & 3\\ -2 & 1 & 4\end{bmatrix}$, $C=\begin{bmatrix}2 & 1 & -1\\ 3 & 1 & -2\end{bmatrix}$

13. Find the additive inverse of the following matrices.

i) $A=\begin{bmatrix}3 & 4\\ 6 & 2\end{bmatrix}$ ii) $B=\begin{bmatrix}a & -a & b\\ -c & a & -b\\ l & m & n\end{bmatrix}$

14. Show that the following matrices are additive inverses of each other.

i) $A=[1 \quad -2 \quad 3]$, $B=[-1 \quad 2 \quad -3]$

ii) $C=\begin{bmatrix}a & -b\\ -c & d\end{bmatrix}$, $D=\begin{bmatrix}-a & b\\ c & -d\end{bmatrix}$

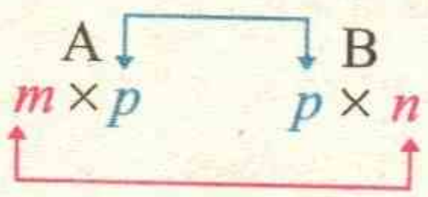
iii) $E=\begin{bmatrix}1 & -2 & -4\\ 2 & 1 & 3\\ -3 & 4 & -2\end{bmatrix}$, $F=\begin{bmatrix}-1 & 2 & 4\\ -2 & -1 & -3\\ 3 & -4 & 2\end{bmatrix}$

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1.4 MULTIPLICATION OF MATRICES

1.4.1 Conformability for multiplication of matrices

Two matrices A and B are said to be conformable for multiplication AB, only when the number of columns of matrix A is equal to the number of rows of matrix B. The product AB, which is not the same as the product BA, is conformable for multiplication only if A is an m × p and matrix B is a p × n matrix. The product AB will then be an m x n matrix. A convenient way to determine whether two matrices are conformable for multiplication, and to determine the order of the resultant matrix, is to write the orders of the two matrices side-by-side as shown below.



This shows that A and B are conformable for multiplication as p = p . It also indicates that the order of the product AB is $m \times n$.
Thus $A_{m\times p} \times B_{p\times n} = AB_{m\times n}$.

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Here, B and A are not conformable for multiplication since $n \neq m$. Thus product BA is not defined.

For matrix multiplication, the operation is row by column. Thus, to obtain the product AB, we multiply each element of a row of A by the corresponding element of a column of B and then we add these products.

Example 8

Suppose A is a 3-by-4 matrix, B is a 4-by-2 matrix and C is a 4-by-3 matrix. Then AB is defined and it is 3-by-2 matrix. AC is defined and it is 3-by-3 i.e. 3-square matrix CA is also defined and it is a 4-by-4 matrix while BA, CB and BA are undefined.

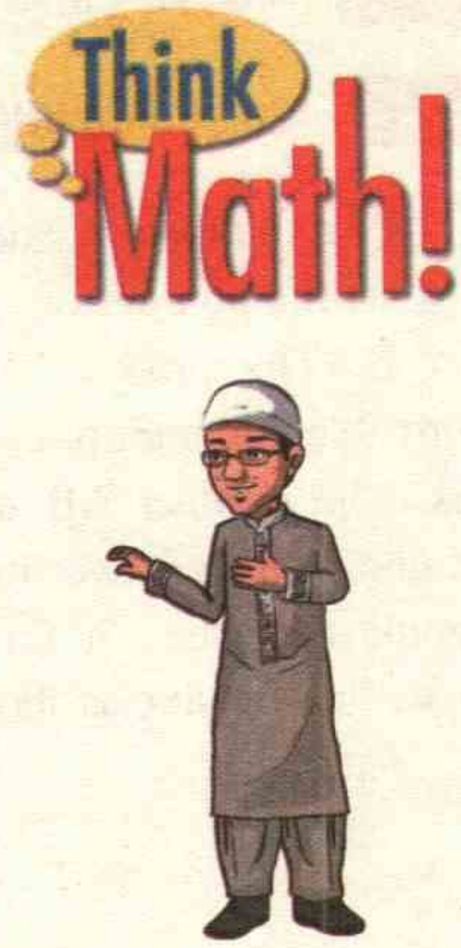
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Example 9

If $A=\begin{bmatrix}2 & 3\\ 1 & 4\end{bmatrix}$ and $B=\begin{bmatrix}3\\ 5\end{bmatrix}$, then

i) is it possible to find both AB & BA.

ii) find the possible product.



Solution:

Given $A=\begin{bmatrix}2 & 3\\ 1 & 4\end{bmatrix}$ and $B=\begin{bmatrix}3\\ 5\end{bmatrix}$

i) First we check whether AB is possible?

Number of columns in A are 2.

Number of rows in B are 2.

$\therefore$ AB is possible. Therefore,

$$AB=\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}\begin{bmatrix}3\\ 5\end{bmatrix}=\begin{bmatrix}2\times3+3\times5\\ 1\times3+4\times5\end{bmatrix}=\begin{bmatrix}6+15\\ 3+20\end{bmatrix}=\begin{bmatrix}21\\ 23\end{bmatrix}$$

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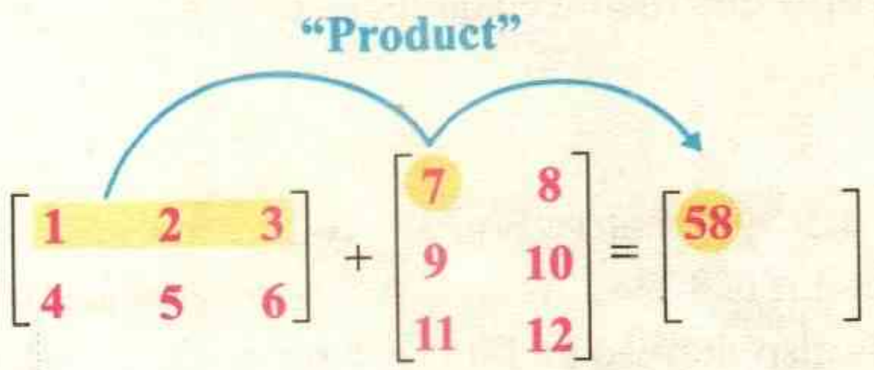
$$\therefore AB=\begin{bmatrix}21\\ 23\end{bmatrix}$$

Now we discuss the existence of the product BA.

Since number of columns in $B = 1$ and numbers of rows in A = 2.

$\because 1 \neq 2$, so BA is not possible.

ii) The only possible product is AB its value is $AB=\begin{bmatrix}21\\ 23\end{bmatrix}$



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1.4.2 Commutative Law of Multiplication of Matrices

Commutative law of multiplication of matrices in general does not hold as shown in the following example.

Example 10

Let $A=\begin{bmatrix}6 & 3\\ 2 & 5\end{bmatrix}$, $B=\begin{bmatrix}-3 & 2\\ 1 & 5\end{bmatrix}$. Determine whether $AB = BA$.

Solution:

$$AB=\begin{bmatrix}6 & 3\\ 2 & 5\end{bmatrix}\begin{bmatrix}-3 & 2\\ 1 & 5\end{bmatrix}=\begin{bmatrix}-15 & 27\\ -1 & 29\end{bmatrix} \text{ -------- (1)}$$

$$BA=\begin{bmatrix}-3 & 2\\ 1 & 5\end{bmatrix}\begin{bmatrix}6 & 3\\ 2 & 5\end{bmatrix}=\begin{bmatrix}-14 & 1\\ 16 & 28\end{bmatrix} \text{ -------- (2)}$$

From (1) and (2), we have $AB \neq BA$.

The above example shows that matrix multiplication is not commutative in general, that is $AB \neq BA$.

Though it can happen that $AB = BA$.

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Now we give an example of matrices for which commutative property of multiplication is true.

Example 11

Let $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$ and $B=\begin{bmatrix}2 & 2\\ 3 & 5\end{bmatrix}$, show that $AB = BA$.

Solution:

$$AB=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}2 & 2\\ 3 & 5\end{bmatrix}=\begin{bmatrix}1\times2+2\times3 & 1\times2+2\times5\\ 3\times2+4\times3 & 3\times2+4\times5\end{bmatrix}$$

$$=\begin{bmatrix}2+6 & 2+10\\ 6+12 & 6+20\end{bmatrix}=\begin{bmatrix}8 & 12\\ 18 & 23\end{bmatrix} \text{.....(1)}$$

Now $$BA=\begin{bmatrix}2 & 2\\ 3 & 5\end{bmatrix}\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}=\begin{bmatrix}2\times1+2\times3 & 2\times2+2\times4\\ 3\times1+5\times3 & 3\times2+5\times4\end{bmatrix}$$

$$=\begin{bmatrix}2+6 & 4+8\\ 3+15 & 6+20\end{bmatrix}=\begin{bmatrix}8 & 12\\ 18 & 26\end{bmatrix} \text{..... (2)}$$

Form (1) and (2) $AB = BA$. So the given matrices commute.

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1.4.3 Associative law under multiplication

If A, B &C are three matrices such that A is m-by-n matrix, B is n-by-p matrix and C is p-by-q matrix.

Then $A(BC)=(AB)C$

This is called the associative law of matrices under multiplication.

Example 12

If $A=\begin{bmatrix}1\\2\end{bmatrix}$, $B=[3 \quad 2]$, $C=\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$, then verify that $A(BC)=(AB)C$.

Solution:

$$BC=[3 \quad 2]\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}=[3\times1+2\times31 \quad 3\times2+2\times4]$$

$$=[3+6 \quad 6+8]=[9 \quad 14]$$

$$A(BC)=\begin{bmatrix}1\\2\end{bmatrix}[9 \quad 14]=\begin{bmatrix}1\times9 & 1\times14\\2\times9 & 2\times14\end{bmatrix}=\begin{bmatrix}9 & 14\\18 & 28\end{bmatrix} \quad \text{..... (i)}$$

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$$AB=\begin{bmatrix}1\\2\end{bmatrix}[3 \quad 2]=\begin{bmatrix}1\times3 & 1\times2\\2\times3 & 2\times2\end{bmatrix}=\begin{bmatrix}3 & 2\\6 & 4\end{bmatrix}$$

$$(AB)C=\begin{bmatrix}3 & 2\\6 & 4\end{bmatrix}\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}=\begin{bmatrix}3\times1+2\times3 & 3\times2+2\times4\\6\times1+4\times3 & 6\times2+4\times4\end{bmatrix}$$

$$=\begin{bmatrix}3+6 & 6+8\\6+12 & 12+6\end{bmatrix}=\begin{bmatrix}9 & 14\\18 & 28\end{bmatrix} \quad \text{..... (ii)}$$

From (i) and (ii), we have $A(BC)=(AB)C$.

This verifies associative law of matrix multiplication.

1.4.4 Distributive laws of multiplication over addition

Let A, B& C are three matrices such that A is a m-by-n matrix, B is a n-by-p matrix and C is also a n-by-p matrix.

$$A(B+C)=AB+AC \qquad \textit{and} \qquad (A+B)C=AC+BC$$

These are called distributive laws of multiplication over addition.

We try to verify these laws by choosing matrices of suitable order in the following example.

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Example 13

If $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$, $B=\begin{bmatrix}5 & 3\\ 2 & 2\end{bmatrix}$, $C=\begin{bmatrix}6 & 2\\ 5 & 1\end{bmatrix}$. Verify the distributive of multiplication over addition.

Solution:

We see A, B and C are 2-by-2 matrices. Hence they are conformable for multiplication and addition.

$$B+C=\begin{bmatrix}5 & 3\\ 2 & 4\end{bmatrix}+\begin{bmatrix}6 & 2\\ 5 & 1\end{bmatrix}=\begin{bmatrix}5+6 & 3+2\\ 2+5 & 4+1\end{bmatrix}=\begin{bmatrix}11 & 5\\ 7 & 5\end{bmatrix}$$

$$A(B+C)=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}11 & 5\\ 7 & 5\end{bmatrix}=\begin{bmatrix}1\times11+2\times7 & 1\times5+2\times5\\ 3\times11+4\times7 & 3\times5+4\times5\end{bmatrix}$$

$$=\begin{bmatrix}11+14 & 5+10\\ 33+28 & 15+20\end{bmatrix}=\begin{bmatrix}25 & 15\\ 61 & 35\end{bmatrix} \quad \ldots\ldots \text{ i)}$$

$$AB=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}5 & 3\\ 2 & 4\end{bmatrix}=\begin{bmatrix}1\times5+2\times2 & 1\times3+2\times4\\ 3\times5+4\times2 & 3\times3+4\times4\end{bmatrix}$$

$$=\begin{bmatrix}5+4 & 3+8\\ 15+8 & 9+16\end{bmatrix}=\begin{bmatrix}9 & 11\\ 23 & 25\end{bmatrix} \quad \ldots\ldots\ldots\ldots \text{ ii)}$$

Now $$AC=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}6 & 2\\ 5 & 1\end{bmatrix}=\begin{bmatrix}1\times6+2\times5 & 1\times2+2\times1\\ 36+4\times5 & 3\times2+4\times1\end{bmatrix}$$

$$=\begin{bmatrix}6+10 & 2+2\\ 18+20 & 6+4\end{bmatrix}=\begin{bmatrix}16 & 4\\ 38 & 10\end{bmatrix} \quad \ldots\ldots\ldots\ldots \text{ ii)}$$

Form (ii) and (iii),

$$AB+AC=\begin{bmatrix}9 & 11\\ 23 & 25\end{bmatrix}+\begin{bmatrix}16 & 4\\ 38 & 10\end{bmatrix}=\begin{bmatrix}9+16 & 11+4\\ 23+38 & 25+10\end{bmatrix}$$

$$=\begin{bmatrix}25 & 15\\ 61 & 35\end{bmatrix} \quad \ldots\ldots\ldots \text{ (iv)}$$

$\therefore$ Form (i) and (iv)

$$A(B+C)=AB+AC$$

Practice

Verify that
$(A+B)C=AC+BC$

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1.4.5 Multiplicative identity of a matrix

Let we have a matrix I and a matrix A. if these two matrices are so related to each other that $IA = AI = A$ i.e. the multiplication of I with A or A with I doesn't change the value of A then I is called the identity matrix of A. For example:

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ then we see that

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times1+0\times4 & 1\times2+0\times5 & 1\times3+0\times6 \\ 0\times1+1\times4 & 0\times2+1\times5 & 0\times3+1\times6 \end{bmatrix} = \begin{bmatrix} 1+0 & 2+0 & 3+0 \\ 0+4 & 0+5 & 0+6 \end{bmatrix}$$

$$IA = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = A$$

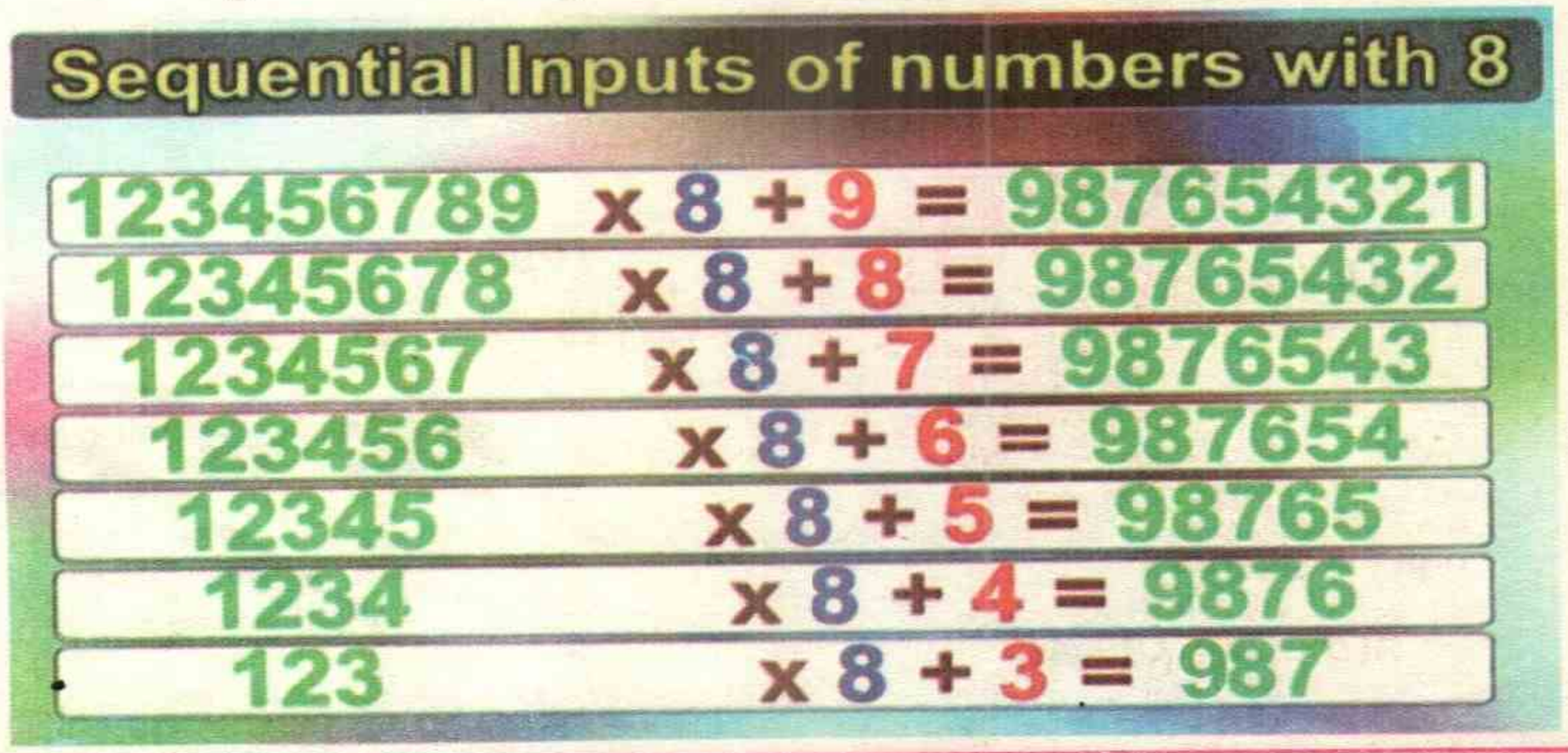
i.e. $IA = A$

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But AI is not defined because number of columns in A = 3

and number of rows in $I = 2$. So $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is not a multiplicative identity of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}.$$



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Example 14

If $I=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ and $A=\begin{bmatrix}9 & -3\\ -1 & 5\end{bmatrix}$, then find IA and AI.

Solution:

$$IA=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}9 & -3\\ -4 & 5\end{bmatrix}=\begin{bmatrix}1\times9+0\times(-4) & 1\times(-3)+0\times5\\ 0\times9+1\times(-4) & 0\times(-3)+1\times5\end{bmatrix}$$

$$IA=\begin{bmatrix}9 & -3\\ -4 & 5\end{bmatrix}=A$$

$$AI=\begin{bmatrix}9 & -3\\ -4 & 5\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}9\times1+(-3)+0 & 9\times0+(-3)\times1\\ (-4)\times1+5\times0 & -4\times0+5\times1\end{bmatrix}$$

$$IA=\begin{bmatrix}9 & -3\\ -4 & 5\end{bmatrix}=A$$

Hence $IA=AI=A$

So $I=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ is the multiplicative identity of $A=\begin{bmatrix}9 & -3\\ -1 & 5\end{bmatrix}$.

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In fact $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ is the multiplicative identity of the system of all 2-square matrices.

Similarly $\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ is the multiplicative identity of the system of all 3-square matrices.

1.4.6 Transpose of a matrix

A matrix which is obtained by interchanging all the rows and columns of a given matrix is called its transpose. The transpose of matrix A is written A^t.

Example 15

If $A=\begin{bmatrix}3 & 4 & 5\\ 2 & 4 & 6\end{bmatrix}$, then $A^t=\begin{bmatrix}3 & 2\\ 4 & 4\\ 5 & 6\end{bmatrix}$.



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1.4.7 Verification of the result $(AB)^t = B^t A^t$

For verifying the result $(AB)^t = B^t A^t$ it is necessary that AB must exists i.e. A & B are conformable for multiplication. This is possible only, when number of columns in A must be equal to the number of rows in B. Now we choose matrices of suitable order/orders so that their multiplication becomes possible and then try to verity the above law, $(AB)^t = B^t A^t$ that is, the transpose of the product of matrices is equal to the product of their transposes but in the reverse order.

Example 16

Let $A = \begin{bmatrix} 3 & -1 \\ 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -5 \\ 6 & -7 \end{bmatrix}$, show that $(AB)^t = B^t A^t$.

Solution:

$$AB = \begin{bmatrix} 3 & -2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 2 & -5 \\ 6 & -7 \end{bmatrix}$$

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$$= \begin{bmatrix} 3\times 2+(-2)\times 6 & 3\times(-5)+(-2)(-7) \\ 1\times 2+4\times 6 & 1\times(-5)+4\times(-7) \end{bmatrix}$$

$$= \begin{bmatrix} 6-12 & -15+14 \\ 2+24 & -5-28 \end{bmatrix} = \begin{bmatrix} -6 & -1 \\ 26 & -33 \end{bmatrix}$$

$$\therefore \quad (AB)^t = \begin{bmatrix} -6 & 26 \\ -1 & -33 \end{bmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

$$B^t A^t = \begin{bmatrix} 2 & 6 \\ -5 & -7 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times 3+6\times(-2) & 2\times 1+6\times 4 \\ -5\times 3+(-7)(-2) & (-5)\times 1+(-7)\times 4 \end{bmatrix}$$

$$= \begin{bmatrix} 6-12 & 2+24 \\ -15+14 & -5-28 \end{bmatrix} = \begin{bmatrix} -6 & 26 \\ -1 & -33 \end{bmatrix} \ldots\ldots \text{(ii)}$$

Form (i) and (ii) we have $(AB)^t = B^t A^t$

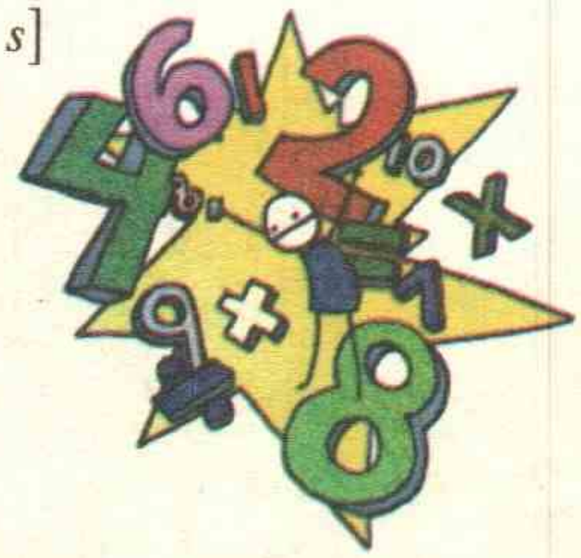
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Exercise 1.4

1. Show that which of the following matrices are conformable for multiplication.

i) $A=\begin{bmatrix} a \\ b \end{bmatrix}$ ii) $B=\begin{bmatrix} p & q \end{bmatrix}$

iii) $C=\begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}$ iv) $D=\begin{bmatrix} p & r & s \end{bmatrix}$



2. If $A=\begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}$, $B=\begin{bmatrix} 3 \\ -2 \end{bmatrix}$

i) Is it possible to find **AB**.

ii) Is it possible to find BA.

iii) Find the possible product/products.

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3. Given that $A=\begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix}$, $B=\begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix}$, $C=\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$, $D=\begin{bmatrix} 1 & -2 \\ -\frac{1}{2} & \frac{2}{3} \end{bmatrix}$.

Find i) AB and ii) CD.

4. Let $A=\begin{bmatrix} 2 & 1 \\ 3 & 0 \\ -1 & 4 \end{bmatrix}$ and $B=\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$. i) **Find AB.** ii) **Does BA exist?**

5. If $A=\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $B=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then show that $AB \neq BA$.

6. If $A=\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, then find $A \times A$.

7. If $A=\begin{bmatrix} -2 & 3 \\ 2 & -1 \end{bmatrix}$, $B=\begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$. Is $AB = BA$.

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8. If $A=\begin{bmatrix}-1\\1\end{bmatrix}$, $B=[1 \quad -2]$, $C=\begin{bmatrix}3 & 1\\-1 & 2\end{bmatrix}$, then find

i) (AB)C and A(BC). ii) Determine whether (AB)C=A(BC).

iii) Interpret which law of multiplication this result shows?

9. Verify that A(B+C)=AB+AC for the following matrices.

i) $A=\begin{bmatrix}1 & 2\\3 & -1\end{bmatrix}$, $B=\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}$, $C=\begin{bmatrix}3 & -1\\0 & 2\end{bmatrix}$

ii) $A=\begin{bmatrix}3 & -1\\0 & 2\end{bmatrix}$, $B=\begin{bmatrix}1\\2\end{bmatrix}$, $C=\begin{bmatrix}-1\\1\end{bmatrix}$

10. Let $I=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$, $A=\begin{bmatrix}5 & -3\\4 & 6\end{bmatrix}$, $B=\begin{bmatrix}-7 & 3\\2 & 8\end{bmatrix}$

Find i) AI ii) BI

11. Prove that

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i) $A=[3 \quad 2 \quad 1]$, $B=[-3 \quad 4 \quad 2]$,

$(A+B)^t=A^t+B^t$ and $(A-B)^t=A^t-B^t$

ii) $C=\begin{bmatrix}7 & -3\\2 & -1\end{bmatrix}$, $D=\begin{bmatrix}1 & 1\\2 & 2\end{bmatrix}$,

$(C+D)^t=C^t+D^t$ and $(C-D)^t=C^t-D^t$

12. i) If $A=\begin{bmatrix}2 & 5\\-3 & 4\end{bmatrix}$, $B=\begin{bmatrix}-1 & 1\\2 & 3\end{bmatrix}$, show that $(AB)^t=B^tA^t$

ii) If $C=\begin{bmatrix}a & b\\c & d\end{bmatrix}$, show that $(C^t)^t=C$

iii) If $A=\begin{bmatrix}1 & 0 & -1\\2 & 0 & 6\end{bmatrix}$, $B=\begin{bmatrix}1 & 7\\-8 & 4\\0 & 1\end{bmatrix}$, show that $(AB)^t=A^tB^t$.

Tid-Bit

The sum of any two odd numbers is even; the product of any two odd numbers is odd.

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1.5 MULTIPLICATIVE INVERSE OF A MATRIX

1.5.1 Determinant of a square matrix

With every square matrix A, a unique real number is associated which is called the determinant of A denoted by $|A|$ or det A and is given by a definite rule.

Thus if $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $|A|=\begin{vmatrix} a & b \\ c & d \end{vmatrix}=ad-bc$

Thus the determinant of a 2-by-2 matrix is obtained by multiplying entries on the main diagonal and subtracting from it the product of entries lying on the secondary diagonal.

Example 17

Find the determinant of the matrix

$A=\begin{bmatrix} 7 & 5 \\ 7 & -12 \end{bmatrix}$.

Solution:

Given $A=\begin{bmatrix} 7 & 5 \\ 7 & -12 \end{bmatrix}$

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$\therefore \quad |A|=\begin{vmatrix} 7 & 5 \\ 7 & -12 \end{vmatrix} = 7\times(-12)-5\times 7 = -84-35 = -119$

Note

$|A|$ is the notation for the determinant of A. The determinant is a number, not a matrix.

1.5.2 Singular and non-singular matrices

A square matrix A is called singular if $|A|=0$ and non-singular if $|A|\neq 0$.

Example 18

Find whether $A=\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$ is a singular matrix.

Solution:

If $A=\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$, then $|A|=\begin{vmatrix} 4 & -2 \\ -2 & 1 \end{vmatrix}=(4)(1)-(-2)(-2)=4-4=0$

Hence A is a singular matrix.

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Example 19

If $P=\begin{bmatrix}-4 & 2\\ 3 & -7\end{bmatrix}$, check whether P is a singular or non-singular matrix.

Solution:

$P=\begin{bmatrix}-4 & 2\\ 3 & -7\end{bmatrix}$, then $|P|=\begin{vmatrix}-4 & 2\\ 3 & -7\end{vmatrix}=(-4)\times(-7)-(3)\times(2)=28-6=22\neq 0$

Since $|P|\neq 0$, therefore P is non-singular matrix.

1.5.3 Adjoint of a matrix

The adjoint of a square matrix $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ is denoted by *adj A* and defined as $adj\,A=\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$. That is, change the places of a and d with each other and change the signs of b and c.

Example 20

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Find adjoint of the following matrices.

i) $A=\begin{bmatrix}3 & -2\\ 1 & 4\end{bmatrix}$ ii) $B=\begin{bmatrix}4 & -2\\ -3 & 1\end{bmatrix}$



Solution:

i) $A=\begin{bmatrix}3 & -2\\ 1 & 4\end{bmatrix}$, then $adj\ A=\begin{bmatrix}4 & 2\\ -1 & 3\end{bmatrix}$

ii) $B=\begin{bmatrix}4 & -2\\ -3 & 1\end{bmatrix}$, then $adj\ B=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$

1.5.4 Multiplicative inverse of a Matrix

Let A be a non-singular square matrix. If there exists another non-singular matrix B such that $AB=BA=I$ is an identity matrix, then B is said to be the multiplicative inverse of A. we denote the inverse of A by A^{-1}. Hence $B=A^{-1}$.

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Example 21

Show that $\begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}$ is the multiplicative inverse of $\begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}$.

Solution:

Let $A = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}$

$$AB = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 3\times3+2(-4) & 3(-2)+2\times3 \\ 4\times3+3(-4) & 4(-2)+3\times3 \end{bmatrix}$$

$$= \begin{bmatrix} 9-8 & -6+6 \\ 12-12 & -8+9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$BA = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} (3\times3)+(-2)4 & 3\times2+(-2)3 \\ (-4\times3)+(3\times4) & (-4)2+3\times3 \end{bmatrix}$$

$$= \begin{bmatrix} 9-8 & -6+6 \\ -12+12 & -8+9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Since $AB = I = BA$, therefore A is the inverse of B.

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1.5.5 Use of Adjoint method to calculate inverse of a non-singular matrix

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $Adj(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ and $|A| = ad - bc$.

To find A^{-1}, we use the formula $A^{-1} = \frac{1}{|A|} Adj\, A$.

Example 22

Find the inverse of $A = \begin{bmatrix} -2 & -1 \\ 3 & 4 \end{bmatrix}$, using the adjoint method.

Solution:

$A = \begin{bmatrix} -2 & -1 \\ 3 & 4 \end{bmatrix}$, $|A| = -2\times4 - 3(-1) = -8+3 = -5$.

Since $|A| \neq 0$, therefore A is non-singular, so we can find A^{-1}.

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$$\text{As } A^{-1} = \frac{1}{|A|} Adj\, A \quad \text{------(1)}$$

Now $Adj\, A = \begin{bmatrix} 4 & 1 \\ -3 & -2 \end{bmatrix}$, putting it in equation (1).

$$\therefore \quad A^{-1} = \frac{1}{-5}\begin{bmatrix} 4 & 1 \\ -3 & -2 \end{bmatrix} = \begin{bmatrix} \frac{-4}{5} & \frac{-1}{5} \\ \frac{3}{5} & \frac{2}{5} \end{bmatrix}$$

1.5.6 Verification of the result $AA^{-1} = I = A^{-1}A$

$$AA^{-1} = \begin{bmatrix} -2 & -1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} \frac{-4}{5} & \frac{-1}{5} \\ \frac{3}{5} & \frac{2}{5} \end{bmatrix} = \begin{bmatrix} -2\times(\frac{-4}{5})+(-1)\times(\frac{3}{5}) & -2\times(\frac{-1}{5})+(-1)\times(\frac{2}{5}) \\ 3\times(\frac{-4}{5})+4\times(\frac{3}{5}) & 3\times(\frac{-1}{5})+4\times\frac{2}{5} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{8}{5}-\frac{3}{5} & \frac{2}{5}-\frac{2}{5} \\ \frac{-12}{5}+\frac{12}{5} & \frac{-3}{5}+\frac{8}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \quad \text{--------(1)}$$

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$$A^{-1}A = \begin{bmatrix} \frac{-4}{5} & \frac{-1}{5} \\ \frac{3}{5} & \frac{2}{5} \end{bmatrix}\begin{bmatrix} -2 & -1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} (\frac{-4}{5})\times(-2)+(\frac{-1}{5})\times(3) & (\frac{-4}{5})\times(-1)+(\frac{-1}{5})\times(4) \\ (\frac{3}{5})\times(-2)+(\frac{2}{5})\times(3) & (\frac{3}{5})\times(-1)+(\frac{2}{5})\times(4) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{8}{5}-\frac{3}{5} & \frac{4}{5}-\frac{4}{5} \\ \frac{-6}{5}+\frac{6}{5} & \frac{-3}{5}+\frac{8}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \quad \text{--------(2)}$$

Thus from (1) and (2),

$$AA^{-1} = I = A^{-1}A.$$



Remember

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

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1.5.7 Verification of the result $(AB)^{-1} = B^{-1}A^{-1}$

The above equation is not true for all types of matrices, but it is true only if A & B are non-singular square matrices of the same order. In such a case, the above equation states that the inverse of the product matrix is equal to the product of inverses but taken in the reverse order.

Now we verify the above result by taking suitable matrices.

Example 23

Let $A=\begin{bmatrix}-2 & 1\\ 1 & 1\end{bmatrix}$ & $B=\begin{bmatrix}2 & 1\\ 3 & 2\end{bmatrix}$

Solution:

$$AB=\begin{bmatrix}-2 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}2 & 1\\ 3 & 2\end{bmatrix}=\begin{bmatrix}-2\times2+1\times3 & -2\times1+1\times2\\ 1\times2+1\times3 & 1\times1+1\times2\end{bmatrix}$$

$$=\begin{bmatrix}-4+3 & -2+2\\ 2+3 & 1+2\end{bmatrix}+\begin{bmatrix}-1 & 0\\ 5 & 3\end{bmatrix}$$

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$$\therefore \det(AB)=\begin{bmatrix}-1 & 0\\ 5 & 3\end{bmatrix}=(-1)\times(3)-(0)\times(5) \quad =-3\neq 0$$

$$\therefore (AB)^{-1} \text{ exists } \& (AB)^{-1}=\frac{1}{\det(AB)}Adj(AB) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

Since $AB=\begin{bmatrix}-1 & 0\\ 5 & 3\end{bmatrix} \Rightarrow Adj(AB)=\begin{bmatrix}3 & 0\\ -5 & -1\end{bmatrix}$

So that (i) becomes

$$(AB)^{-1}=-\frac{1}{3}\begin{bmatrix}3 & 0\\ -5 & -1\end{bmatrix}=\begin{bmatrix}-\frac{1}{3}\times3 & -\frac{1}{3}\times0\\ -\frac{1}{3}\times-5 & -\frac{1}{3}\times(-1)\end{bmatrix}=\begin{bmatrix}-1 & 0\\ \frac{5}{3} & \frac{1}{3}\end{bmatrix} \quad \ldots\ldots (ii)$$

Again $A=\begin{bmatrix}-2 & 1\\ 1 & 1\end{bmatrix} \Rightarrow \det(A)=\begin{vmatrix}-2 & 1\\ 1 & 1\end{vmatrix}=(-2)(1)-(1)(1)$

$$\det A=-2-1=-3\neq 0$$

$$\therefore Adj(A)=\begin{bmatrix}1 & -1\\ -1 & -2\end{bmatrix}$$

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$$\therefore\ A^{-1}=\frac{1}{\det A}Adj(A)=-\frac{1}{3}\begin{bmatrix}1 & -1\\ -1 & -2\end{bmatrix}$$

$$\text{i.e. } A^{-1}=\begin{bmatrix}-\frac{1}{3}\times 1 & -\frac{1}{3}\times(-1)\\ -\frac{1}{3}\times(-1) & -\frac{1}{3}\times(-2)\end{bmatrix}=\begin{bmatrix}-\frac{1}{3} & \frac{1}{3}\\ \frac{1}{3} & \frac{2}{3}\end{bmatrix}$$

Again $B=\begin{bmatrix}2 & 1\\ 3 & 2\end{bmatrix}$, so that $|B|=\begin{vmatrix}2 & 1\\ 3 & 2\end{vmatrix}=2(2)-1(3)$, so B^{-1} exists and

$B^{-1}=\frac{1}{|B|}Adj(B)$. But $Adj(B)=\begin{bmatrix}2 & -1\\ -3 & 2\end{bmatrix}$

$$\therefore\ B^{-1}=\frac{1}{1}\begin{bmatrix}2 & -1\\ -3 & 2\end{bmatrix}=\begin{bmatrix}2 & -1\\ -3 & 2\end{bmatrix}$$

$$\therefore B^{-1}A^{-1}=\begin{bmatrix}2 & -1\\ -3 & 2\end{bmatrix}\begin{bmatrix}-\frac{1}{3} & \frac{1}{3}\\ \frac{1}{3} & \frac{2}{3}\end{bmatrix}=\begin{bmatrix}2\times-\frac{1}{3}+(-1)\times\frac{1}{3} & 2\times\frac{1}{3}+(-1)\times\frac{2}{3}\\ -3\times\left(-\frac{1}{3}\right)+2\times\frac{1}{3} & -3\times\frac{1}{3}+2\times\frac{2}{3}\end{bmatrix}$$

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$$B^{-1}A^{-1}=\begin{bmatrix}-\frac{2}{3}-\frac{1}{3} & \frac{2}{3}-\frac{2}{3}\\ 1+\frac{2}{3} & -1\times\frac{4}{3}\end{bmatrix}=\begin{bmatrix}\frac{-2-1}{3} & 0\\ \frac{3+2}{3} & \frac{-3+4}{3}\end{bmatrix}$$

$$=\begin{bmatrix}\frac{-3}{3} & 0\\ \frac{5}{3} & \frac{1}{3}\end{bmatrix}=\begin{bmatrix}-1 & 0\\ \frac{5}{3} & \frac{1}{3}\end{bmatrix}$$

$$\text{i.e. } B^{-1}A^{-1}=\begin{bmatrix}-1 & 0\\ \frac{5}{3} & \frac{1}{3}\end{bmatrix}\ \ldots\ldots \text{(iii)}$$

Comparing (ii) and (iii) we get $(AB)^{-1}=B^{-1}A^{-1}$.



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Exercise 1.5

1. Find the determinant of following matrices and evaluate them.

i) $A = \begin{bmatrix} 5 & 6 \\ -4 & 1 \end{bmatrix}$

ii) $B = \begin{bmatrix} 4 & -2 \\ 5 & 13 \end{bmatrix}$

iii) $C = \begin{bmatrix} 11 & 7 \\ -6 & 5 \end{bmatrix}$

iv) $D = \begin{bmatrix} 5 & 6 \\ -8 & -9 \end{bmatrix}$

v) $E = \begin{bmatrix} 2p & -3q \\ r & -s \end{bmatrix}$

vi) $F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

vii) $G = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 2 & 3 \\ -2 & -3 & 4 \end{bmatrix}$

viii) $H = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

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2. Find which of the following matrices are singular and which are non-singular.

i) $A = \begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}$

ii) $B = \begin{bmatrix} 3 & -6 \\ -2 & 4 \end{bmatrix}$

iii) $C = \begin{bmatrix} 3a & -2b \\ 2a & b \end{bmatrix}$

iv) $D = \begin{bmatrix} -3 & 6 \\ 2 & -4 \end{bmatrix}$

3. Find the adjoint of the following matrices.

i) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

ii) $B = \begin{bmatrix} -3 & -1 \\ 2 & 3 \end{bmatrix}$

iii) $C = \begin{bmatrix} 2 & -4 \\ 3 & 1 \end{bmatrix}$

iv) $D = \begin{bmatrix} -3 & 6 \\ 2 & -4 \end{bmatrix}$

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4. Find the multiplicative inverses of the following matrices if they exist.

i) $A=\begin{bmatrix}4 & 1\\ 3 & 1\end{bmatrix}$

ii) $B=\begin{bmatrix}3 & 4\\ 1 & 2\end{bmatrix}$

iii) $C=\begin{bmatrix}4 & -3\\ -1 & 2\end{bmatrix}$

iv) $D=\begin{bmatrix}0 & -3\\ 2 & 4\end{bmatrix}$

v) $I=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

5. If $A=\begin{bmatrix}2 & 0\\ -3 & 1\end{bmatrix}$, $B=\begin{bmatrix}1 & -1\\ -1 & 3\end{bmatrix}$ find

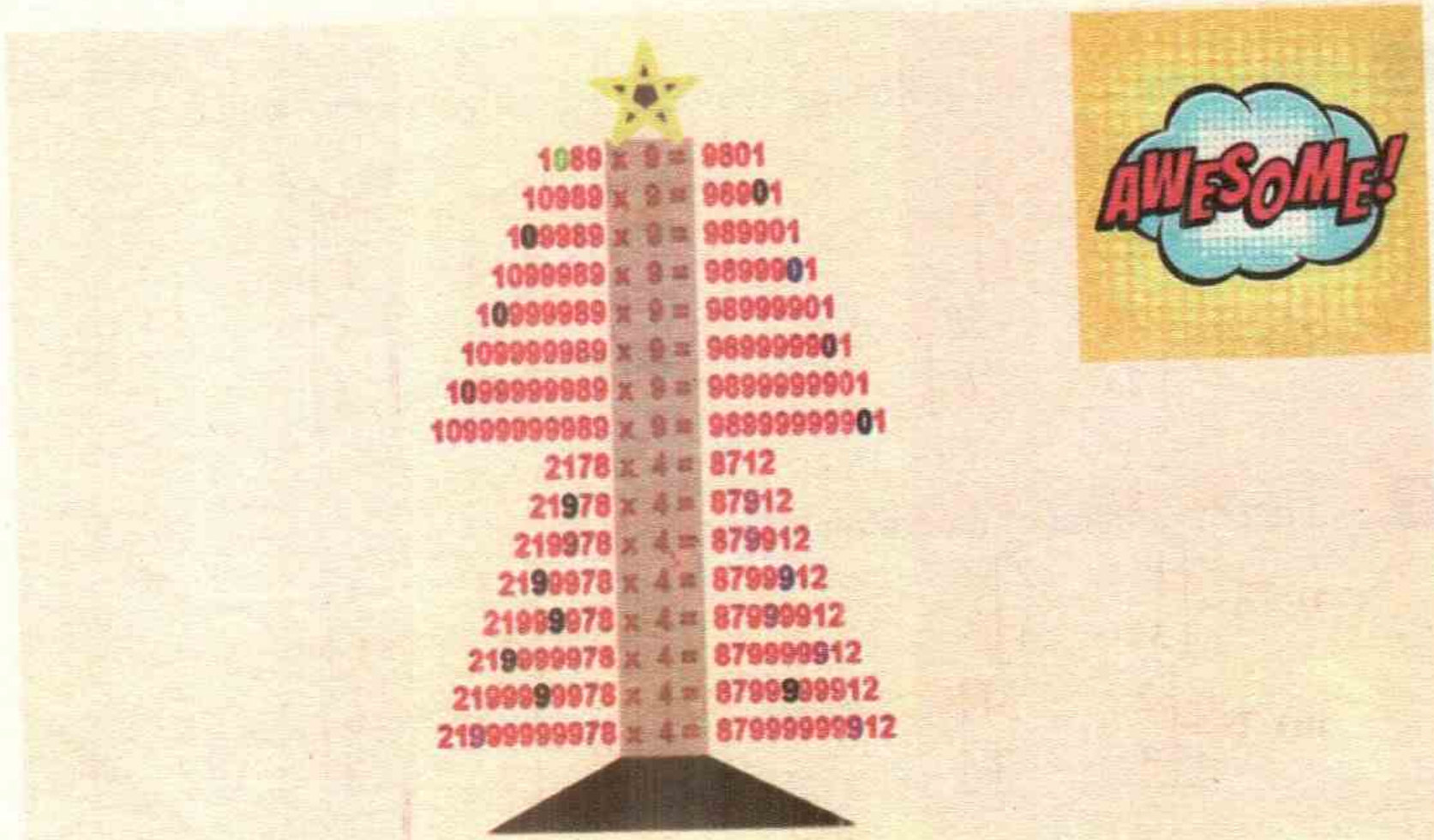
i) AB ii) BA iii) A^{-1} & B^{-1}

then show that $(AB)^{-1}=B^{-1}A^{-1}$ & $(BA)^{-1}=A^{-1}B^{-1}$

6. If $A=\begin{bmatrix}0 & -1\\ 2 & 1\end{bmatrix}$, $B=\begin{bmatrix}2 & 3\\ 1 & 0\end{bmatrix}$, then show that

$(AB)^{-1}=B^{-1}A^{-1}$ & $(BA)^{-1}=A^{-1}B^{-1}$

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1.6 SOLUTION OF SIMULTANEOUS LINEAR EQUATIONS

System of two linear equations in variable x and y in general form is shown below.

$$ax+by=m \quad \ldots\ldots \text{(i)}$$
$$cx+dy=n \quad \ldots\ldots \text{(ii)}$$

or $$AX=B \quad \ldots\ldots \text{(iii)}$$

where $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $X=\begin{bmatrix} x \\ y \end{bmatrix}$, $B=\begin{bmatrix} m \\ n \end{bmatrix}$

if $|A|\neq 0$, we can find A^{-1}. Multiplying both sides of equation (iii) by A^{-1}, we get

$$A^{-1}(AX)=A^{-1}B \Rightarrow (A^{-1}A)X=A^{-1}B \Rightarrow IX=A^{-1}B \Rightarrow X=A^{-1}B \quad \ldots\ldots \text{(iv)}$$

From equation (iv),

$$\begin{bmatrix} x \\ y \end{bmatrix}=\frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} m \\ n \end{bmatrix}$$

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$$\therefore \quad A^{-1}=\frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

or $$\begin{bmatrix} x \\ y \end{bmatrix}=\frac{1}{|A|}\begin{bmatrix} dm-bn \\ -cm+an \end{bmatrix}$$

$$\Rightarrow x=\frac{dm-bn}{ad-bc}, \quad y=\frac{-cm+an}{ad-bc}$$



$$\therefore \quad |A|=ad-bc$$

Hence the solution of given system of equations (i) and (ii) is

$$x=\frac{dm-bn}{ad-bc}, \quad y=\frac{-cm+an}{ad-bc}$$

This method is known as matrix inversion method.

It may be remembered that if $|A|=0$, A^{-1} does not exist. Hence in such a case solution set of these equations cannot be found.

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Example 24

Solve the following system of equations with the help of matrices.

$x-3y=0, \quad 2x+y=7$

Solution:

These equations can be written in the form of matrices as

$$\begin{bmatrix}1 & -3\\ 2 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}0\\ 7\end{bmatrix}$$

Let $A=\begin{bmatrix}1 & -3\\ 2 & 1\end{bmatrix}$, $X=\begin{bmatrix}x\\ y\end{bmatrix}$, $B=\begin{bmatrix}0\\ 7\end{bmatrix}$

$|A|=1\times1-2(-3)-1+6=7\neq0$.

Hence A^{-1} exists. By using formula, we get $A^{-1}=\frac{1}{7}\begin{bmatrix}1 & 3\\ -2 & 1\end{bmatrix}$

Now $AX=B\Rightarrow X=A^{-1}B\Rightarrow\begin{bmatrix}x\\ y\end{bmatrix}=\frac{1}{7}\begin{bmatrix}1 & 3\\ -2 & 1\end{bmatrix}\begin{bmatrix}0\\ 7\end{bmatrix}$

$\Rightarrow\begin{bmatrix}x\\ y\end{bmatrix}=\frac{1}{7}\begin{bmatrix}1\times0+3\times7\\ -2\times0+1\times7\end{bmatrix}=\frac{1}{7}\begin{bmatrix}21\\ 7\end{bmatrix}=\begin{bmatrix}3\\ 1\end{bmatrix}$

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$\therefore \quad x=3, \quad y=1$

Example 25

Is the following system of equations solvable?

$3x-6y=9, \quad 2x-4y=-3$

Solution:

Here $A=\begin{bmatrix}3 & -6\\ 2 & -4\end{bmatrix}$,

then $|A|=\begin{vmatrix}3 & -6\\ 2 & -4\end{vmatrix}=3(-4)-2(-6)=-12+12=0$

Hence the given equations are non-solvable.

DID YOU KNOW?

Some odd multiplications:

$37037037037\times9=333333333333$

$13717421\times9=123456789$

NOTE FOR SALE

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1.6.1 Cramer's Rule

We can easily solve simultaneous equations by applying Cramer's rule. The method is explained as:

Let the system of equations be

$$ax + by = m$$
$$cx + dy = n$$

In terms of matrices, we write these equations as;

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} m \\ n \end{bmatrix}, \text{ or } A\begin{bmatrix} x \\ y \end{bmatrix} = B$$

Where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} m \\ n \end{bmatrix}$

$$|A| = ad - bc$$

We replace the coefficients of x in A (that is a, c) by m, n of B and denote as A_x.

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$$\therefore \quad A_x = \begin{bmatrix} m & b \\ n & d \end{bmatrix}, |A_x| = md - nb, \text{ then}$$

$$x = \frac{|A_x|}{|A|} = \frac{md - nb}{ad - bc}$$

Likewise, for y, we replace the coefficients y in A (that is b, d) by m, n of B and denote it by A_y.

$$A_y = \begin{bmatrix} a & m \\ c & n \end{bmatrix} \text{ where } |A_y| = an - mn, \text{ then } y = \frac{|A_y|}{|A|} = \frac{an - cm}{ad - bc}$$

The method explained above is known as Cramer's Rule.

Tid-Bit

If you add up the numbers 1 - 100 consecutively (1+2+3+4...) the total is 5050.

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Example 26

Solve the following system of equations by using Cramer's rule.

$$x-2y=1, \qquad 3x+y=10$$

Solution:

In terms of matrices we can write the above system as:

$$\begin{bmatrix}1 & -2\\ 3 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}1\\ 10\end{bmatrix} \Rightarrow AX=B,$$

Where $A=\begin{bmatrix}1 & -2\\ 3 & 1\end{bmatrix}$, $X=\begin{bmatrix}x\\ y\end{bmatrix}$, $B=\begin{bmatrix}1\\ 10\end{bmatrix}$

$$|A|=1\times1-3(-2)=1+6=7\neq0$$

Replacing the coefficients of x in A (that is 1, 3) by 1, 10 of B to find A_x.

$$A_x=\begin{bmatrix}1 & -2\\ 10 & 1\end{bmatrix}, \text{ giving } |A|=1\times1-10(-2) \ =1+20 \ =21$$

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$$\therefore \qquad x=\frac{|A_x|}{A}=\frac{21}{7}=3$$

For A_y, replace the coefficients of y in A (that is -2, 1) by 1, 10 of B.

Then $A_y=\begin{bmatrix}1 & 1\\ 3 & 10\end{bmatrix}$, giving $|A_y|=1\times10-3(1)=10-3=7$

$$\therefore \qquad y=\frac{|A_y|}{A}=\frac{7}{7}=1$$

Hence the solution set $=\{(3, 1)\}$

1.6.2 Real life problems leading to simultaneous equations

Many real life problems require for their solution the finding of two unknown quantities. The general method of solution is similar to that used when there is one unknown, but with the important difference that when there are two unknowns to be found, two equations must be formed form the data, which can be solved by either matrix inversion method or Cramer's rule as shown in the following examples.

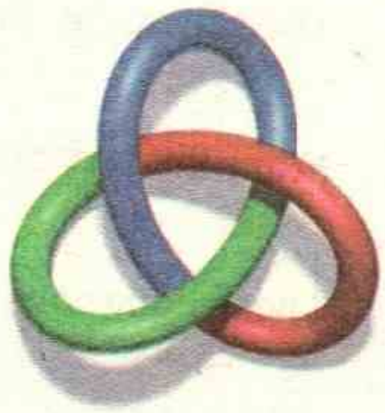
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Example 27

My friend asked me this question. There are two numbers such that the sum of the first and three times the second is 53, while the difference between 4 times the first and twice the second is 2. Can you help me out in finding the numbers?

Solution:



Let x = one number.

y = the second.

Then from the first set of facts

$$x+3y=53 \ldots\ldots \text{(i)}$$

From the second set of facts

$$4x-2y=2 \ldots\ldots \text{(ii)}$$

These equations can be written in the form of matrices as:

$$\begin{bmatrix}1 & 3\\ 4 & -2\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}35\\ 2\end{bmatrix}$$

Let $A=\begin{bmatrix}1 & 3\\ 4 & -2\end{bmatrix}$, $X=\begin{bmatrix}x\\ y\end{bmatrix}$, $B=\begin{bmatrix}53\\ 2\end{bmatrix}$

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Now $|A|=1\times(-2)-3\times4=-2-12=-14\neq0$. Hence A^{-1} exists. By using formula we get

$$A^{-1}=\frac{1}{-14}\begin{bmatrix}-2 & -3\\ -4 & 1\end{bmatrix}=-\frac{1}{14}\begin{bmatrix}-2 & -3\\ -4 & 1\end{bmatrix}$$

Now $AX=B \Rightarrow A^{-1}B \Rightarrow \begin{bmatrix}x\\ y\end{bmatrix}=-\frac{1}{14}\begin{bmatrix}-2 & -3\\ -4 & 1\end{bmatrix}\begin{bmatrix}53\\ 2\end{bmatrix}$

$$\Rightarrow\begin{bmatrix}x\\ y\end{bmatrix}=-\frac{1}{14}\begin{bmatrix}-2\times53+(-3)\times2\\ -4\times53+1\times2\end{bmatrix}$$

$$=-\frac{1}{14}\begin{bmatrix}-106-6\\ -212+2\end{bmatrix}=-\frac{1}{14}\begin{bmatrix}-112\\ -210\end{bmatrix}=\begin{bmatrix}8\\ 15\end{bmatrix}$$

$\therefore \quad x=8, \quad y=15$

Therefore, the numbers are 8 and 15.

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Example 28

The cost of 1 rubber and 7 sharpeners are 8 rupees, while that of 3 rubbers and 1 sharpener are 3 rupees. What are the prices of a rubber and a sharpener respectively?

Solution:

Let x = rubber price.

y = sharpener price.

Then from the first set of facts

$$x + 7y = 15 \ldots\ldots\ldots(1)$$

From the second,

$$3x + y = 5 \ldots\ldots\ldots(2)$$

These equations can be written in the form of matrices as:

$$\begin{bmatrix} 1 & 7 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 15 \\ 5 \end{bmatrix}$$

Let $A = \begin{bmatrix} 1 & 7 \\ 3 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $B = \begin{bmatrix} 15 \\ 5 \end{bmatrix}$

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Now $D = |A| = 1\times1 - 3\times(7) = 1 - 21 = -20 \neq 0.$

Here $A_x = \begin{bmatrix} 15 & 7 \\ 5 & 1 \end{bmatrix}$, giving $D_1 = |A_x| = 15\times1 - 5(7) = 15 - 35 = -20$

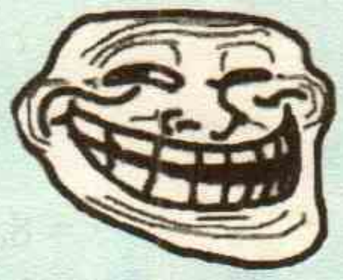
$$\therefore \quad x = \frac{D_1}{D} = \frac{-20}{-20} = 1$$

$A_y = \begin{bmatrix} 1 & 15 \\ 3 & 5 \end{bmatrix}$, giving $D_2 = |A_y| = 1\times5 - 3(15) = 5 - 45 = -40$

$$\therefore \quad x = 1, y = 2$$

Thus price of one rubber = $\therefore \quad y = \frac{D_2}{D} = \frac{-40}{-20} = 2$ 1 rupee and price of one sharpener = 2 rupees.

1) Choose a number between 1-1000000.
2) Add three.
3) Subtract three.
And You've got the number you started with.



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Exercise 1.6

1. Solve the following system of linear equations using inversion method.

   i) $2x+3y=-1,\ \ x-y=2$

   ii) $x+2y=-13,\ \ 3x+6y=11$

   iii) $x+2y=1,\ \ 2x+3y=\frac{5}{2}$

   iv) $x-2y-1=0,\ \ 2x+y+3=0$

2. Solve the following system of linear equations using Cramer's Rule.

   i) $x-2y=5,\ \ 2x-y=6$

   ii) $4x+3y=-2,\ \ x-2y=5$

   iii) $5x+7y=3,\ \ 3x+y=5$

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3. Amjad thought of two numbers whose sum is 12 and whose difference is 4. Find the numbers.

4. The length of a rectangular playground is twice its width. The perimeter is 30. Find its dimensions.

5. 3 bags and 4 pens together cost 257 rupees whereas 4 bags and 3 pens together cost 324 rupees. Find the cost of a bag and 10 pens.

6. If twice the son's age in years is added to the father's age, the sum is 70. But if twice the father's age is added to the son's age, the sum is 95. Find the ages of father and son.

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REVIEW EXERCISE 1

1. Choose the correct answer in each of the following problems.

(i) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is

(a) an identity matrix w.r.t multiplication
(b) a column matrix
(c) an identity matrix w.r.t addition
(d) a row matrix

(ii) The matrix $\begin{bmatrix} 4 & 0 \\ 0 & -12 \end{bmatrix}$ is

(a) a scalar matrix
(b) 2×3 matrix
(c) a diagonal matrix
(d) None of these

(iii) If $A = \begin{bmatrix} -1 & -2 \\ 3 & 1 \end{bmatrix}$, then *adj* A is equal to

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(a) $\begin{bmatrix} -1 & -2 \\ 3 & 1 \end{bmatrix}$
(b) $\begin{bmatrix} 1 & 2 \\ -3 & -1 \end{bmatrix}$
(c) $\begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}$
(d) $\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$

(iv) If $A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$, then A^{-1} equals

(a) $\begin{bmatrix} 4 & 3 \\ -3 & 2 \end{bmatrix}$
(b) $\begin{bmatrix} 4 & -3 \\ -3 & 2 \end{bmatrix}$
c) $\begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix}$
(d) $\begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix}$

(v) For what value of d is the 2×2 matrix $\begin{bmatrix} 5 & 1.5 \\ 2 & d \end{bmatrix}$ NOT invertible?

(a) –0.6
(b) 0
(c) 0.6
(d) 3

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(vi) Suppose A and B are 2×5 matrices. Which of the following are the dimensions of the matrix $A + B$?

(a) 2×5 (b) 10×10 (c) 7×1 (d) 7×7

(vii) Which of the following is the multiplicative inverse of $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

(a) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}$

(viii) Evaluate the determinant of the matrix $\begin{bmatrix} 4 & -1 \\ -9 & 2 \end{bmatrix}$

(a) 17 (b) 1 (c) -1 (d) -17

2. Find x and y.

$$\begin{bmatrix} x-1 & 4 \\ y+3 & -7 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ -2 & -7 \end{bmatrix}$$

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3. Find the product if possible.

$$\begin{bmatrix} 1 \\ -5 \\ 3 \end{bmatrix} \begin{bmatrix} -6 & 5 & 8 \\ 0 & 4 & -1 \end{bmatrix}$$



4. Find the inverse of the matrix $A = \begin{bmatrix} 6 & -3 \\ 5 & -2 \end{bmatrix}$

5. Solve the system.

$$2x + 5y = 9$$
$$5x - 2y = 8$$

Think

6. Qasim and Farzana are selling fruit for a school fundraiser. Customers can buy small boxes of oranges and large boxes of oranges. Qasim sold 3 small boxes of oranges and 14 large boxes of oranges for a total of Rs.203. Farzana sold 11 small boxes of oranges and 11 large boxes of oranges for a total of Rs.220. Find the cost each of one small box of oranges and one large box of oranges.

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SUMMARY

- A matrix is a rectangular array of real numbers enclosed in brackets.
- The rows of a matrix run horizontally, and the columns of a matrix run vertically.
- A matrix with m rows and n columns has order $m \times n$ (read "m by n").
- Each number in a matrix is called an element or entry of the matrix.
- Two matrices of the same order are equal if their corresponding elements are equal.
- A matrix is said to be a row matrix if it has only one row.
- A matrix is said to be a column matrix if it has only one column.
- A matrix in which the number of rows is equal to the number of columns is called a square matrix.
- A matrix in which the number of rows is not equal to the number of columns is called a rectangular matrix.
- A matrix is said to be a zero matrix or null matrix if all its elements are zero.
- If all the elements of a square matrix except the diagonal elements are zero, then the matrix is called a diagonal matrix.
- A diagonal matrix, whose all the diagonal elements are equal, is called a scalar matrix.
- A scalar matrix of order n in which each diagonal element is 1 (unity) is called an identity matrix of order n.
- A matrix obtained by interchanging rows and columns of a matrix A is called the transpose of the matrix A and is denoted by A^t.
- If a square matrix $A = A^t$, then A is called a symmetric matrix.
- A square matrix A is said to be skew-symmetric if $A^t = -A$.
- $(A+B)^t = A^t + B^t, (A-B)^t = A^t - B^t$.
- Two matrices are said to be conformable for addition/subtraction if they are of the same order.
- The null matrix is the additive identity for the matrix addition.
- If A is a matrix then $-A$ is the additive inverse of A.

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- Two matrices A and B are said to be conformable for multiplication AB, only when the number of columns of matrix A is equal to the number of rows of matrix B.
- $A_{m\times p} \times B_{p\times n} = AB_{m\times n}$
- If A, B and C are three matrices then $A(BC) = (AB)C$. This property is called associative law of matrices w.r.t multiplication.
- Commutative law of multiplication does not hold in matrices in general.
- Let A, B and C be three matrices, then

  (i) $A(B+C) = AB + AC$ (Left Distributive Law)

  (ii) $(A+B)C = AC + BC$ (Right Distributive Law)

  These are called distributive laws under multiplication over addition.
- $\left(A^t\right)^t = A.$
- $(AB)^t = B^t A^t$
- With every square matrix A, a unique real number is associated which is called the determinant of A denoted by $|A|$ or det A.
- If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.
- A square matrix A is called singular if $|A| = 0$ and non-singular if $|A| \neq 0$.
- The adjoint of a square matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is denoted by $adj\ A$ and defined as $adj\ A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
- If A and B are square matrices such that AB = BA = I where I is the identity matrix. Then B is called the inverse of A, denoted as $B = A^{-1}$.
- $AA^{-1} = I = A^{-1}A$
- $A^{-1} = \frac{1}{|A|} adj\ A$
- $(AB)^{-1} = B^{-1}A^{-1}$.

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UNIT 02

REAL AND COMPLEX NUMBERS

After reading this unit, the students will be able to:

- Recall the set of real numbers as a union of sets of rational and irrational numbers.
- Depict real numbers on the number line.
- Demonstrate a number with terminating and non-terminating recurring decimals on the number line.
- Give decimal representation of rational and irrational numbers.
- Know the properties of real numbers.
- Explain the concept of radicals and radicands.
- Differentiate between radical form and exponential form of an expression.
- Transform an expression given in radical form to an exponential form and vice versa.
- Recall base, exponent and value.
- Apply the laws of exponents to simplify expressions with real exponents.
- Define complex number z represented by an expression of the form $z = a + ib$, where a and b are real numbers and $i = \sqrt{-1}$.
- Recognize a as real part and b as imaginary part of $z = a + ib$.
- Define *conjugate* of a complex number.
- Know the condition for equality of complex numbers.
- Carry out basic operations (i.e. addition, subtraction, multiplication and division) on complex numbers.

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2.1 REAL NUMBERS

2.1.1 Recall

A rational number is a number that can be expressed in the form $\frac{p}{q}$ where p and q are integers and $q \neq 0$.

For example $-17, 1\frac{3}{4}, 1.25$ are rational numbers.

The set of Rational Numbers is denoted by Q. Set of Irrational Numbers, denoted by Q', consists of all those numbers which are not rational.

For example $\sqrt{3}, \pi$ are irrational numbers.

The set of real numbers, denoted by R, is the union of sets of rational and irrational numbers. Thus $Q \cup Q' = R$.

The relationship between the various set of numbers is as under.

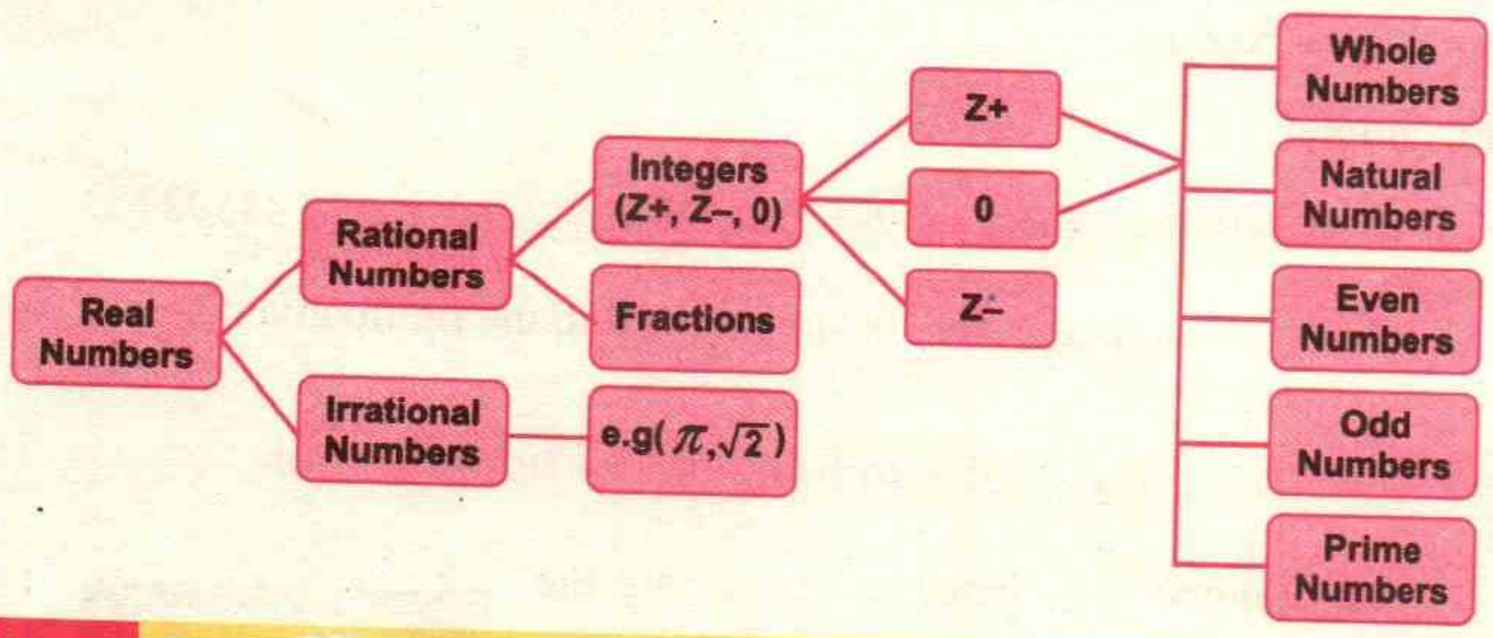
Tid-Bit

The word Rational comes from the word ratio. Thus irrational means "not ratio".

Tid-Bit

All the numbers on the number line are real numbers.

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2.1.2 Decimal representation of rational and irrational numbers

Every rational number can be written as either a terminating decimal or a repeating decimal. All terminating and repeating decimals are rational numbers. For example,

$$\frac{3}{8} = 0.375, \quad \frac{2}{15} = 0.133\ldots = 0.1\bar{3}$$

where the bar over the digit 3 means that this digit is repeated forever.

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Example 1

Write $\frac{3}{8}$ and $\frac{2}{15}$ as a decimal fractions.

Solution:

$$\begin{array}{r} 0.375 \\ 8\overline{)\,3.000} \\ -24 \\ \hline 60 \\ -56 \\ \hline 40 \\ -40 \\ \hline 0 \end{array}$$

← This is a terminating decimal

← The remainder is zero.

$\frac{3}{8} = 0.375$

$$\begin{array}{r} 0.133 \\ 15\overline{)\,2.000} \\ -15 \\ \hline 50 \\ -45 \\ \hline 50 \\ -45 \\ \hline 5 \end{array}$$

← This is a non-terminating decimal

← The remainder is never zero.

$\frac{2}{15} = 0.133\ldots = 0.1\bar{3}$

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An irrational number is a number whose decimal representation never terminates or repeats.

For example:

$0.01001000100001\ldots$, $\sqrt{7} = 2.6457513\ldots$, and $\pi = 3.1415927\ldots$

The real number system is summarized in the figure given below.

2.1.3 Depicting Real Numbers on the Number line

The number line helps in visualizing the set of real numbers. We assume that for any point on a line there is a real number, and that for every real number there is a point on the line. The number associated with a point is called the coordinate of that point on the line, and the point

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Zero is neither positive nor negative.

A number > 0 is positive.

A number < 0 is negative.

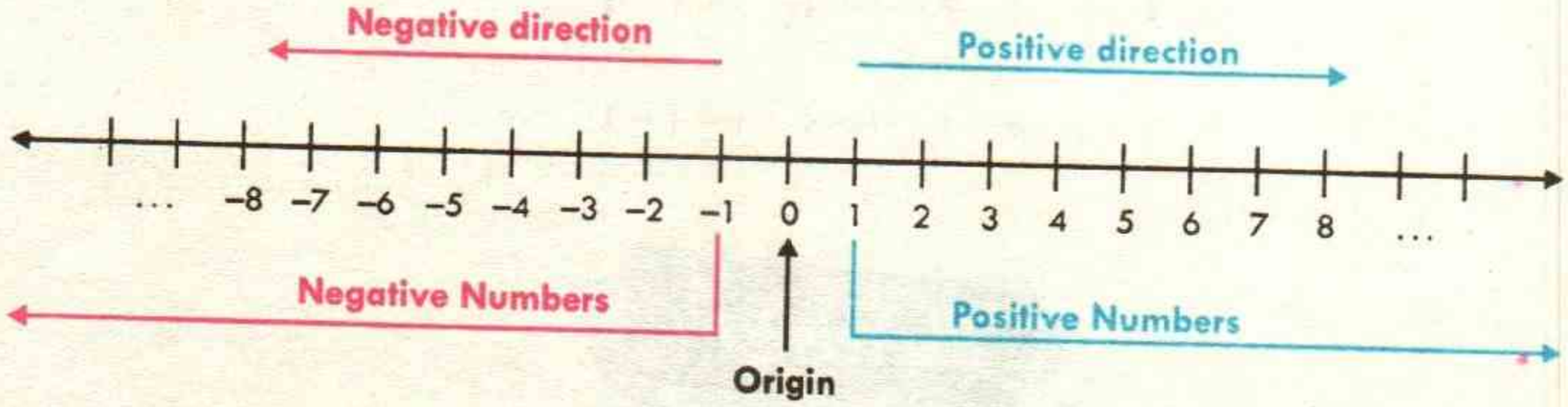
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is called the graph of the number. Numbers to the right of 0 are called positive, while those to the left are negative. The graph of the value zero is called the origin.

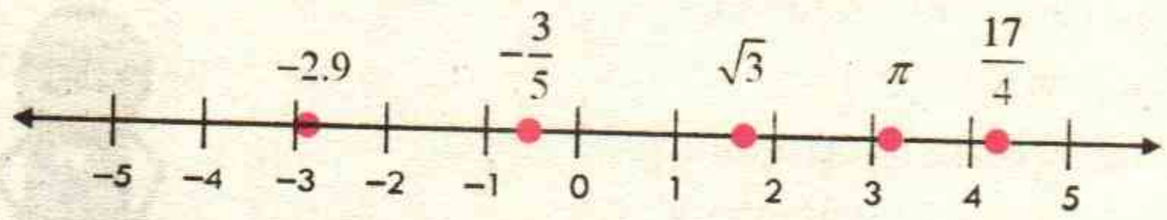
In order to draw a real number line, we first draw a straight line and label a convenient point on the line with 0. Then we mark off equally spaced distances in both directions from 0. Label the points to the right of 0 with the numbers 1, 2, 3,... The points to the left of 0 we label, in order, $-1, -2, -3, \ldots$.

Here is what it looks like:



The following figure shows the graph for the coordinate values

$-2.9, -\frac{3}{5}, \sqrt{3}, \pi$ and $\frac{17}{4}$.



2.1.4 Demonstrating a number with terminating and non-terminating recurring decimals on the number line

There is an end to the division process when we divide 1 by 8 , i.e. $1 \div 8 = 0.125$.

Therefore, 0.125 is called a terminating decimal.

Some numbers such as $\frac{5}{27}$ (= 0.185 185 185 ...) cannot be written as decimal with a finite number of decimal places, but result in a number of digits that repeat infinitely. These are called non-terminating recurring decimals.

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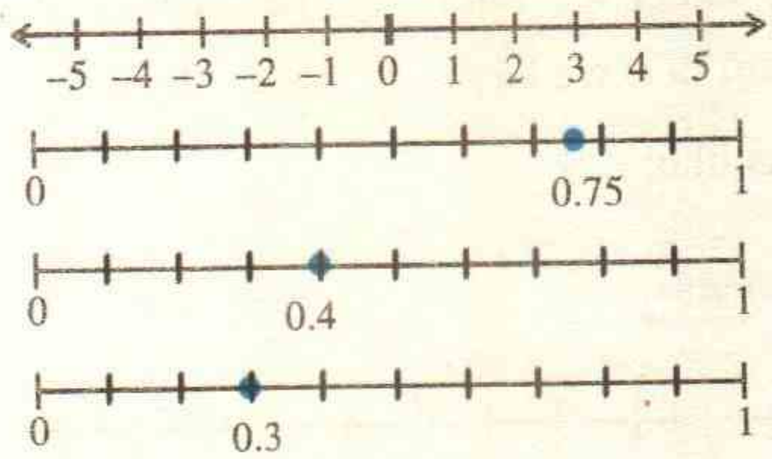
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Example 2

Represent the following real numbers on the number line $\frac{3}{4}, \frac{2}{5}, \frac{1}{3}$.

Solution:

$$\frac{3}{4} = .75, \frac{2}{5} = .4, \quad \frac{1}{3} = 0.333\ldots \approx 0.3$$



Exercise 2.1

In Questions 1-10, consider the numbers

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$$2, 5, 3, \frac{5}{7}, -1.96, \sqrt{36}, -\frac{7}{6}, \sqrt{3}, -9, 1, \sqrt{7}, -\sqrt{14}, \pi, 4\frac{2}{3}, 0.333\ldots$$

1. Which are whole numbers?
2. Which are integers?
3. Which are irrational numbers?
4. Which are natural numbers?
5. Which are rational numbers?
6. Which are real numbers?
7. Which are rational numbers but not integers?
8. Which are integers but not whole numbers?
9. Which are integers but not natural numbers?
10. Which are real numbers but not integers?
11. Write the decimal representation of each of the following numbers.

$$\frac{1}{6}, \frac{5}{6}, \frac{2}{9}, \frac{1}{8}$$

12. Depict each number on a number line. $\frac{1}{3}, \frac{1}{4}, \frac{1}{9}, \frac{1}{10}$



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2.2 PROPERTIES OF REAL NUMBERS

At this stage, we are aware of that the set R of real numbers is the union of two disjoint sets. The set Q of rational numbers and the set Q' of irrational numbers i.e. $R = Q \cup Q'$ and $Q \cap Q' = \varnothing$. Now we want to specify a set of properties which the set R will describe.

2.2.1 Real Number System

2.2.1.1 Addition Properties

i. **Closure Property w.r.t addition**

$a + b \in R$,if $a, b \in R$

i.e. The sum of two real numbers is again a real number.

ii. **Associative Property w.r.t addition**

$a+(b+c)=(a+b)+c$,if $a, b, c \in R$

iii. **Additive Identity**

if $a \in R$, there exists $0 \in R$ such that

$$a + 0 = 0 + a = a$$

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iv. **Additive Inverse**

$\forall a \in R$, there exists an element a', such that

$a+a'=a'+a=0$, a' is called the additive inverse of a and is denoted by $-a$.

Thus $a+(-a)=(-a)+a=0$

v. **Commutative Property w.r.t addition**

$a+b=b+a$,if $a, b \in R$

2.2.1.2 Multiplication Properties

i. **Closure Property w.r.t multiplication**

$a \cdot b \in R$,if $a, b \in R$.

i.e. The product of two real numbers is again a real number.

ii. **Associative Property w.r.t multiplication**

$a(bc) = (ab)c$, if $a, b, c \in R$.

iii. **Multiplicative Identity**

There exists a number $1 \in R$ such that

$a \cdot 1 = 1 \cdot a = a$, if $a \in R$.

The element '1' is called multiplicative identity.

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iv. **Multiplicative Inverse**

For each $a \in R$, $a \neq 0$, there exists $a^{-1} \in R$ such that

$a \cdot a^{-1} = a^{-1} \cdot a = 1$

v. **Commutative property w.r.t to multiplication.**

$a.b = b.a \ , \forall \ a, b \in R$

2.2.1.3 Distributive Property of Multiplication over addition

if $a, b, c \in R$,

$a(b+c) = ab + ac$,if $a, b \in R$

and $(b+c)a = ba + ca$,if $a, b \in R$

2.2.1.4 Properties of Equality of Real Numbers

i. **Reflexive Property**

$a = a$,if $a \in R$

ii. **Symmetric Property**

$a = b \Rightarrow b = a$,if $a, b \in R$

iii. **Transitive Property**

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$\forall \ a, b, c \in R \ \ a = b$ and $b = c \Rightarrow a = c$, if $a, b, c \in R$

iv. **Additive Property**

$a = b \Rightarrow a + c = b + c$,if $a, b, c \in R$

v. **Multiplicative Property**

$a = b \Rightarrow ac = bc$,if $a, b \in R$

vi. **Cancellation Property w.r.t Addition**

$a, b, c \in R$

$a + c = b + c \Rightarrow a = b$

vii. **Cancellation Property w.r.t Multiplication**

if $ac = bc \Rightarrow a = b$,if $a, b, c \in R$

2.2.2 Properties of Inequality of Real Numbers

i. **Trichotomy Property**

if $a, b \in R$, exactly one of the following is true

$a < b$, $a = b$, $a > b$

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ii. Transitive Property

if $a,b,c \in R$, $a > b$ and $b > c \Rightarrow a > c$

iii. Additive Property

if $a,b,c \in R$,

if $a < b \Rightarrow a + c < b + c$ and $a > b \Rightarrow a + c > b + c$

iv. Multiplicative Property

if $a,b,c \in R$,

For $c > 0$,	For $c < 0$,
if $a < b \Rightarrow ac < bc$	$a < b \Rightarrow ac > bc$

Example 3

Name the property.

1. $5 \cdot \frac{1}{5} = 1$
2. $19 + 6 \in R$
3. If $2x = 8$ then $x = 4$
4. If $7 = n$ then $n = 7$
5. $2 + (6 + 3) = (2 + 6) + 3$
6. $\sqrt{7} + 0 = \sqrt{7}$
7. $\frac{1}{5} = \frac{1}{5}$
8. $4 + (-4) = 0$
9. $2x + 5x = (2 + 5)x$

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Solution:

1. $5 \cdot \frac{1}{5} = 1 \Rightarrow 5$ & $\frac{1}{5}$ are multiplicative inverses of each other.
2. $19 + 6 \in R \Rightarrow$ This is closure property w.r.t addition.
3. If $2x = 8$ then $x = 4 \Rightarrow$ Here multiplicative property has been used.
4. If $7 = n$ then $n = 7 \Rightarrow$ This is symmetric property of equality i.e. if $a = b$ then $b = a$.
5. $2 + (6 + 3) = (2 + 6) + 3 \Rightarrow$ This is due to associative property of addition.
6. $\sqrt{7} + 0 = \sqrt{7} \Rightarrow$ Property of additive identity.
7. $\frac{1}{5} = \frac{1}{5} \Rightarrow$ Reflexive property of equality.
8. $4 + (-4) = 0 \Rightarrow 4$ & -4 are additive inverses of each other.
9. $2x + 5x = (2 + 5)x \Rightarrow$ This is distributive property of equality.

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Example 4

Solve the following equation using properties of real numbers.

$2x-5=3x+4$

Solution:

Given $2x-5=3x+4$

$\Rightarrow \quad 2x-5+5=3x+4+5$ $\quad \because a=b \Rightarrow a+c=b\pm c$

$\Rightarrow \quad 2x+5-5=3x+9$ Commutative property for addition.

$\Rightarrow \quad 2x+0=3x+9$ $\because$ 5 & -5 are additive inverses of each other.

$\Rightarrow \quad 2x=3x+9$ $\because$ 0 is the additive identity.

$\Rightarrow \quad 3x+9=2x$ Symmetric property.

$\Rightarrow \quad 3x+9+(-2x)=2x+(-2x)$ $\because a=b \Rightarrow a+c=b+c$

$\Rightarrow \quad 3x+9+(-2x)=0$ $\because 2x$ & $-2x$ are additive inverses of each other.

$\Rightarrow \quad 3x+(-2x)+9=0$ Commutative property.

$\Rightarrow \quad [3+(-2)]x+9=0$ Distributive property .

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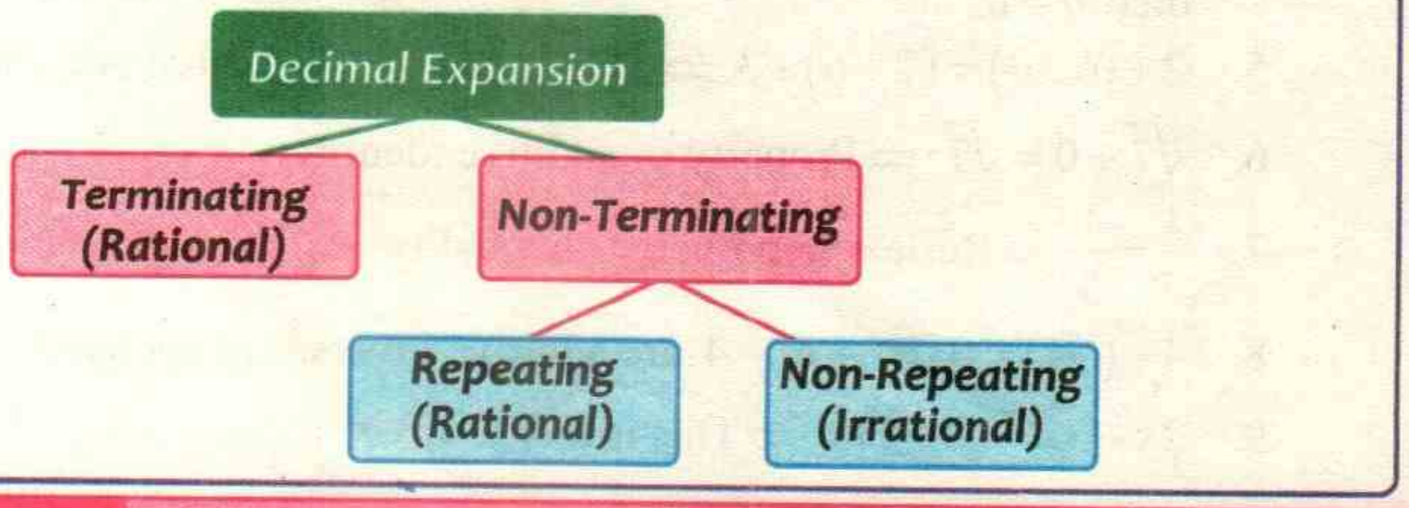
$\Rightarrow \quad 1\cdot x+9=0$ or $x+9=0$ $\because$ 1 is the multiplicative identity.

$\Rightarrow \quad x+9+(-9)=0+(-9)$ $\because \quad a=b \Rightarrow a+c=b+c$

$\Rightarrow \quad x+0=-9$ $\because$ 9 & -9 are additive inverses of each other and 0 is the additive identity.

$\Rightarrow \quad x=-9$ $\because$ 0 is the additive identity.

Remember



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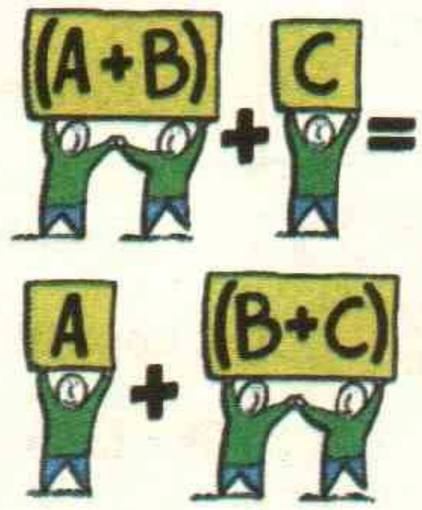
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Exercise 2.2

1. Name the properties used in the following equations.

i) $1+(4+3)=(1+4)+3$ ii) $5(a+b)=5a+5b$

iii) $a+0=0+a=a$ iv) $5\times\frac{1}{5}=\frac{1}{5}\times5=1$



2. Write the missing number.

i) $2+(\text{----}+4)=(2+6)+4$

ii) $7+(4+2)=13$, so $(7+4)+2=$ ----

iii) $9\times(3\times4)=108$, so $(9\times3)\times4=$ ----

iv) $5\times(8\times9)=(5\times\text{----})\times9$

3. Choose the correct option.

i. $8\times(6\times7)$ is equal to:

a. $8\times6-7$ b. $8-(6-7)$ c. 8×12 d. $(8\times6)\times7$

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ii. Which one of the following illustrates the Associative Law of addition?

a. $3+(2+4)=(4+4)+1$ b. $3+(2+4)=(3+2)+4$

c. $3+(2+4)=(5+2)+2$ d. $3+(2+4)=(2+6)+1$

iii. Which of the following illustrates the Associative Law of multiplication?

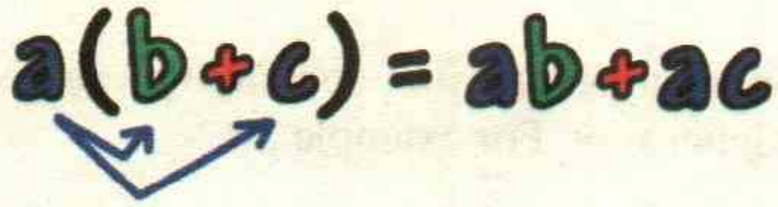
a. $4\times(3\times6)=(6\times6)\times2$ b. $4\times(3\times6)=(3\times12)\times2$

c. $4\times(3\times6)=(4\times3)\times6$ d. $4\times(3\times6)=(3\times8)\times3$

4. Do this with and without using distributive property.

i) $39\times63+39\times37$ ii) $81\times450+81\times550$

iii) $50\times161-50\times81$ iv) $827\times60-327\times60$



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Activity

Mahnoor wanted to work $\frac{40}{9}$ but she didn't have a calculator. She did the following calculations:

$\frac{40}{9} = \frac{40}{4+5}$ ⇒Line 1 $= \frac{40}{4} + \frac{40}{5}$ ⇒Line 2

$= 10 + 8$ ⇒Line 3 $= 18$ ⇒Line 4

Mahnoor's friend Tahira told her that she thought Mahnoor's answer was wrong, and the correct answer should be 4.44. Which line of Mahnoor's working was wrong?

Who was correct?

2.3 RADICALS AND RADICANDS

2.3.1 Concept of Radicals and Radicands

If $\sqrt[n]{a}$ is a real number and n is a positive integer greater than 1 then.

$$\sqrt[n]{a} = a^{\frac{1}{n}} \quad \text{.....(1)}$$

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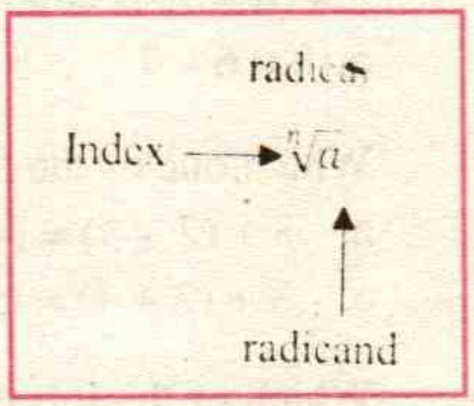
where n is called the index of the radical,

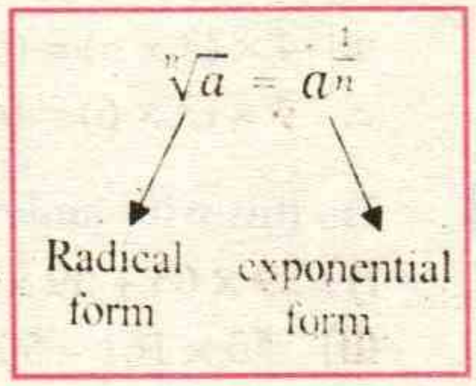
$\sqrt{\ }$ is called the radical symbol,

a is called the radicand,

$\sqrt[n]{a}$ is the radical form of the nth root of a,

$a^{\frac{1}{n}}$ is the exponential form of the nth root of a.





If $n = 2$, then it becomes square root and in this case we simply write $\sqrt{a}$ instead of $\sqrt[2]{a}$. So if n is not given it will be taken as 2.

If $n = 3$, then it is called cube root . For example $\sqrt[3]{7}$.

If $n = 4$, then it is called 4th root . For example $\sqrt[4]{16}$.

And so on.

A radical expression is any expression containing a radical, whether it is a square root, cube root, or higher root. For example ,

$\sqrt{3}, \sqrt[3]{5}, \sqrt{2} + \sqrt{3}, 5\sqrt{a} - 3\sqrt[4]{b}, \frac{3 - \sqrt[4]{3}}{\sqrt[5]{4}}$ are all radical expressions.

The following points should be noted about Eq.(1).

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Some frequently used radicals are given in the following table.

Square Roots	Cube Roots	Fourth Root
$\sqrt{1}=1$	$\sqrt[3]{1}=1$	$\sqrt[4]{1}=1$
$\sqrt{4}=2$	$\sqrt[3]{8}=2$	$\sqrt[4]{16}=2$
$\sqrt{9}=3$	$\sqrt[3]{27}=3$	$\sqrt[4]{81}=3$
$\sqrt{16}=4$	$\sqrt[3]{64}=4$	$\sqrt[4]{256}=4$
$\sqrt{25}=5$	$\sqrt[3]{125}=5$	$\sqrt[4]{625}=5$
$\sqrt{36}=6$	$\sqrt[3]{216}=6$	$\sqrt[4]{1296}=6$

The following example shows an important concept about radicals.

Example 5

What is the difference between $x^2=16$ and $x=\sqrt{16}$?

Solution:

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In the first, we are asked what numbers squared are 16, so x can be 4 or -4 as $4^2=16$ and $(-4)^2=16$. We write this as $x=\pm 4$.

In the second, $x=4$, because x is the principal square root of 16, which has always a positive value.

Another form of radicals:

If $\sqrt[n]{a}$ represents a real number and $\frac{m}{n}$ is a positive rational number, $n \geq 2$, then

$$a^{\frac{m}{n}}=\sqrt[n]{a^m} \text{ or } \left(\sqrt[n]{a}\right)^m \quad \ldots\ldots\ldots\ldots(2)$$

where $a^{\frac{m}{n}}$ is the exponential form of the expression, and

$\sqrt[n]{a^m}$ is the radical form of the expression.

If $m=n$ then (2) becomes

$$\sqrt[n]{a^n}=a \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

The following points should be noted about Eq.(1).

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a) **If a is positive, then the nth root of a is also positive.**

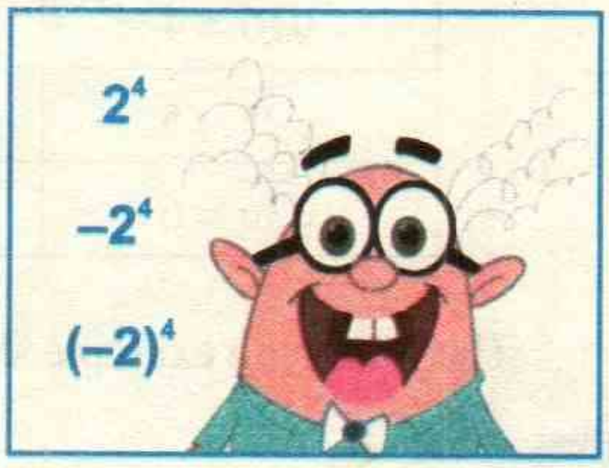
Example 6

$$\sqrt[3]{64} = \sqrt[3]{(4)^3} = 4.$$

b) **If a is negative, then n must be odd for the nth root of a to be a real number.**

Example 7

$$\sqrt[3]{-64} = \sqrt[3]{(-4)^3} = -4.$$



c) **If a is zero, then $\sqrt[n]{0} = 0$.**

Properties of radicals:

Product rule of radicals: $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$.

This rule is used for simplifying as well as for multiplying radicals.

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Example 8

Simplify $\sqrt{6x}\sqrt{6y^2}$.

Solution:

$$\sqrt{6x}\sqrt{6y^2}$$

$$= \sqrt{(6x)(6y^2)} = \sqrt{(36y^2)(x)} = \sqrt{36}\sqrt{y^2}\sqrt{x} = 6y\sqrt{x} = \sqrt{6^2y^2.x} = 6y\sqrt{x}$$

Quotient rule of radicals: $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

Example 9

Simplify $2\sqrt{\frac{150xy}{3x}}$

Solution:

$$\frac{2\sqrt{150xy}}{\sqrt{3x}} = 2 \cdot \sqrt{\frac{150xy}{3x}} = 2 \cdot \sqrt{50y} = 2 \cdot \sqrt{25 \cdot 2y} = 2 \cdot \sqrt{25} \cdot \sqrt{2y} = 2 \cdot 5\sqrt{2y} = 10\sqrt{2y}$$

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2.3.2 Difference between radical form and exponential form of an expression

(a) Radical form of an expression

The number or quantity that is written under a radical sign($\sqrt{\ }$ or $\sqrt[n]{\ }$) is called radical form of an expression.

For example $\sqrt{9}$ is the radical form of 3.

(b) Exponential form of an expression.

The number or quantity that is written in the form of exponent is called exponential form of an expression.

For example 3^2 is exponential form of 9.

2.3.3 How to transform an expression given in radical form to exponential form

Using $\sqrt[n]{a} = a^{\frac{1}{n}}$ and $a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a^m}\right)$ we can transform radical form to exponential form.

Example 10

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Write each expression in exponential form.

i. $\sqrt[8]{c^3}$ ii. $\sqrt[3]{16}$ iii. $(\sqrt[4]{9-4b})^5$

Solution:

i. $\sqrt[8]{c^3} = c^{\frac{3}{8}}$ ii. $\sqrt[3]{16} = \sqrt[3]{2^4} = 2^{\frac{4}{3}}$ iii. $(\sqrt[4]{9-4b})^5 = (9-4b)^{\frac{5}{4}}$

2.3.4 How to transform an expression given in exponential form to radical form

Using Eq(1) and Eq(2) we can transform exponential form of an expression to radical form of an expression.

Example 11

Write each expression in radical form.

i. $c^{\frac{3}{4}}$ ii. $(2x+5)^{\frac{2}{3}}$ iii. $(9-4b)^{\frac{5}{4}}$

Solution:

i. $c^{\frac{3}{4}} = \sqrt[4]{c^3}$ ii. $(2x+5)^{\frac{2}{3}} = \sqrt[3]{(2x+5)^2}$ iii. $(9-4b)^{\frac{5}{4}} = \sqrt[4]{(9-4b)^5}$

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Exercise 2.3

1. Write down the index and radicand for each of the following expressions.

(i) $\sqrt{\frac{11}{y}}$ (ii) $\sqrt[3]{\frac{13}{3x}}$ (iii) $\sqrt[5]{ab^2}$

2. Transform the following radical forms into exponential forms. Do not simplify.

(i) $\sqrt{36}$ (ii) $\sqrt{1000}$ (iii) $\sqrt[3]{8}$

(iv) $\sqrt[n]{q}$ (v) $\sqrt{(5-6a^2)^3}$ (vi) $\sqrt[3]{-64}$

3. Transform the following exponential form of an expression into radical form.

(i) $-7^{\frac{1}{3}}$ (ii) $x^{-\frac{3}{2}}$ (iii) $(-8)^{\frac{1}{5}}$

(iv) $y^{\frac{3}{4}}$ (v) $b^{\frac{4}{5}}$ (vi) $(3x)^{\frac{1}{q}}$

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4. Simplify.

(i) $\sqrt[3]{125x}$ (ii) $\sqrt[3]{\frac{8}{27}}$ (iii) $\sqrt{\frac{625x^3y^4}{25xy^2}}$

(iv) $\sqrt{(3y-5)^2}$ (v) $6\sqrt{18}$ (vi) $\sqrt[3]{54x^3y^3z^2}$

Sometimes you can simplify in order to get common radicands

$5\sqrt{3}+\sqrt{12}$

$5\sqrt{3}+\sqrt{4\cdot 3}$

$5\sqrt{3}+2\sqrt{3}$

$7\sqrt{3}$

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2.4 LAWS OF EXPONENTS / INDICES

2.4.1 Base, Exponent and Value

If n is a positive integer, a^n represents the product of n factors each of which is a. Thus $a^4 = a \cdot a \cdot a \cdot a$.

In a^n, a is called the base and n the exponent or index. We may read a^n as the "nth power of a" or "a to the nth." If n = 2 we read a^2 as "a squared"; a^3 is read "a cubed".

Example 12

$x^4 = xxxx$, $2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32$, $(-3)^3 = (-3)(-3)(-3) = -27$

2.4.2 Laws of Exponents

If $a, b \in R$ and $m, n \in N$, the following laws hold.

i) Multiplication Law of Indices.

To multiply powers of the same base, keep the base the same and add the exponents. $a^m \cdot a^n = a^{m+n}$

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Example 13

$$3^3 \cdot 3^2 = 3^{3+2} = 3^5$$

$$5^2 \cdot 5^4 = 5^{2+4} = 5^6$$

ii) Division Law of Indices

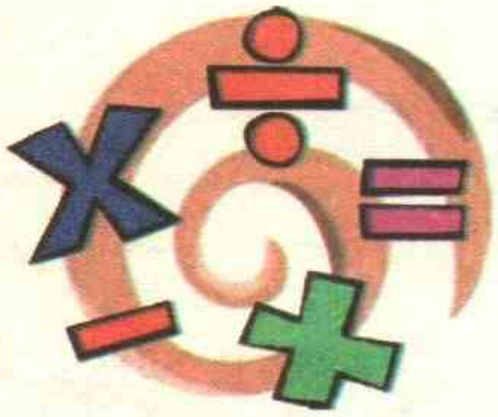
To divide two expressions with the same base and different exponents, keep the base the same and subtract the exponents.

$$\frac{a^m}{a^n} = a^{m-n}, \text{ where } m > n \text{ and } a \neq 0$$

Example 14

$$\frac{4^7}{4^2} = 4^{7-2} = 4^5$$

$$\frac{5^7}{5^3} = 5^{7-3} = 5^4$$



iii) Power Law of Indices

To raise an exponential expression to a power keep the base the same and multiply the exponents $(a^m)^n = a^{mn}$.

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Example 15

$$(3^3)^2 = 3^{3\times2} = 3^6$$

$$(4^5)^3 = 4^{5\times3} = 4^{15}$$

iv) More Laws of Indices

To multiply different bases with the same exponent, keep the exponent constant and multiply the expressions with the same exponent. $(ab)^n = a^n b^n$

Example 16

$$(4\times2)^3 = 4^3 \times 2^3$$

$$(3\times5)^4 = 3^4 \times 5^4$$

v) To divide the two expressions with the same exponent, keep the exponent the same and divide the two expressions. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}, b \neq 0$

Example 17

$$\left(\frac{3}{2}\right)^5 = \frac{3^5}{2^5}$$

$$\left(\frac{4}{5}\right)^2 = \frac{4^2}{5^2}$$

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vi) Zero and Negative Indices

Any non-zero number raised to the zero power equals one.

$a \neq 0$ then $a^0 = 1$

Example 18

$5^0 = 1, \quad 2^0 = 1, \quad 100^0 = 1$ If n is a positive integer, we define

$a^{-n} = \frac{1}{a^n}$, where $a \neq 0$.

Example 19

$$2^{-4} = \frac{1}{2^4} = \frac{1}{16}, \frac{1}{2^{-3}} = 3^3 = 27, -4x^{-2} = \frac{-4}{x^2}, (a+b)^{-1} = \frac{1}{(a+b)}$$

If n is a positive integer and if a and b are such that $a^n = b$, then a is said to be an nth root of b.

If b is positive, there is only one positive number such that $a^n = b$. We write this positive number as $a^n = b$ and call it the principal nth root of b.

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Exercise 2.4

1. Write the base, exponent and value of the following.

i) $(2)^{-9} = \frac{1}{512}$ ii) $\left(\frac{a}{b}\right)^p = \frac{a^p}{b^p}$ iii) $(-4)^2 = 16$

2. If a, b denote the real numbers then simplify the following.

i) $a^3 \times a^5$ ii) $\left(\frac{b}{a}\right)^{\frac{3}{2}}\left(\frac{b}{a}\right)^{-\frac{2}{3}}$ iii) $(-a)^4 \times (-a)^3$

iv) $\left(-2a^2b^3\right)^3$ v) $a^3(-2b)^2$ vi) $(a^2b)(a^2b)$

vii) $\frac{a^0b^0}{2}$ viii) $\left(-3a^2b^2\right)^2$ ix) $\left(\frac{a^2}{b^4}\right)^{\frac{3}{2}}$

3. Simplify the following.

i) $\frac{7^6}{7^4}$ ii) $\frac{2^4 \cdot 5^3}{10^2}$ iii) $\left\{\frac{(a+b)^2 \cdot (c+d)^3}{(a+b) \cdot (c+d)^2}\right\}^3$

iv) $\left(\sqrt[3]{a}\right)^{\frac{1}{2}}$ v) $\sqrt[5]{x^5} \cdot \sqrt[4]{x^4}$

4. Simplify the following in such a way that no answers should contain fractional or negative exponents.

i) $\left(\frac{25}{81}\right)^{\frac{1}{2}}$ ii) $\frac{(ab)^{\frac{1}{b}}}{\left(\frac{1}{ab}\right)^{\frac{1}{a}}}$

iii) $\frac{2^{p+1} \cdot 3^{2p-q} \cdot 5^{p+q} \cdot 6^q}{6^p \cdot 10^{q+2} \cdot 15^p}$ iv) $\left(\frac{x^p}{x^q}\right)^{p+q} \cdot \left(\frac{x^q}{x^r}\right)^{q+r} \cdot \left(\frac{x^r}{x^p}\right)^{r+p}$

5. Prove that $\left(\frac{4^5 \cdot 64^3 \cdot 2^3}{8^5 \cdot (128)^2}\right)^{\frac{1}{2}} = 2.$

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2.5 COMPLEX NUMBERS

Complex numbers play a very important role in Mathematics and Sciences. The use of complex numbers is indispensable in Physics, Aeronautical and Electrical Engineering especially in the analysis of Electric circuits.

2.5.1 Definition of Complex Number

A number of the form $a+bi$ where a and b are real numbers is called a complex number. The real number a is called the real part of $a + bi$ and the real number b is called the imaginary part of $a+bi$.

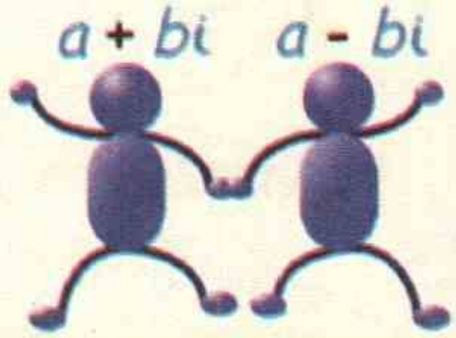
Examples:

$2+3i$, $4+5i$, $\sqrt{2}+\sqrt{3}i$ etc are complex numbers.

2.5.2 Conjugate of a Complex Numbers

The conjugate (or complex conjugate) of the complex number $a+ib$ is $a-bi$. We denote the conjugate of $a+ib$ by $\overline{a+bi}=a-bi$.

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Thus,

Conjugate of $Z=a+bi$ is $\overline{Z}=a-bi$

For example: If $Z=5+8i$, then $\overline{Z}=5-8i$

2.5.3 Equality of Two Complex Numbers

Let $Z_1=a+ib$ and $Z_2=c+id$ be two complex numbers then $Z_1=Z_2$ if and only if $a=c$ and $b=d$ i.e. the two complex numbers will be equal if their real parts are equal and imaginary parts are equal.

2.6 OPERATIONS ON COMPLEX NUMBERS

i) Addition of complex numbers

Let $Z_1=a+ib$ and $Z_2=c+id$ then the addition of Z_1 and Z_2 denoted by Z_1+Z_2 and define d as $Z_1 + Z_2 = a + ib + c + id$

$$=(a+c)+i(b+d)$$

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ii) Subtraction of complex numbers

Let $Z_1 = a + ib$ and $Z_2 = c + id$ then subtraction of Z_1 and Z_2 denoted by $Z_1 - Z_2$ is given as $Z_1 - Z_2 = (a + ib) - (c + id)$

$$= a + ib - c - id$$

$$= (a - c) + i(b - d)$$

iii) Multiplication of complex numbers

Let $Z_1 = a + ib$ and $Z_2 = c + id$. If $Z_1 Z_2$ denotes the product of Z_1 and Z_2 then $Z_1 Z_2 = (a + ib)(c + id)$

$$= (ac + iad + ibc + i^2 bd)$$

$$= ac + i(ad + bc) + (-1)bd \qquad \text{as } i^2 = -1$$

$$= (ac - bd) + i(ad + bc)$$

iv) Division of complex numbers

Let $Z_1 = a + ib$ and $Z_2 = c + id$ then the quotient $\frac{Z_1}{Z_2}$ is given as

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$$\frac{Z_1}{Z_2} = \frac{a + ib}{c + id} = \frac{a + ib}{c + id} \times \frac{c - id}{c - id} \qquad \text{(Multiplying and dividing by } c - id \text{)}$$

$$= \frac{ac + ibc - iad - i^2 bd}{c^2 - i^2 d^2}$$

$$= \frac{ac + i(bc - ad) + bd}{c^2 + d^2}$$

$$= \frac{ac + bd}{c^2 + d^2} + \frac{i(bc - ad)}{c^2 + d^2}$$



Example 20

Add the following numbers $5 + 2i,\ 3 + i$.

Solution:

Let $Z_1 = 5 + 2i$ and $Z_2 = 3 + i$

then $Z_1 + Z_2 = 5 + 2i + 3 + i = (5 + 3) + (2 + 1)i = 8 + 3i$

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Example 21

Subtract $7+3i$ from $5-8i$.

Solution:

$$5-8i-(7+3i) = 5-8i-7-3i = (5-7)-(8+3)i = -2-11i$$

Example 22

Let $Z_1 = 2-i$ and $Z_2 = 3+i$, then find Z_1Z_2.

Solution:

$$Z_1Z_2 = (2-i)(3+i)$$
$$= 6+2i-3i-i^2$$
$$= 6+(2-3)i-(-1) \qquad as\ i^2=-1$$
$$= 6-i+1 = 7-i$$

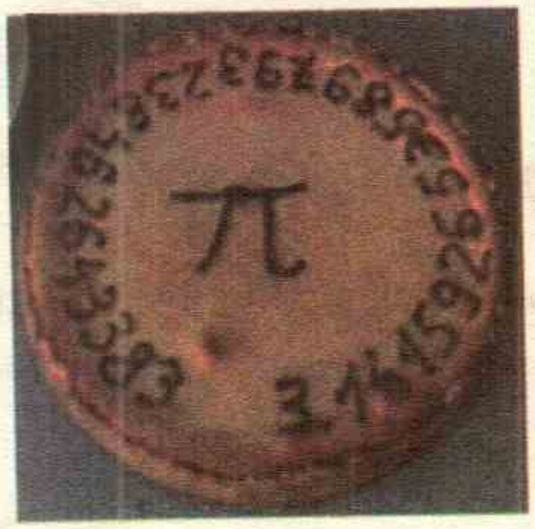
Example 23

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Let $Z_1 = 3+4i$ and $Z_2 = 3-2i$ find the quotient $\frac{Z_1}{Z_2}$.

Solution:

$$\frac{Z_1}{Z_2} = \frac{3+4i}{3-2i}$$
$$= \frac{(3+4i)(3+2i)}{(3-2i)(3+2i)} \qquad \text{(Multiplying and dividing by } 3+2i\text{)}$$
$$= \frac{9+6i+12i+8i^2}{(3)^2-(2i)^2}$$
$$= \frac{9+18i-8}{9+4}$$
$$= \frac{1+18i}{13}$$
$$= \frac{1}{13}+\frac{18}{13}i$$



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Exercise 2.5

1. Add the following complex numbers.

i) $8+9i,\ 5+2i$

ii) $6+3i,\ 3-5i$

iii) $2i+3,\ 8-5\sqrt{-1}$

iv) $\sqrt{3}+\sqrt{2}\,i,\ 3\sqrt{3}-2\sqrt{2}\,i$

2. Subtract.

i) $-2+3i$ from $6-3i$

ii) $9+4i$ from $9-8i$

iii) $1-3i$ from $8-i$

iv) $6-7i$ from $6+7i$.

3. Multiply the following complex numbers.

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i) $1+2i,\ 3-8i$

ii) $2i,\ 4-7i$

iii) $5-3i,\ 2-4i$

iv) $\sqrt{2}+i, 1-\sqrt{2}i$

4. Divide the first complex number by the second.

i) $Z_1=2+i,\ \ Z_2=5-i$

ii) $Z_1=3i+4,\ \ Z_2=1-i$

5. Perform the indicated operations and reduce to the form a + bi.

i) $(4-3i)+(2-3i)$

ii) $(5-2i)-(4-7i)$

iii) $2i(4-5i)$

iv) $(2-3i)\div(4-5i)$

6. Find the complex conjugate of the following complex numbers.

$-8-3i,\ -4+9i, 7+6i,\ \ \sqrt{5}-i$

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REVIEW EXERCISE 2

1. Tell whether the following are true or false.

i) $3^{\frac{1}{3}} = \sqrt{3}$ ii) $2^{\frac{2}{3}} = \sqrt[3]{4}$ iii) $\sqrt{49} = \sqrt{7}$ iv) $\sqrt[3]{27} = x^3$

2. Select the correct answer.

i) The additive inverse of $\sqrt{5}$ is

a) $-\sqrt{5}$ b) $\frac{1}{\sqrt{5}}$ c) $\sqrt{-3}$ d) -5

ii) $2(3+4) = 2\times3+2\times4$, here the property used is

a) Commutative b) Associative

c) Distributive d) Closure

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iii) $\sqrt{-1}\times\sqrt{-1} =$

a) 1 b) i c) -1 d) 0

iv) Which of the following represents numbers greater than -3 but less than 6?

a) $\{x: -3 > x > 6\}$ b) $\{x: -3 \le x \le 6\}$

c) $\{x: -3 < x < 6\}$ d) $\{x: -3 \ge x \ge 6\}$

v) If $n = 8$ and $16\times2^m = 4^{n-8}$, then $m = ?$

a) -4 b) -2 c) 0 d) 8

vi) $(i)\cdot(-i) =$

a) 1 b) -1 c) $-i$ d) i

vii) The multiplicative identity of real numbers is

a) 0 b) 1 c) -1 d) R

viii) 0 is

a) a positive integer b) a negative integer

c) neither positive nor negative d) not an integer

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ix) For $i=\sqrt{-1}$, if $3i(2+5i)=x+6i$, then $x=?$

a) 5 b) -15 c) $5i$ d) $15i$

x) $\sqrt{0}=$

a) 0 b) 1 c) -1 d) not defined

xi) $\sqrt{-(-9)^2}=?$ (Note: $i=\sqrt{-1}$) =

a) 9 b) $9+i$

c) $9-i$ d) $9i$

3. Simplify each of the following.

i) $\left(-\frac{2}{3}\right)^3$ ii) $(-2)^3\cdot(3)^2$ iii) $-3\sqrt{48}$ iv) $\frac{5}{\sqrt[3]{9}}$

4. Multiply $8i,-8i$.

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5. Divide $2-5i$ by $1-6i$.

6. Name the property used $7\times\frac{1}{7}=\frac{1}{7}\times7=1$.

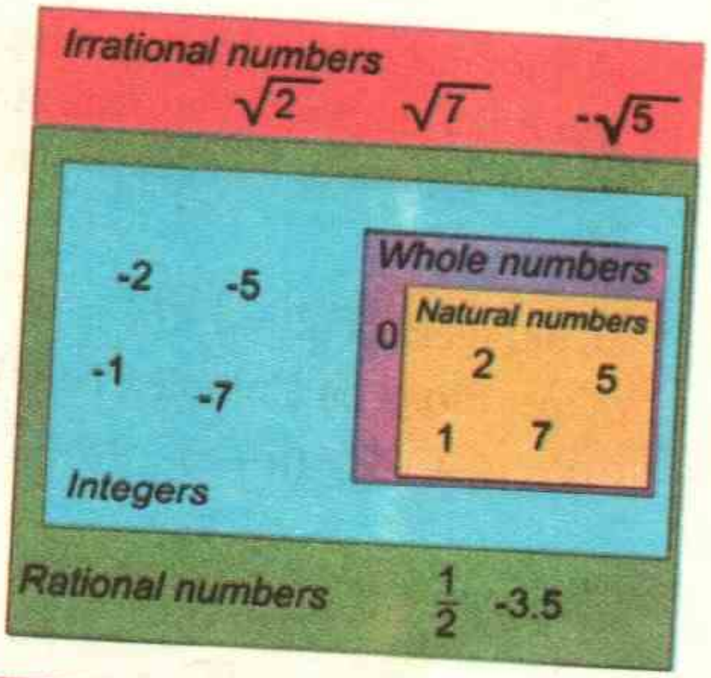
1089 x 9 = 9801

Think

7. Use laws of exponents to simplify.

$$\frac{(81)^n\cdot3^5+(3)^{4n-1}(243)}{(9^{2n})(3^3)}$$

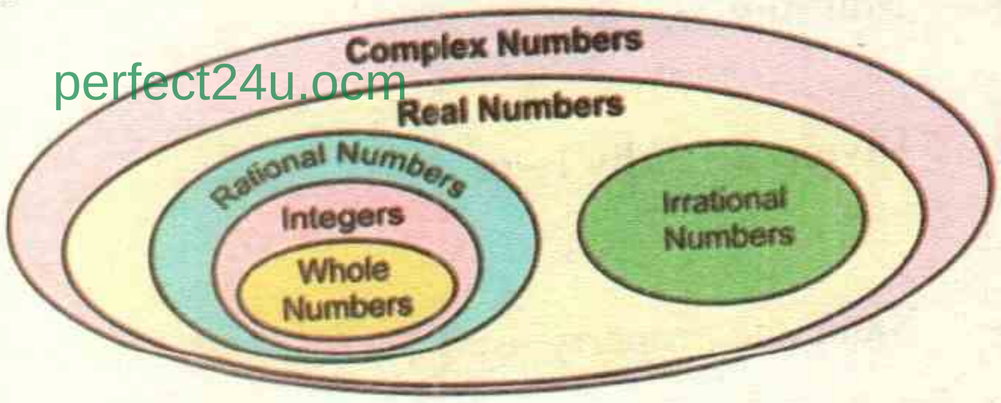
Real numbers



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SUMMARY

- Some important sets :
  - **Set of Natural Numbers** $N = \{1, 2, 3, ...\}$
  - **Set of Whole Numbers** $W = \{0, 1, 2, 3, ...\}$
  - **Set of Integers** $Z = \{0, \pm 1, \pm 2, \pm 3, ...\}$
  - **Set of Rational Numbers** $Q = \left\{ \frac{p}{q} \mid p, q \in Z, q \neq 0 \right\}$
- Set of Irrational Numbers, denoted by Q', consists of all those real numbers which are not rational.
- $Q \cup Q' = R$
- Laws of Exponents / Indices:
- $a^m \cdot a^n = a^{m+n}$
- $(ab)^n = a^n b^n$
- $(a^m)^n = a^{mn}$
- $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}, b \neq 0$



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- $\frac{a^m}{a^n} = a^{m-n}$, where $m > n$ and $a \neq 0$
- $a \neq 0$ then $a^0 = 1$
- A number of the form $a + bi$ where a and b are real numbers is called a complex number. The real number a is called the real part of $a + bi$ and the real number b is called the imaginary part of $a + bi$.
- $i^2 = -1$
- Conjugate of $Z = a + bi$ is $\overline{Z} = a - bi$
- Operations on Complex Numbers

i) $Z_1 + Z_2 = (a + c) + i(b + d)$

ii) $Z_1 - Z_2 = (a - c) + i(b - d)$

iii) $Z_1 Z_2 = (ac - bd) + i(ad + bc)$

iv) $\frac{Z_1}{Z_2} = \frac{ac + bd}{c^2 + d^2} + \frac{i(bc - ad)}{c^2 + d^2}$

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UNIT 03

LOGARITHM

After reading this unit, the students will be able to:

- Express a number in standard form of scientific notation and vice versa.
- Define logarithm of a number to the base a as the power to which a must be raised to give the number (i.e. $a^x = y \Leftrightarrow \log_a y = x, a > 0, y > 0$ *and* $a \neq 1$.
- Define a common logarithm, characteristic and mantissa of log of a number.
- Use tables to find the log of a number.

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- Give concept of antilog and use tables to find the antilog of a number.
- Differentiate between common and natural logarithm.
- Prove the following laws of logarithm.
  - $\log_a (mn) - \log_a m + \log_a n$,
  - $\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n$,
  - $\log_a m^n = n \log_a m$,
  - $\log_a m \log_m n = \log_a n$.



- Apply laws of logarithm to convert lengthy processes of multiplication, division and exponentiation into easier processes of addition and subtraction etc.

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Introduction

The word Logarithm is a combination of two Greek words "Logos" which means to calculate and "Arithmos" which stands for word number. Thus logarithm means a sort of calculation involving numbers. Logarithms were first invented by Scotsman John-Napier (or John Neper). He designed this system for one purpose. The simplification in the computation of multiplication, division, extraction of square roots of very large numbers and very small numbers.

When John Napier published his work composed of 147 pages in 1614, it was greatly admired by Henry Briggs, a professor at Gresham College, London. Later on Briggs computed Logarithms to base 10. Now-a-days the tables of common logarithms is the logarithms to the base 10 were invented by Henry Briggs. Logarithms to the base e (where $e = 2.71828...$) are today known as **Natural Logarithms**.

During the last fifty years, the invention of different types scientific calculators the idea of logarithm is still very important and is being used in many areas of science and mathematics.

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3.1 SCIENTIFIC NOTATION

While studying science, it is not uncommon to find ourselves working with very large numbers and very small numbers having a lot of zeros. Writing of such numbers is difficult, tedious and time consuming. To overcome such difficulties mathematicians have developed a technique using which we can write very large and very small numbers precisely & concisely.

We say that a positive number x is written in scientific notation if it is written as the product of a number "a" where a is real number greater than or equal to 1 but less than 10 and integer power n of 10 i.e.

$$x = a \times 10^n$$



Example 1

Write 7800 in scientific notation.

Solution:

In scientific notation 7800 is written as:

$$7800 = 7.8 \times 10^3$$

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Example 2

Write 0.00729 in scientific notation.

Solution:

In scientific notation 0.00729 is written as:

$0.00729 = 7.29 \times 10^{-3}$

3.1.1 HOW TO WRITE A NUMBER x FROM STANDARD TO SCIENTIFIC NOTATION

Here we will discuss the steps necessary for converting a number from its standard notation to scientific notation. In this connection, we must remember that any number say 5149 can be written as 5149.0

Suppose x is a number in standard form and we are to convert it into scientific notation.

a) How to write a number x in Scientific Notation when $x > 10$.

Step 1. We move the decimal point to the left, so that it occupies a position with only one non-zero digit lying to the left of it. The number obtained after this step is always greater than or equal to 1 but less than 10 and it gives the value of "a" in the definition.

Step 2. We count the number of places the decimal point had been moved. This number determines "n" of definition.

Step 3. Since the decimal point has been moved to the left, n will be positive.

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Example 3

Write 674000000 in scientific notation.

Solution:

Let us write $x = 674000000.0$

Step-1: We move the decimal to the left, so that it occupies a position with only one non-zero digit lying to the left as such we get
$6.74000000 = a$ which can be written as
$a = 6.74$

Step-2: The number of places the decimal had been moved $= 8$.

Step-3: Since the decimal has been moved to the left so $n = +8$.

$\therefore \quad x = a \times 10^n$

i.e. $674000000 = 6.74 \times 10^8$ which is the required scientific notation.

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b) How to write a number x in Scientific Notation when $0 < x < 1$.

Step 1. We move the decimal point to the right, so that it occupies a position with only one non-zero digit lying to the left of it. The number obtained after this step is always greater than or equal to 1 but less than 10 and it gives the value of "a" in the definition.

Step 2. We count the number of places the decimal point had been moved. This number determines "n" of definition.

Step 3. Since the decimal point has been moved to the right, n will be negative.

Example 4

Write 0.00000327 in scientific notation.

Solution:

Let $x = 0.00000327$

Step-1: We move the decimal to the right, so that it occupies a position with only one non-zero digit lying to the left and as such we get

000003.27 *i.e.* $a = 3.27$

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Step-2: The number of places the decimal had been moved $= 6$.

Step-3: Since the decimal has been moved to the right so $n = -6$.

$\therefore \quad x = a \times 10^n$

i.e. $0.00000327 = 3.27 \times 10^{-6}$ which is the required scientific notation.

Example 5

Write the following numbers in scientific notation.

i) 0.25×10^{-2} ii) 42.5×10^{2}

Solution:

i) 0.25×10^{-2}

Given 0.25×10^{-2}. To convert it in scientific notation, we take 0.25 and suppose $x = 0.25$

Step-1: Moving the decimal point to the right so that only one digit lies to the left of it and as such we get $a = 2.5$.

Step-2: The decimal has been moved to one place.

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Step-3: Since the decimal is moved to the right, so $n = -1$

i.e. $0.25 = 2.5\times10^{-1}$ .....(*)

Now we take the original number i.e.

0.25×10^{-2} and using (*) we can write

$0.25\times10^{-2} = 2.5\times10^{-1}\times10^{-2}$

$0.25\times10^{-2} = 2.5\times10^{-1+(-2)}$ (using law of exponents)

$0.25\times10^{-2} = 2.5\times10^{-1-2}$

$0.25\times10^{-2} = 2.5\times10^{-3}$

ii) 42.5×10^{2}

Given 42.5×10^{2} . To convert it into scientific notation, we take $x = 42.5$.

Step-1: Moving the decimal point to the left so that we have $a = 4.25$.

Step-2: The decimal has been moved to one place.

Step-3: Since the decimal is moved to the left, so $n = +1 = 1$

$\therefore \quad x = 4.25 = a\times10^{n} = 4.25\times10^{1}$

i.e. $4.25 = 4.25\times10^{1}$ .....(*)

Therefore the original number becomes

$4.25\times10^{2} = 4.25\times10^{1}\times10^{2}$ (using x)

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$4.25\times10^{2} = 4.25\times10^{1+2}$ (using law of exponents)

$4.25\times10^{2} = 4.25\times10^{3}$

3.1.2 HOW TO WRITE A NUMBER x FROM ITS SCIENTIFIC NOTATION TO STANDARD FORM

Suppose we are given a number in scientific notation. To change it into its standard form, we move the decimal point to the same number of places as the exponent of 10.

i) If the exponent of 10 is positive, we move the decimal to the right and

ii) If the exponent on 10 is negative, we move the decimal point to the left.

Example 6

Write the following in standard notation.

i) 4.56×10^{7} ii) 8.92×10^{-5}

Solution:

i) 4.56×10^{7}

Given 4.56×10^{7} . The exponent is positive and is equal to 7, so we move the decimal point 7 places to the right. It is important to note that while moving the decimal point extra zeros can be placed if necessary.

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Therefore,

$4.56\times10^{7} = 45600000.0$ (the decimal has been moved to 7 places right)

Thus $4.56\times10^{7} = 45600000$

ii) 8.92×10^{-5}

Given 8.92×10^{-5}. We see that exponent is negative, so we will move the decimal point 5-places to left.

Therefore,

$8.92\times10^{-5} = 0.0000892$ (the decimal has been moved to 5 places left)

Thus $8.92\times10^{-5} = 0.0000892$

In the following example we give an application of scientific notation.

Example 7

How many miles does light travel in 1 day? The speed of light is 186,000 mi/sec. Write the answer in scientific notation.

Solution:

Here, $v = 186000 = 1.86\times10^{-5}$ mi/sec and

$t = 24\times60\times60 = 86400 = 8.64\times10^{4}\ sec$

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$\because \quad S = vt$

$= 1.86\times10^{5}\times8.64\times10^{4} = 1.86\times8.64\times10^{5+4} = 16.0704\times10^{9}$

$S = 1.60704\times10^{10}$

Light travels 1.60704×10^{10} miles in 1 day.

Exercise 3.1

1. Write each number in scientific notation.

i) 405,000 ii) 1, 670,000 iii) 0.00000039

iv) 0.00092 v) 234,600,000,000 vi) 8,904,000,000

vii) 0.00104 viii) 0.00000000514 ix) 0.05×10^{-3}

2. Write each number in standard notation:

i) 8.3×10^{-5} ii) 4.1×10^{6} iii) 2.07×10^{7}

iv) 3.15×10^{-6} v) 6.27×10^{-10} vi) 5.41×10^{-8}

vii) 7.632×10^{-4} viii) 9.4×10^{5} ix) -2.6×10^{9}

3. How long does it take light to travel to Earth from the sun? The sun is 9.3×10^{7} mi from Earth, and light travels 1.86×10^{5} mi/s.

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3.2 LOGARITHM

Suppose we are asked to find such a value of x which makes $3^x = 9$ a true statement. We can answer at once that $x = 2$. But if we have problem $3^x = 10$ and we are to find x then it is not easy to find x. there are infinite number of such a type of problems we encounter, when we study mathematics. If we do not use calculators, logarithms is the tool which can solve such a type of problems for us.

Definition:

If for a positive real number a (but $a \neq 1$) $a^x = y$ then the index x is called the logarithm of y to the base a. We write this as $\log_a y = x$.

"log" is the abbreviation of the word logarithm. Since $\log_a y = x$ and $y = a^x$ are two different ways of expressing the same relation among the numbers a, y & x, so we can write

$\log_a y = x$ if and only if $a^x = y$.

or $\log_a y = x \Leftrightarrow a^x = y$ "$\Leftrightarrow$" is a symbol for "if and only if"

$a^x = y$ is called the exponential form and $\log_a y$ is the logarithmic form. In both the above expressions a is the base.

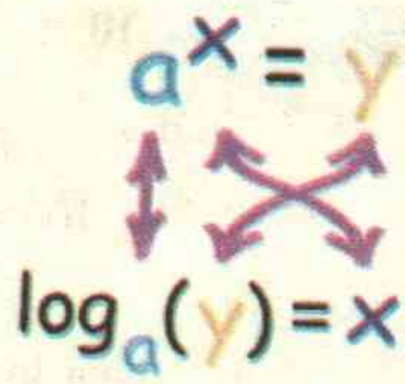
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$2^4 = 16$ can be written as $\log_2 16 = 4$.

Example 8

Write the following in logarithmic form.

i) $2^4 = 16$ ii) $4^{-3} = \frac{1}{64}$



Solution:

We know that $a^x = y \Leftrightarrow \log_a y = x$ ..... (1)

i) $2^4 = 16$

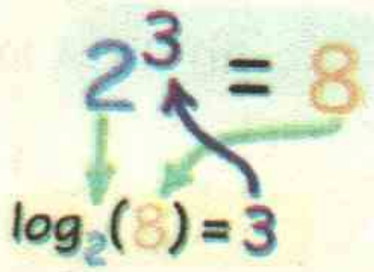
$\therefore a = 2,\ x = 4,\ y = 16$, so using the above result (1)

$2^4 = 16$ is equivalent to $\log_2 16 = 4$.

ii) $4^{-3} = \frac{1}{64}$

Here $a = 4,\ x = -3,\ y = \frac{1}{64}$, so using the result (1)

$4^{-3} = \frac{1}{64}$ is equivalent to $\log_4\left(\frac{1}{64}\right) = -3$.



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Example 9

Write the following in exponential form.

i) $\log_8(64)=2$ ii) $\log_3\left(\frac{1}{9}\right)=-2$

Solution:

We know that $a^x=y \Leftrightarrow \log_a y=x$ ..... (*)

i) $\log_8(64)=2$

Here $a=8,\ y=64$ *and* $x=2$, so using the result (*)

$\therefore \log_8(64)=2$ is equal to $8^2=64$.

ii) $\log_3\left(\frac{1}{9}\right)=-2$

Here $a=3,\ y=\frac{1}{9}$ *and* $x=-2$, so using the result (*)

$\therefore \log_3\left(\frac{1}{9}\right)=-2$ is equal to $3^{-2}=\frac{1}{9}$.

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Example 10

Find x.

i) $\log_x 100=2$. ii) $\log_{12}\left(12^3\right)=x$ iii) $\log_x\frac{4}{9}=2$

Solution:

i) Given $\log_x 100=2$. This can be written in exponential form as

$x^2=100$ $\because \log_a y=x \Leftrightarrow a^x=y$

$x^2=10^{10} \Rightarrow x=10$

ii) Given $\log_{12}12^3=x$. This can be written in exponential form as

$12^x=12^3$ $\because \log_a y=x \Leftrightarrow a^x=y$

$x=3$

iii) Given $\log_x\frac{4}{9}=2$

$\Rightarrow x^2=\frac{4}{9}=\left(\frac{2}{3}\right)^2$ Comparing we get $x=\frac{2}{3}$.

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Exercise 3.2

1. Write the following in logarithmic form.

i) $4^4 = 256$

ii) $2^{-6} = \frac{1}{64}$

iii) $10^0 = 1$

iv) $x^{\frac{3}{4}} = y$

v) $3^{-4} = \frac{1}{81}$

vi) $64^{\frac{2}{3}} = 16$

2. Write the following in exponential form.

i) $\log_a\left(\frac{1}{a^2}\right) = -2$

ii) $\log_2 \frac{1}{128} = -7$

iii) $\log_b 3 = 64$

iv) $\log_a a = 1$

v) $\log_a 1 = 0$

vi) $\log_4 \frac{1}{8} = \frac{-3}{2}$

3. Solve.

i) $\log_{\sqrt{5}} 125 = x$

ii) $\log_4 x = -3$

iii) $\log_{81} 9 = x$

iv) $\log_3 (5x+1) = 2$

v) $\log_2 x = 7$

vi) $\log_x 0.25 = 2$

vii) $\log_x (0.001) = -3$

viii) $\log_x \frac{1}{64} = -2$

ix) $\log_{\sqrt{3}} x = 16$

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PRODUCT RULE FOR RADICALS



$$\sqrt[n]{x} \cdot \sqrt[n]{y} = \sqrt[n]{xy}$$

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3.2.1 Defining Common Logarithm, Characteristic And Mantissa

a) Common Logarithm

Logarithms having base 10 are called common logarithms or Briggs logarithms. This logarithm was invented by a British Mathematician Prof. Henry Briggs (1560 – 1631). Tables of common logarithm make calculations very easy. The base of logarithm is not written because it is considered to be 10.

For example:

$\log_{10} x$ is written as $\log x$.

$\log_{10} 100 = 2$ is written as $\log 100 = 2$



b) Characteristic and Mantissa

Consider the following table.

Number N	0.0001	0.001	0.01	0.1	1	10	100	1000	10000
Exponential form of N	10^{-4}	10^{-3}	10^{-2}	10^{-1}	10^{0}	10	10^{2}	10^{3}	10^{4}
$\log N$	–4	–3	–2	–1	0	1	2	3	4

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It is obvious that $10^{1.5377}$ will give some number greater than 10^{1} but smaller than 10^{2}. Actually, $10^{1.5377} = 34.49$; hence $\log 34.49 = 1.5377$.

Logarithm of a number consists of two parts. The digit before the decimal point is the characteristic of the log, and the decimal fraction part is the mantissa of the log. In the above example, the characteristic is 1 and the mantissa is .5377.

The mantissa of the log of a number is found in tables, ignoring the decimal point of the number. Each mantissa in the tables is understood to have a decimal point preceding it, and the mantissa is always considered positive.

Rules for finding characteristic of $\log x$

1. If $x > 1$, characteristic = (No. of digits on the left side of decimal point in x) – 1.
2. If $x < 1$, characteristic = –[(No. of zeros immediately following the decimal point) + 1].

If the characteristic is negative, it is written as $-a$ or $\bar{a}$.

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Example 11

Write the characteristics of the following logarithms:

(i) log 4350 (ii) log 435 (iii) log 43.5

(iv) log 4.35 (v) log 0.435 (vi) log 0.0435

(vii) log 0.00435 (viii) log 0.000435

Solution:

(i) Characteristic of log 4350 = 3 ($\because$ 4 – 1 = 3)

(ii) Characteristic of log 435 = 2 ($\because$ 3 – 1 = 2)

(iii) Characteristic of log 43.5 = 1 ($\because$ 2 – 1 = 1)

(iv) Characteristic of log 4.35 = 0 ($\because$ 1 – 1 = 0)

(v) Characteristic of log 0.435 = –1 ($\because$ – (0 + 1) = –1)

(vi) Characteristic of log 0.0435 = –2 ($\because$ – (1 + 1) = –2)

(vii) Characteristic of log 0.00435 = –3 ($\because$ – (2 + 1) = –3)

(viii) Characteristic of log 0.000435 = –4 ($\because$ – (3 + 1) = –4)

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3.2.2 The use of Logarithmic Table to find the log of a number

To find the logarithm of a number we have to find the characteristic and the mantissa. The characteristic is found by using the above two rules. The mantissa is found from the *logarithmic table.*

To show how the logarithm of a number can be obtained, consider the following example.

Example 12

Find the logarithm of 763.5

Solution:

As the number under consideration is greater than one and has 3 digits in its integral part. Hence by the first rule the characteristic of log (763.5) is 2.

To find the mantissa of the logarithm of the number, we refer to the logarithmic tables given at the end of the book. Following is the required extract of the table for ready reference.

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	0	1	2	3	4	5	6	7	8	9	Mean Difference 123 / 456 / 789		
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	112	233	455
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	112	233	455
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	112	233	445

Choose a row from the first column containing the first two digits of the number, namely 76. Proceed along this row until we come to the column headed by the third digit 3 of the number. We get the figure 8825. We next add to this figure the number 3 to be found in the same row in the mean differences column headed by the fourth digit 5 of the number. Thus we get the number (8825+3 = 8828). This number with the decimal point prefixed gives the mantissa of log (763.5) as .8828.

Hence log (763.5) = 2.8828.

Similarly it can be verified that log (7.529) = 0.8767; log (0.0077) = 3.8865;

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log (0.7605) = 1.8811 and log (75) = 1.8751.

Exercise 3.3

1. Find the characteristic of the common logarithm of each of the following numbers.

i) 57 ii) 7.4 iii) 5.63 iv) 5.63

v) 982.5 vi) 7824 vii) 186000 viii) 0.71

2. Find the following.

i) log 87.2 ii) log 37300 iii) log 753 iv) log 9.21

v) log 0.00159 vi) log 0.0256 vii) log 6.753

3. Find logarithms of the following numbers.

i) 2476 ii) 2.4 iii) 92.5 iv) 482.7

v) 0.783 vi) 0.09566 vii) 0.006735 viii) 700

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3.2.3 Anti-logarithm

In many cases we are required to find the number, the logarithm of which is known.

If $y = \log_{10} x$, then x is called the antilogarithm of y to the base 10; and this is shortly written as x = antilog y.

If we have an equation, where y is to be known. To find the value of y, we use the table of Anti-logarithm. i.e. y = Antilog x .

To explain this concept, consider the following example.

Example 13

Find the numbers whose logarithms are (i) 2.3456 (ii) $\bar{2}.1576$

Solution:

(i) 2.3456

To find the sequence of digits in the required number, we look up the table of antilogarithms given at the end of the book.

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Since the mantissa is .3456, we look out for the row in the first column containing the first two digits of the mantissa .34, and in this row we select the number 2213 in the column headed by the third digit 5 of the mantissa. Then, add to this the figure 3 which is the fourth digit value from 6th column of mean differences in the same row .

The required sequence of digits in the number is (2213+3 = 2216). As the characteristic of the given logarithm is 2, there must be 2 + 1 i.e. 3 digits in the integral part of the number.

Hence the number whose logarithm is 2.3456 must be 221.6.

(ii) $\bar{2}.1576$

As in above, from the table of anti-logarithms, the sequence of digits in the number, corresponding to the mantissa .1576, is found to be 1437. Now the characteristic is –2. Therefore, there must be one zero between the decimal point and the first significant digit in the number.

Hence the number whose logarithm is $\bar{2}.1576$ must be .01437.

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Exercise 3.4

1. Find anti-logarithm of the following numbers.

i) 1.2508 ii) 0.8401 ii) 2.540

iv) $\bar{2}.2508$ v) $\bar{1}.5463$ vi) 3.5526

2. Find the values of x from the following equations:

i) $\log x = \bar{1}.8401$ ii) $\log x = 2.1931$ iii) $\log x = 4.5911$

iv) $\log x = \bar{3}.0253$ v) $\log x = 1.8716$ vi) $\log x = \bar{2}.8370$

3.3 COMMON AND NATURAL LOGARITHM

We have seen in the definition of logarithm that the base "a" of logarithms may be any positive real number other than 1. As the value of the base changes, the logarithmic value of a number also changes. For example

$\log_4(64) = 3 \quad \therefore$ if we let $\log_2 64 = x$, then $2^x = 64 = 2^6 \Rightarrow x = 6$

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$\log_4(64) = 3$ or $\log_8(64) = 2$.

If fact every base define a logarithm system. Here we will discuss about two logarithmic used in mathematics and as such have their special names. There systems are the following.

i) System of common logarithms

ii) System of natural logarithms

3.3.1 Difference between common and natural logarithms

i) System of Common Logarithms

This system is constructed for the base $a = 10$ is also called the Briggs System on the name of its inventor Henry Briggs.

Hereafter in this book, when we speak of logarithm of a number without specifying the base, it must be understood that the base of the logarithm is 10. Thus in common logarithm system $\log_{10} x$ is also written as $\log x$.

Therefore if it is given $\log(204)$, it means $\log_{10}(204)$ and vice-versa.

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ii) System of Logarithms Common natural

Natural logarithm system is the logarithm which uses "e" as a baxe. "e" was introduced by Euler, it is an irrational number. Here we will take its approximate value as $e = 2.71828$. The natural logarithm of a number x is written as $\log_e x$ but it is shortly abbreviated as Inx. Thus $\log_e x = Inx$ and $Inx = y \Leftrightarrow e^y = x$

The relation between common logarithm and natural logarithm is given by

$$\log_{10} x = \frac{\log_e x}{\log_e 10} = \frac{\ln x}{\ln 10}. \quad \text{But } \ln 10 = 2.30258$$

$$\therefore \quad \log_{10} x = \frac{1}{2.30258} \ln x$$

3.4 LAWS OF LOGARITHM

There are four laws of logarithms. We shall prove them one by one as:

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1. $\log_a mn = \log_a m + \log_a n$

Proof:

Let $\log_a m = x \Rightarrow a^x = m$ _____ (i) (By definition)

and $\log_a n = y \Rightarrow a^y = n$ _____ (ii) (By definition)

Now, $mn = a^x \times a^y = a^{x+y} \Rightarrow a^{x+y} = mn \Rightarrow \log_a mn = x + y$

Using (i) and (ii), we get $\log_a mn = \log_a m + \log_a n$



2. $\log_a \frac{m}{n} = \log_a m - \log_a n$

Proof:

Here, $\frac{m}{n} = \frac{a^x}{a^y} = a^{x-y} \Rightarrow a^{x-y} = \frac{m}{n} \Rightarrow \log_a \frac{m}{n} = x - y$

Using (i) and (ii),we get $\log_a \frac{m}{n} = \log_a m - \log_a n$

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3. $\log_a m^n = n \log_a m$

Proof:

By (i), we have $a^x = m$

Taking power n both sides.

$$(a^x)^n = (m)^n \quad \Rightarrow \quad a^{xn} = m^n$$

$$\Rightarrow \quad \log_a m^n = nx$$

Using (i), we get $\log_a m^n = n \log_a m$

Tid-Bit

$n \log_a m = \log_a m^n$

4. $\log_a m \log_m n = \log_a n$

Proof:

Let $\log_a m = x \Rightarrow a^x = m$ -------- (iii) (By definition)

and $\log_m n = y \Rightarrow m^y = n$ -------- (iv) (By definition)

Now, $a^{xy} = (a^x)^y = m^y = n \quad \Rightarrow \quad a^{xy} = n \quad \Rightarrow \quad \log_a n = xy$

using (i) and (ii), we get

$$\log_a m \log_m n = \log_a n$$

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Note

This law can also be written as:

$$\log_m n = \frac{\log_a n}{\log_a m}$$

This is called Change of Base Law.

Example 14

Express each of the following as a single logarithm.

(i) $\log 2 + \log 3$ (ii) $\log 6 - \log 2$

(iii) $-1 + logy$ (iv) $\log_2 3 \cdot \log_3 5$

(v) $\log_2 8 + \log_2 32$ (vi) $\log 3 + \log 5 + \log 6 - \log 25$

Solution:

(i) $\log 2 + \log 3 = \log(2 \times 3) = \log 6$

(ii) $\log 6 - \log 2 = \log \frac{6}{3} = \log 2$

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(iii) $-1+logy$

$$=-log10+logy=log10^{-1}+logy=log\frac{1}{10}+logy=log0.10+logy=log0.10y$$

(iv) $\log_2 3\cdot\log_3 5=\log_2 5$

(v) $\log_2 8+\log_2 32=\log_2(8\times32)=\log_2 256$

(vi) $\log 3+\log 5+\log 6-\log 25=\log\frac{3\times5\times6}{25}=\log\frac{90}{25}=\log\frac{18}{5}$

Exercise 3.5

1. Use logarithm properties to simplify the expressions.

i) $\log_7\sqrt{7}$ ii) $\log_8\frac{1}{2}$

iii) $\log_{10}\sqrt{1000}$ iv) $\log_9 3+\log_9 27$

v) $\log\frac{1}{(0.0035)^{-4}}$ vi) $\log 45$

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2. Express each of the following as a single logarithm.

i) $3\log2-4\log3$ ii) $2\log3+4\log2-3$

iii) $\log5-1$ iv) $\frac{1}{2}\log x-2\log2y+3\log z$

3. Find the value of 'a' from the following equations.

i) $\log_2 6+\log_2 7=\log_2 a$ ii) $\log_{\sqrt{3}} a=\log_{\sqrt{3}}5+\log_{\sqrt{3}}8-\log_{\sqrt{3}}2$

iii) $\frac{\log_7 r}{\log_7 t}=\log_a r$ iv) $\log_6 25-\log_6 5=\log_6 a$

4. Find $\log_2 3\cdot\log_3 4\cdot\log_4 5\cdot\log_5 6\cdot\log_6 7\cdot\log_7 8$

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3.5 APPLICATION OF LOGARITHM

In the following examples we shall explain how lengthy processes of multiplication and division are converted into easier processes of addition and subtraction by applying the laws of logarithm.

Example 15

Simplify (238.2)(9.506) with the help of logarithm.

Solution:

Let $x = (238.2)(9.506)$

Taking log on both sides,

$\log x = \log(238.2)(9.506)$

$\Rightarrow \quad \log x = \log(238.2) + \log(9.506)$ $\quad (\because \log mn = \log m + \log n)$

$\Rightarrow \quad \log x = 2.3770 + 0.9786 \quad = 3.3550$



Taking Antilog on both sides.

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Antilog $(\log x)$ = Antilog (3.3550)

$x = 2265$

Example 16

Simplify $\dfrac{2.83}{(6.52)^2}$ with the help of logarithm.



Solution:

Let $x = \dfrac{2.83}{(6.52)^2}$

Taking log on both sides

$\log x = \log \dfrac{2.83}{(6.52)^2}$

$= \log 2.83 - \log(6.52)^2 \quad (\because \log \dfrac{m}{n} = \log m - \log n)$

$= \log 2.83 - 2\log 6.52 \quad (\because \log m^n = n \log m)$

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$$= 0.4518 - 2(0.8142)$$
$$= 0.4518 - 1.6284$$
$$= -1.766$$

Add and subtract 2 to make the number positive.

$$\log x = -1.1766 + 2 - 2$$
$$= 0.8234 - 2$$
$$= \bar{2}.8234$$

Taking antilog on both sides

$$\text{Antilog}(\log x) = \text{Antilog}\ \bar{2}.8234$$
$$x = 0.06659$$

Exercise 3.6

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1. Simplify with the help of logarithm.

i) 3.81×43.4

ii) $73.42 \times 0.00462 \times 0.5143$

iii) $\dfrac{784.6 \times 0.0431}{28.23}$

iv) $\dfrac{0.4932 \times 653.7}{0.07213 \times 8456}$

v) $\dfrac{(78.41)^3 \sqrt{142.3}}{\sqrt[4]{0.1562}}$

2. Find the following if $\log 2 = 0.3010$, $\log 3 = 0.4771$, $\log 5 = 0.6990$, $\log 7 = 0.8451$.

i) $\log 105$

ii) $\log 108$

iii) $\log \sqrt[3]{72}$

iv) $\log 2.4$

v) $\log 0.0081$

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REVIEW EXERCISE 3

1. Select the correct answer.

(i) $\log_9 \frac{1}{81} =$

a) -1 b) -2 c) 2 d) Does not exist

(ii) If $\log_2 8 = x$ then $x =$

a) 64 b) 3^2 c) 3 d) 2^8

(iii) Base of common log is

a) 10 b) e c) π d) 5

(iv) $\log \sqrt{10} =$

a) -1 b) $-\frac{1}{2}$ c) $\frac{1}{2}$ d) 2

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(v) for any non-zero value of x, $x^0 =$

a) 2 b) 1 c) 0 d) 10

(vi) Rewrite $t = \log_b m$ as an exponential equation

a) $t = m^b$ b) $b^m = t$ c) $m = b^t$ d) $m^t = b$

(vii) $\log_{10} 10 =$

a) 2 b) 3 c) 0 d) 1

(viii) Characteristic of log 0.000059 is

a) -5 b) 5 c) -4 d) 4

(ix) Evaluate $\log_7 \frac{1}{\sqrt{7}}$

a) -1 b) $-\frac{1}{2}$ c) $\frac{1}{2}$ d) 2

(x) Base of natural log is

a) 10 b) e c) π d) 1

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(xi) $\log m + \log n =$

a) $\log m \log n$

b) $\log m - \log n$

c) $\log mn$

d) $\log \frac{m}{n}$

(xii) 0.069 can be written in scientific notation as

a) 6.9×10^{3}

b) 6.9×10^{-2}

c) 0.69×10^{3}

d) 69×10^{2}

(xiii) $\ln x - 2\ln y$

a) $\ln \frac{x}{y}$

b) $\ln xy^2$

c) $\ln \frac{x^2}{y}$

d) $\ln \frac{x}{y^2}$



2. Write 9473.2 in scientific notation.

3. Write 5.4×10^{6} in standard notation.

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4. Write in logarithmic form: $3^{-3} = \frac{1}{27}$.

5. Write in exponential form: $\log_5 1 = 0$

6. Solve for x: $\log_4 16 = x$

7. Find the characteristic of the common logarithm 0.0083 .

8. Find log 12.4

9. Find the value of 'a',

$\log_{\sqrt{5}} 3a = \log_{\sqrt{5}} 9 + \log_{\sqrt{5}} 2 - \log_{\sqrt{5}} 3$

Think

10. $\frac{(63.28)^3 (0.00843)^2 (0.4623)}{(412.3)(2.184)^5}$

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SUMMARY

- Scientific notation is used to write conveniently very large and very small positive numbers and to perform computations.
- Scientific notation for a number is an expression of the type $N \times 10^m$ where $1 \leq N < 10$, N is in decimal notation and m is an integer.
- Logarithm was discovered by the famous Muslim mathematician Musa Al-Khwarizmi. Afterwards John Napier developed the concept of logarithms and prepared different tables which are very helpful in calculations.
- The basic use of logarithm is to convert lengthy operations of multiplication and division into simple operations of addition and subtraction.
- If y is any positive real number and $y = a^x$, (where $a > 0, \neq 1$) then x is called the logarithm of the number y to the base a, and we briefly write this as $x = \log_a y$.
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- The logarithm of a number y (> 0) to any base a ($> 0, \neq 1$) is the index of the power to which a must be raised to make that power equal to y.
- $y = a^x$ and $x = \log_a y$.
- Logarithms having base 10 are called Common logarithms or Briggs logarithms.
- Logarithm of a number consists of two parts. The digit before the decimal point is the characteristic of the log, and the decimal fraction part is the mantissa of the log.
- Logarithms having base e are called Natural logarithms.
- Natural logarithm is invented by John Napier (1550 — 1617).
- Approximate value of e is 2.71828.
- Laws of logarithms.
  - $\log_a mn = \log_a m + \log_a n$.
  - $\log_a \frac{m}{n} = \log_a m - \log_a n$.
  - $\log_a m^n = n \log_a m$.
  - $\log_a m \log_m n = \log_a n$.



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UNIT 04

ALGEBRAIC EXPRESSIONS AND ALGEBRAIC FORMULAS

After reading this unit, the students will be able to:

- Know that a rational expression behaves like a rational number.
- Define a rational expression as the quotient $\frac{p(x)}{q(x)}$ of two polynomials p(x) and q(x) where q(x) is not the zero polynomial.
- Examine whether a given algebraic expression is a
  - polynomial or not,
  - rational expression or not.
- Define $\frac{p(x)}{q(x)}$ as a rational expression in its lowest terms if $p(x)$ and $q(x)$ are polynomials with integral coefficients and having no common factor.
- Examine whether a given rational algebraic expression is in lowest form or not.
- Reduce a given rational expression to its lowest terms.
- Find the sum, difference and product of rational expressions.
- Divide a rational expression with another and express the result in its lowest terms.
- Find value of algebraic expression at some particular real number.
  Know the formulas
  - $(a+b)^2+(a-b)^2=2(a^2+b^2)$
  - $(a+b)^2-(a-b)^2=4ab.$
- Find the value of a^2+b^2 and of ab when the values of $a+b$ and $a-b$ are known.
- Know the formula $(a+b+c)^2=a^2+b^2+c^2+2ab+2bc+2ca.$

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- Find the value of $a^2+b^2+c^2$ when the values of $a+b+c$ and $ab+bc+ca$ are given.
- Find the value of $a+b+c$ when the values of $a^2+b^2+c^2$ and $ab+bc+ca$ are given.
- Find the value of $ab+bc+ca$ when the values of $a^2+b^2+c^2$ and $a+b+c$ are given.
- Know the formulas
  - $(a+b)^3 = a^3+3ab(a+b)+b^3$
  - $(a-b)^3 = a^3-3ab(a-b)-b^3$.
- Find the value of $a^3 \pm b^3$ when the values of $a \pm b$ and ab are given.
- Find the value of $x^3 \pm \frac{1}{x^3}$ when the value of $x \pm \frac{1}{x}$ is given.
  - Know the formula $a^3 \pm b^3 = (a \pm b)(a^2 \mp ab+b^2)$.
  - Find the product of $x+\frac{1}{x}$ and $x^2+\frac{1}{x^2}-1$.
  - Find the product of $x-\frac{1}{x}$ and $x^2+\frac{1}{x^2}+1$.
  - Find the continued product of $(x+y)(x-y)(x^2+xy+y^2)(x^2-xy+y^2)$.
- Recognize the surds and their application.
- Explain the surds of second order. Use basic operations on surds of second order to rationalize the denominators and evaluate it.
- Explain rationalization (with precise meaning) of real numbers of the types $\frac{1}{a+b\sqrt{x}}, \frac{1}{\sqrt{x}+\sqrt{y}}$ and their combinations where x and y are natural numbers and a and b are integers.

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Introduction

In 820 AD., the Muslim mathematician, Al-Khwarizmi, wrote the algebraic text entitled Al-jabr Wa-Al Muqabala (the science of cancellation and reduction). A Latin translation of this text became known in Europe under the title Al-Jabr, became the word algebra. Al-Khwarizmi is known as the "Father of Algebra"

Algebra uses special types of mathematical processes called algebraic operations. These operations include addition, subtraction, multiplication, division, extraction of roots and raising to integral or fractional powers.

4.1 ALGEBRAIC EXPRESSIONS

When variables and constants are connected by the algebraic operations of addition, subtraction, multiplication, division, root extraction & rising integral or factional powers, the resulting expression so obtained is called an algebraic expression. For example

$3x^2+5x-7$, $\sqrt{x}+\frac{1}{\sqrt{x}}$ where $x>0$.

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$4x+3y+\frac{5}{z}$ where $z=0$, $x^2-4xy+6$ are all algebraic expressions.

Different parts of an algebraic expression which are combined together by the operations of addition and subtraction are called terms of the algebraic expression. For example in the algebraic expression $3x^2+5\sqrt{x}-7$, the terms are $3x^2, 5\sqrt{x}, -7$.

4.1.1 Rational Expressions

An algebraic expression of the form $\frac{p(x)}{q(x)}$ where $p(x)$ & $q(x)$ are polynomials and $q(x)\neq 0$ is called a rational expression. For example;

i) $\frac{x^2-6x+1}{x+9}$ is a rational expression if $x+9\neq 0$.

ii) $x+\frac{3}{x}$ is a rational expression if $x\neq 0$ because it can be written as $\frac{x^2+3}{x}$.

iii) $\frac{x^2+4x+5}{3}$ is a rational expression.

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Note

Every polynomial $p(x)$ can be regarded as a rational expression, since we can write $p(x)$ as $\frac{p(x)}{1}$. Thus every polynomial is a rational expression but every rational expression need not be a polynomial.

4.1.2 How to examine whether a given algebraic expression is a a) Polynomial or not b) Rational expression or not

a) Polynomial or not

Every algebraic expression may not be a polynomial. For example $2x^2+7x+\frac{6}{x}$ with $x\neq 0$ is obviously an algebraic expression but it is not a polynomial. Similarly $\frac{1}{x+3}$ with $x\neq -3$ is an algebraic expression but it is not a polynomial.

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b) Rational expression or not

We know that a rational expression is always in the form $\frac{p(x)}{q(x)}$ where $p(x)$ & $q(x)$ are polynomials and $q(x)\neq 0$. Now $\sqrt{\frac{3x^2-5x-2}{10-x}}$ is an algebraic expression with $x\neq 10$. But it is not a rational expression because $\sqrt{\frac{3x^2-5x-2}{10-x}}=\frac{\sqrt{3x^2-5x-2}}{\sqrt{10-x}}$. Here $\sqrt{3x^2-5x-2}$ and $\sqrt{10-x}$ are not polynomials. Thus every algebraic expression is not a rational expression.

Tid-Bit

Zero (0) is the only number which can not be represented by Roman numerals.

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4.1.3 Rational Expressions behave like a rational number

Let a and b be two integers, then $\frac{a}{b}$ may or may not be an integer (for example 2 and 5 are integers but $\frac{5}{2}$ is not an integer). Therefore, the set of integers is extended and $\frac{a}{b}$ where $b \neq 0$ and $a, b \in Z$ is defined as a rational number and thus we have a new set, the set of rational numbers.

Similarly if $p(x)$ & $q(x)$ where $q(x) \neq 0$ are polynomials then $\frac{p(x)}{q(x)}$ is not necessarily a polynomial (for example $x+1$ and x^2 are polynomials but $\frac{x+1}{x^2}$ is not a polynomial). Therefore, similar to the concept of rational numbers, the concept of rational expressions is developed). Rational numbers and rational expressions behave similarly. Thus they have similar properties.

4.1.4 Lowest terms of a Rational Expression $\frac{p(x)}{q(x)}$

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A rational expression $\frac{p(x)}{q(x)}$ is said to be in its lowest terms if $p(x)$ & $q(x)$ are polynomials with integral coefficients and having no common factor. For example,

$\frac{8}{y^2-4y}$ is in its lowest terms while $\frac{4x+20}{(x+5)(x+1)(x-8)}$ is not.

4.1.5 How to examine whether a given algebraic expression is in its lowest terms or not

Given the rational expression $\frac{p(x)}{q(x)}$, factorize both the polynomials in the numerator and denominator. If there is any common factor between them then it is not in its lowest terms otherwise it is.

For example, $\frac{x^2+1}{x-1}$ is in its lowest terms as there is no common factor in numerator and denominator.

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Example 1

Examine whether the following rational expressions are in their lowest terms or not:

i) $\frac{9}{x-3}$ ii) $\frac{x^2-7x}{x+4}$ iii) $\frac{x^2-1}{x-1}$

Solution:

i) $\frac{9}{x-3}$ is in its lowest terms, because it has no factor in common in the numerator and denominator.

ii) $\frac{x^2-7x}{x+4}$ is in its lowest terms, because it has no factor common in the numerator and denominator.

iii) $\frac{x^2-1}{x-1}$ is not in its lowest terms as $(x-1)$ is a common factor in the numerator and denominator, because we can write it as $\frac{x^2-1}{x-1}=\frac{(x-1)(x+1)}{x-1}$.

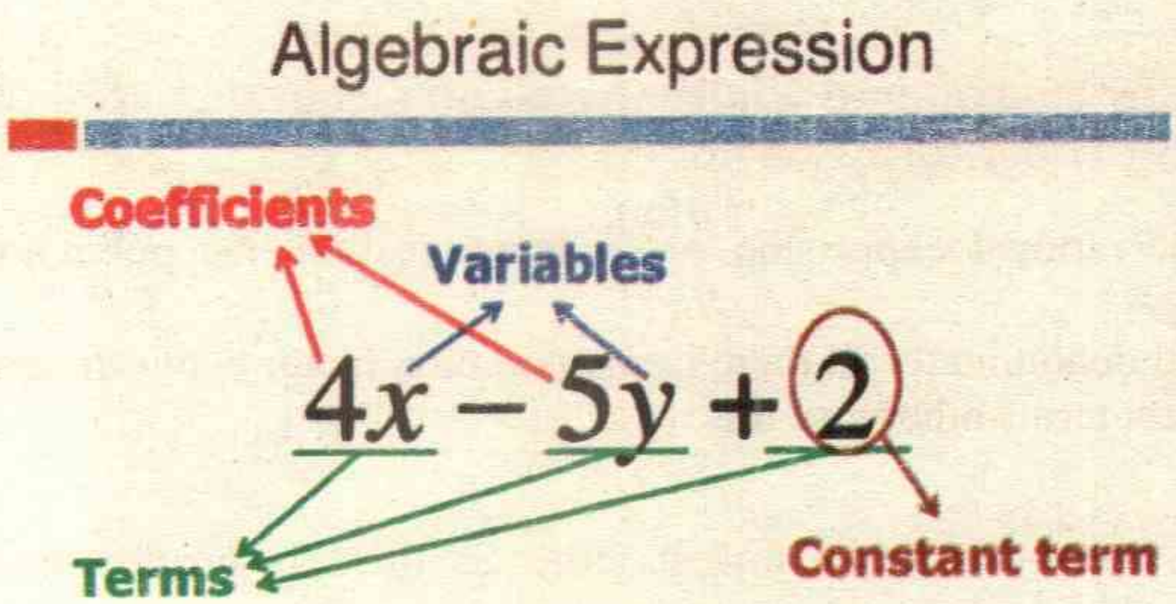
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4.1.6 Working Rule to express a rational expression in its lowest terms

Let the given rational expression be $\frac{p(x)}{q(x)}$.

Step 1: Factorize both the polynomials in the numerator and denominator.

Step 2: Cancel common factors between them.



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Example 2

Reduce the following rational expressions to their lowest terms:

i) $\dfrac{x+3}{(x+3)(x-2)}$ ii) $\dfrac{x^2-b^2}{(x+b)x^2}$

Solution:

i) $\dfrac{x+3}{(x+3)(x-2)}$

$= \dfrac{1}{x-2}$ ..... (Cancelling common factors)

ii) $\dfrac{x^2-b^2}{(x+b)x^2}$

$= \dfrac{(x+b)(x-b)}{(x+b)x^2} = \dfrac{x-b}{x^2}$

Which is in lowest terms.

4.1.7 Finding Sum and Difference of Rational Expressions

Example 3

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Add $\dfrac{x+2}{x+3}$ & $\dfrac{x+5}{x+3}$.

Solution:

Here the denominator in both the rational expressions is the same, so LCM is $x+3$.

$\therefore \quad \dfrac{x+2}{x+3}+\dfrac{x+5}{x+3}=\dfrac{x+2+x+5}{x+3}=\dfrac{2x+7}{x+3}$.

Example 4

Add $\dfrac{4}{x-3}$ *and* $\dfrac{6}{x+3}$.

Solution:

$\dfrac{4}{x-3}+\dfrac{6}{x+3}=\dfrac{4(x+3)+6(x-3)}{(x-3)(x+3)}$ ($\because$ LCM of $x-3$ & $x+3$ is $(x-3)(x+3)$)

$= \dfrac{4x+12+6x-18}{x^2-9} = \dfrac{10x-6}{x^2-9}$

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Example 5

Subtract $\frac{2}{(x-2)(x-3)}$ from $\frac{x}{(x+3)(x-2)}$.

Solution:

$$\frac{x}{(x+3)(x-2)} - \frac{2}{(x-2)(x-3)}$$

$$= \frac{x(x-3)-2(x+3)}{(x-2)(x+3)(x-3)} \quad (\because \text{ LCM of denominators is } (x-2)(x+3)(x-3))$$

$$= \frac{x^2-3x-2x-6}{(x-2)(x+3)(x-3)} \quad \text{(Simplifying)}$$

$$= \frac{x^2-5x-6}{(x-2)(x+3)(x-3)} \quad \text{(Simplifying)}$$

$$= \frac{x^2-5x-6}{(x-2)(x^2-9)} \quad (\because \text{ LCM of denominators is } (x+3)(x-3)=(x^2-9))$$

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Example 6

$$\frac{1}{x+y} + \frac{1}{x-y} - \frac{1}{(x+y)(x-y)}$$



Solution:

$$\frac{1}{x+y} + \frac{1}{x-y} - \frac{1}{(x+y)(x-y)}$$

$$= \frac{(x-y)+(x+y)-1}{(x+y)(x-y)} \quad (\because \text{ LCM of denominators is } (x+y)(x-y))$$

$$= \frac{x-\cancel{y}+x+\cancel{y}-1}{x^2-y^2} \quad \text{(Simplifying)}$$

$$= \frac{2x-1}{x^2-y^2} \quad \text{(Simplifying)}$$

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4.1.8 Product of Rational Expressions

Example 7

Simplify $\frac{x}{x+2}\cdot\frac{x+3}{x-3}$.

Solution:

$$\frac{x}{x+2}\cdot\frac{x+3}{x-3}=\frac{x(x+3)}{(x+2)(x-3)}=\frac{x^2+3x}{x(x-3)+2(x-3)}=\frac{x^2+3x}{x^2-x-6}$$

Example 8

Multiply $\frac{m-2}{3m+9}$ & $\frac{2m+6}{2m-4}$.

Solution:

$$\frac{m-2}{3m+9}\times\frac{2m+6}{2m-4}=\frac{\cancel{m-2}}{3\cancel{(m+3)}}\cdot\frac{\cancel{2}(m+3)}{\cancel{2}(\cancel{m-2})}=\frac{1}{3}$$

Tid-Bit

A polynomial having three terms is called trinomial.

4.1.9 Dividing a Rational Expression with another and expressing the result in the lowest term

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To find the quotient of two fractions, multiply the first fraction by the reciprocal of the second fraction.

Example 9

Divide $\frac{x^2+x-2}{3x^2+9x+6}\div(x-1)$.

Solution:

$$\frac{x^2+x-2}{3x^2+9x+6}\div(x-1)=\frac{x^2+x-2}{3x^2+9x+6}\times\text{Reciprocal of }(x-1).$$

$$=\frac{x^2+x-2}{3x^2+9x+6}\times\frac{1}{x-1}=\frac{x^2+2x-x-2}{3(x^2+3x+2)}\times\frac{1}{x-1}$$

$$=\frac{x(x+2)-1(x+2)}{3(x^2+2x+x+2)}\times\frac{1}{x-1}=\frac{(x+2)(x-1)}{3[x(x+2)+1(x+2)]}\times\frac{1}{x-1}$$

$$=\frac{\cancel{(x+2)}(x-1)}{3\cancel{(x+2)}(x+1)}\times\frac{1}{x-1}=\frac{1}{3(x+1)}$$

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Exercise 4.1

1. Which of the following expressions are polynomials:

i) $1-5y+8y^2+6y^3$ ii) $\frac{5}{x^2}+\frac{3}{4x+1}$ iii) $\frac{\sqrt{x}}{6x-1}$

2. Which the following rational expressions are in their lowest terms:

i) $\frac{5y^2-5}{y-1}$ ii) $\frac{x^2-9}{x-2}$ iii) $\frac{x+y}{x^2-y^2}$

3. Reduce the following rational expressions to their lowest terms:

i) $\frac{x-5}{x^2-5x}$ ii) $\frac{t^3(t-3)}{(t-3)(t+5)}$ iii) $\frac{x^4+\frac{1}{x^4}}{x^2-\frac{1}{x^2}}$ iv) $\frac{2a+6}{a^2-9}$

4. Add the following rational expressions:

i) $4x^2-5x-10,\ 2x^2+5x+10$ ii) $\frac{y+9}{y^2+3},\ \frac{-7y+7}{y^2+3}$

iii) $\frac{y}{y+4},\frac{2y}{y-4}$ iv) $\frac{t}{t^2-25},\ \frac{3t}{t+5}$

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5. Subtract the first expression from the second in the following:

i) $y^2+4y-15,\ 8y^2+2$ ii) $\frac{8x^2-7}{x^2+1},\frac{8x^2+7}{x^2+1}$

iii) $\frac{1}{a-3},\frac{2a}{a^2-9}$ iv) $\frac{x}{3x-6},\frac{x+2}{x-2}$

6. Simplify the following.

i) $\frac{2x}{6x-9}\cdot\frac{4x-6}{x^2+x}$ ii) $\frac{x+4}{3-x}\cdot\frac{x^2-9}{x^2-16}$ iii) $\frac{3x-15}{2x+6}\cdot\frac{x^2-9}{x^2-25}$

7. Simplify the following.

i) $\frac{2y-10}{3y}\div(y-5)$ ii) $\frac{p}{q}\div\frac{r}{q}\cdot\frac{p}{q}$ iii) $\frac{a^2-9}{(a-6)(a+4)}\div\frac{a-3}{a-6}$

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4.1.10 Value of an Algebraic Expression at Some Particular Real Number

Suppose we are given an algebraic expression which is a polynomial in x of any order. The value of the polynomial $p(x)$ for any real number can be obtained just by replacing x of $p(x)$ by that real number. For example, let $P(x)=3x^2+4x-9$ and we are to find the value of the polynomial at $x=3$ then

$$p(3)=3(3)^2+4\times3-9$$

$$=27+12-9=39-9=30$$

30 is the value of the polynomial $p(x)=3x^2+4x-9$ at $x=3$.

If the algebraic expression is a rational expression $\frac{p(x)}{q(x)}$ then the value of such an algebraic expression can be determined for any real number x provided $q(x)\neq0$. If $q(x)\neq0$ for some real number x then $\frac{p(x)}{q(x)}$ is undefined for that value of x.

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Example 10

When $x=-3$, find the value of:

i) $2x^2-3x$ ii) $(3x)^2-x^2$

Solution:

i) $2x^2-3x=2(-3)^2-3(-3)$ (By putting $x=-3$)

$=2(9)+9=18+9=27$

ii) $(3x)^2-x^2=(3(-3))^2-(-3)^2=(-9)^2-9$ (By putting $x=-3$)

$=81-9=72$



Every time you do this:



A kitten dies.

$$\frac{x^2+2x+1}{x^2+3}$$

$$=\frac{2x+1}{3}.$$

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Example 11

When $x=2,\ y=-3$, evaluate the following:

i) x^2-y^2 ii) $(2x)^2-(3y)^2$

Solution:

i) $x^2-y^2=2^2-(-3)^2=4-9=-5$

ii) $(2x)^2-(3y)^2=(2\times2)^2-\big(3\times(-3)\big)^2$ (By putting $x=2,\ y=-3$)

$=4^2-(-9)^2$

$=16-81=-65$

Exercise 4.2

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1. Evaluate the following when $a=3,\ b=-1,\ c=2$.

i) $5a-10$ ii) $3b+5c$ iii) $2a-3b+2c$

2. Evaluate the following for $x=-5$ and $y=2$

i) $7-3xy$ ii) x^2+xy+y^2 iii) $(3x)^2-(4y)^2$

3. Evaluate the following, when $k=-2,\ \ell=3,\ m=4$.

i) $k^2(2\ell-3m)$ ii) $5m\sqrt{k^2+\ell^2}$ iii) $\dfrac{k+\ell+m}{k^2+\ell^2+m^2}$

4. Evaluate $\dfrac{a+1}{4a^2+1}$ when $a=\dfrac{1}{2}$ and $a=-\dfrac{1}{2}$.

5. If $a=9,\ b=12,\ c=15$ *and* $S=\dfrac{a+b+c}{2}$. Find the value of $\sqrt{S(S-a)(S-b)(S-c)}$.

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4.2 ALGEBRAIC FORMULAS

4.2.1 Establish the Formulas

a) $(a+b)^2+(a-b)^2=2(a^2+b^2)$ b) $(a+b)^2-(a-b)^2=4ab$

c) Finding the values of a^2+b^2 & ab when the values of $a+b$ & $a-b$ are given.

In the previous class we learned about basic algebraic formulas

$$(a+b)^2=a^2+2ab+b^2 \text{ ---------- (1)}$$
$$(a-b)^2=a^2-2ab+b^2 \text{ ---------- (2)}$$
$$a^2-b^2=(a+b)(a-b) \text{ ---------- (3)}$$

a) Adding equation (1) and (2) we get

$$(a+b)^2+(a-b)^2=(a^2+2ab+b^2)+(a^2-2ab+b^2)$$
$$=2a^2+2b^2=2(a^2+b^2)$$

$\therefore\ (a+b)^2+(a-b)^2=2(a^2+b^2)$ - - - - - - - (4)

b) Subtracting equation (2) from (1)

$$(a+b)^2-(a-b)^2=(a^2+2ab+b^2)-(a^2-2ab+b^2)$$
$$=2ab+2ab=4ab$$

$\therefore\ (a+b)^2-(a-b)^2=4ab$ - - - - - - - - (5)

The formulas (4) and (5) are also useful for simplification purposes.

DID YOU KNOW?

$888+88+8+8+8=1000$

$\frac{8+8}{8}(8\times 8\times 8-8)-8=1000$

$\frac{8888-888}{8}=1000$

$8(8\times 8+8\times 8)-8-8-8=1000$

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c) Finding the values of a^2+b^2 & ab when the values of $a+b$ & $a-b$ are given.

Example 12

Find the values of a^2+b^2 and ab, when $a+b=5$, $a-b=2$.

Solution:

$$a+b=5 \Rightarrow (a+b)^2=5^2 \Rightarrow a^2+2ab+b^2=25 \text{ -------(1)}$$

$$a-b=2 \Rightarrow (a-b)^2=2^2 \Rightarrow a^2-2ab+b^2=4 \text{ --------(2)}$$

Adding equations (1) and (2), we get

$$(a^2+2ab+b^2)+(a^2-2ab+b^2)=25+4$$

$$2a^2+2b^2=29 \text{ or } a^2+b^2=\frac{29}{2}$$

From equation (1) $2ab=25-(a^2+b^2)=25-\frac{29}{2}$ (Putting $a^2+b^2=\frac{29}{2}$)

$$2ab=16 \text{ or } ab=8$$

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Exercise 4.3

1. Find the values of x^2+y^2 and xy, when

   i) $x+y=8$, $x-y=3$. ii) $x+y=10$, $x-y=7$.

   iii) $x+y=11$, $x-y=5$. iv) $x+y=7$, $x-y=4$.

2. i) Find the values of a^2+b^2 and ab, when $a+b=7$, $a-b=3$.

   ii) Find the values of a^2+b^2 and ab, when $a+b=9$, $a-b=1$.

3. i) If $a+b=10$, $a-b=6$,then find the value of a^2+b^2.

   ii) If $a+b=5$, $a-b=\sqrt{17}$,then find the value of ab.

4. Find the value of $4xy$ when $x+y=17$ and $x-y=5$.

5. If $x+y=11$ and $x-y=3$, find $8xy\left(x^2+y^2\right)$.

6. If $u+v=7$ and $uv=12$, find $u-v$.

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4.2.2 Derivation of the formula $(a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca$.

Let $b+c=x$, then

$$(a+b+c)^2 = (a+x)^2 = a^2+2ax+x^2 \text{ -------(1)}$$

Putting back the values of x in equation (1), we get

$$(a+b+c)^2 = a^2+2a(b+c)+(b+c)^2$$

$$= a^2+2ab+2ac+b^2+2bc+c^2$$

$\therefore$ $(a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca$ $\quad [\because ac=ca]$

a) Finding value of $a^2+b^2+c^2$ when the values of $a+b+c$ & $ab+bc+ca$ are given

Example 13

Find the value of $a^2+b^2+c^2$, when $a+b+c=9$, $ab+bc+ca=20$.

Solution:

Given that $a+b+c=9$ squaring both side we get

$\therefore$ $(a+b+c)^2 = 9^2 \Rightarrow a^2+b^2+c^2+2ab+2bc+2ca = 81$ ------- (1)

Putting the values of $ab+bc+ca=20$ in equation (1), we get

$a^2+b\ +c^2+2(20)=81$

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or $a^2+b^2+c^2 = 81-2(20) = 81-40 = 41$

b) Finding value of $a+b+c$ when the values of $a^2+b^2+c^2$ & $ab+bc+ca$ are given

Example 14

Find the value of $a+b+c$, when $a^2+b^2+c^2=29$, and $ab+bc+ca=26$.

Solution:

$(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca)$ ------- (1)

Given that $a^2+b^2+c^2=29$, and $ab+bc+ca=26$

Putting these values in equation (1), we get

$(a+b+c)^2 = 29+2(26) = 29+52 = 81$

$\therefore$ $a+b+c = \sqrt{81} = 9$

c) Finding value of $ab+bc+ca$ when the values of $a^2+b^2+c^2$ & $a+b+c$ are given

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Example 15

Find the value of $ab+bc+ca$, where $a+b+c=14$ and $a^2+b^2+c^2=78$.

Solution:

$(a+b+c)^2=12^2 \Rightarrow a^2+b^2+c^2+2(ab+bc+ca)=144$ ------(1)

Putting the value of $a^2+b^2+c^2=78$ in equation (1), we get

$78+2(ab+bc+ca)=144$

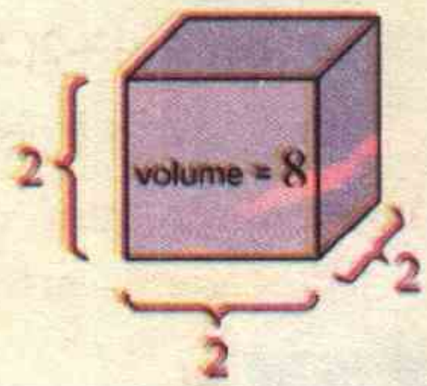
$2(ab+bc+ca)=144-78=66$

$\therefore \quad ab+bc+ca=\frac{66}{2}=33$

Exercise 4.4

1. Find the values of $a^2+b^2+c^2$, when
   i) $a+b+c=5$ and $ab+bc+ca=-4$
   ii) $a+b+c=5$ and $ab+bc+ca=-2$
2. Find the value of $a+b+c$, when
   i) $a^2+b^2+c^2=38$ and $ab+bc+ca=-1$
   ii) $a^2+b^2+c^2=10$ and $ab+bc+ca=11$
3. Find the value of $ab+bc+ca$, when
   i) $a^2+b^2+c^2=56$ and $a+b+c=12$
   ii) $a^2+b^2+c^2=12$ and $a+b+c=5$
4. Prove that $x^2+y^2+z^2-xy-yz-zx=\left(\frac{x-y}{\sqrt{2}}\right)^2+\left(\frac{y-z}{\sqrt{2}}\right)^2+\left(\frac{z-x}{\sqrt{2}}\right)^2$.
5. Write $2\left[x^2+y^2+z^2-xy-yz-zx\right]$ as the sum of three squares.
6. Find the value of $a^2+b^2+c^2-ab-bc-ca$ when $a-b=2, b-c=3, c-a=4$.

$2=\sqrt[3]{8}$



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4.2.3 Derivation of the formulas

i) $(a+b)^3 = a^3 + 3ab(a+b) + b^3$

ii) $(a-b)^3 = a^3 - 3ab(a-b) - b^3$

i) $(a+b)^3 = a^3 + 3ab(a+b) + b^3$

We have $(a+b)^3 = (a+b)(a+b)^2$

$= (a+b)(a^2 + 2ab + b^2)$

$= a \cdot a^2 + a \cdot 2ab + a \cdot b^2 + b \cdot a^2 + b \cdot 2ab + b^2 \cdot b$

$= a^3 + 2a^2b + ab^2 + ba^2 + 2ab^2 + b^3$

$= a^3 + 3a^2b + 3ab^2 + b^3$

$\therefore \quad (a+b)^3 = a^3 + 3ab(a+b) + b^3$

ii) $(a-b)^3 = a^3 - 3ab(a-b) - b^3$

We have $(a-b)^3 = (a-b)(a-b)^2$

$= (a-b)(a^2 - 2ab + b^2)$

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$= a \cdot a^2 - a \cdot 2ab + ab^2 - ba^2 + 2a \cdot b \cdot b - b \cdot b^2$

$= a^3 - 2a^2b + ab^2 - ba^2 + 2ab^2 - b^3$

$= a^3 - 3a^2b + 3ab^2 - b^3$

$\therefore \quad (a-b)^3 = a^3 - 3ab(a-b) - b^3$

a) Find the values of $a^3 \pm b^3$ when the values of $a \pm b$ and ab are given.

Example 16

Find the value of $a^3 + b^3$, when $a+b=5$ and $ab=10$.

Solution:

$(a+b)^3 = a^3 + 3ab(a+b) + b^3$

$5^3 = a^3 + b^3 + 3 \times 10(5)$ (Putting the values of $a+b$ and ab)

$125 = a^3 + b^3 + 50$

$\therefore \quad a^3 + b^3 = 125 - 50 = 75.$

NOTE FOR SALE

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Example 17

Find the value of $a^3 - b^3$, when $a - b = 2$ and $ab = 15$.

Solution:

$$(a-b)^3 = a^3 - b^3 - 3ab(a-b) \text{ -------- (1)}$$

Putting $a - b = 2$ and $ab = 15$ in equation (1), we get

$$2^3 = a^3 - b^3 - 3(15)(2)$$

$$8 = a^3 - b^3 - 90 \quad \Rightarrow a^3 - b^3 = 90 + 8$$

or $\quad a^3 - b^3 = 98$

b) Find the values of $x^3 \pm \frac{1}{x^3}$ when the values of $x \pm \frac{1}{x}$ is given.

i) Derivation of the formula $\left(x + \frac{1}{x}\right)^3 = x^3 + \frac{1}{x^3} + 3\left(x + \frac{1}{x}\right)$.

We have

$$(a+b)^3 = a^3 + 3ab(a+b) + b^3$$

$$= a^3 + b^3 + 3ab(a+b)$$

$$\left(x + \frac{1}{x}\right)^3 = x^3 + \frac{1}{x^3} + 3x \cdot \frac{1}{x}\left(x + \frac{1}{x}\right)$$

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$$\therefore \left(x + \frac{1}{x}\right) = x^3 + \frac{1}{x^3} + 3\left(x + \frac{1}{x}\right) \text{ -------(1)}$$

Hence it is clear that for given value of $x + \frac{1}{x}$, the value of $x^3 + \frac{1}{x^3}$ can be found out using formula (1).

ii) Derivation of the formula $\left(x - \frac{1}{x}\right)^3 = x^3 - \frac{1}{x^3} - 3\left(x - \frac{1}{x}\right)$.

We have $(a-b)^3 = a^3 - 3ab(a-b) - b^3$

$$= a^3 - b^3 - 3ab(a-b)$$

$$\therefore \quad \left(x - \frac{1}{x}\right)^3 = x^3 - \frac{1}{x^3} - 3x \cdot \frac{1}{x}\left(x - \frac{1}{x}\right)$$

$$\therefore \quad \left(x - \frac{1}{x}\right)^3 = x^3 - \frac{1}{x^3} - 3\left(x - \frac{1}{x}\right) \text{ -------(2)}$$

Hence the value of $x^3 - \frac{1}{x^3}$ can be found out using formula (2) having given the value of $x - \frac{1}{x}$.

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Example 18

Find the value of $x^3+\frac{1}{x^3}$, when $x+\frac{1}{x}=5$.

Solution:

$$\left(x+\frac{1}{x}\right)^3 = x^3+\frac{1}{x^3}+3\left(x+\frac{1}{x}\right) \text{ ------- (1)}$$

Putting the value of $x+\frac{1}{x}=5$ in equation (1), we get

$$(5)^3 = x^3+\frac{1}{x^3}+3\cdot 5 \text{ or } x^3+\frac{1}{x^3} = 125-15 = 110$$

Exercise 4.5

1. Find the value of a^3+b^3, when
   i) $a+b=4$ and $ab=5$. ii) $a+b=3$ and $ab=20$.
   iii) $a+b=4$ and $ab=2$.

2. Find the value of a^3-b^3, when
   i) $a-b=5$ and $ab=7$. ii) $a-b=2$ and $ab=15$.
   iii) $a-b=7$ and $ab=6$.

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3. Find the value of $x^3+\frac{1}{x^3}$, when
   i) $x+\frac{1}{x}=\frac{5}{2}$ ii) $x+\frac{1}{x}=2$

4. Find the value of $x^3-\frac{1}{x^3}$, when
   i) $x-\frac{1}{x}=\frac{3}{2}$ ii) $x-\frac{1}{x}=\frac{7}{3}$ iii) $x-\frac{1}{x}=\frac{15}{4}$

5. If $3a+\frac{1}{a}=4$, find $27a^3+\frac{1}{a^3}$.

6. If $x-\frac{1}{2x}=6$, find $x^3-\frac{1}{8x^3}$.

7. If $a+b=6$, show that $a^3+b^3+18ab=216$.

8. If $u-v=3$ then prove that $u^3-v^3-9uv=27$.

9. If $a+\frac{1}{a}=2$, find the values of $a^2+\frac{1}{a^2}$, $a^4+\frac{1}{a^4}$, $a^3+\frac{1}{a^3}$.

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4.2.4 Derivation of the formula $a^3 \pm b^3 = (a \pm b)(a^2 \pm ab + b^2)$.

$$\text{R.H.S} = (a+b)(a^2 - ab + b^2)$$
$$= a \cdot a^2 - a \cdot ab + ab^2 + ba^2 - b \cdot ab + b \cdot b^2$$
$$= a^3 - a^2b + ab^2 + a^2b - ab^2 + b^3$$
$$= a^3 + b^3 \quad = \text{L.H.S}$$

a) Derivation of the formula $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$.

$$\text{R.H.S} = (a-b)(a^2 + ab + b^2)$$
$$= a \cdot a^2 + a \cdot ab + ab^2 - ba^2 - b \cdot ab - b \cdot b^2$$
$$= a^3 + a^2b + ab^2 - a^2b - ab^2 - b^3$$
$$= a^3 - b^3 \quad = \text{L.H.S}$$

Example 19

Find the value of $a^3 + b^3$, when $a+b=4$, $ab=3$.

Solution:

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We know that

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$
$$= (a+b)\{(a+b)^2 - 3ab\} \quad \text{----(1)}$$

Putting $a+b=4$, $ab=3$ in equation (1), we get

$$a^3 + b^3 = (4)(4^2 - 3 \cdot 3) = 4(16-9) = 4 \times 7 = 28$$

Example 20

Find the value of $a^3 - b^3$, when $a-b=2$, $ab=10$.

Solution:

We know that

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$
$$= (a-b)(a^2 + b^2 - 2ab + 2ab + ab)$$
$$= (a-b)\{(a-b)^2 + 3ab\} \quad \text{----(1)}$$

Putting $a-b=2$ and $ab=10$ in equation (1), we get

$$a^3 - b^3 = (2)(2^2 + 3 \times 10) = 2(4+30) = 2 \times 34 = 68$$

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b) **To find the product** $\left(x+\frac{1}{x}\right)\left(x^2-1+\frac{1}{x^2}\right)$.

We know that $a^3+b^3=(a+b)(a^2-ab+b^2)$

Here $a=x$, and $b=\frac{1}{x}$

$$\therefore\quad x^3+\frac{1}{x^3}=\left(x+\frac{1}{x}\right)\left(x^2-x\cdot\frac{1}{x}+\frac{1}{x^2}\right)$$

$$=\left(x+\frac{1}{x}\right)\left(x^2-1+\frac{1}{x^2}\right)$$

Hence, the required product $= x^3+\frac{1}{x^3}$

c) **To find the product** $\left(x-\frac{1}{x}\right)\left(x^2+\frac{1}{x^2}+1\right)$

We know that

$a^3-b^3=(a-b)(a^2+ab+b^2)$

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Here $a=x$, and $b=\frac{1}{x}$

$$\therefore\quad x^3-\frac{1}{x^3}=\left(x-\frac{1}{x}\right)\left(x^2+x\cdot\frac{1}{x}+\frac{1}{x^2}\right)$$

$$=\left(x-\frac{1}{x}\right)\left(x^2+1+\frac{1}{x^2}\right)$$

Hence, the required product $= x^3-\frac{1}{x^3}$

d) **To find the continued product of**

$(x+y)(x-y)(x^2+xy+y^2)(x^2-xy+y^2)$

We have $(x+y)(x-y)(x^2+xy+y^2)(x^2-xy+y^2)$

$=(x+y)(x^2-xy+y^2)\cdot(x-y)(x^2+xy+y^2)$

$=(x^3+y^3)\cdot(x^3-y^3)$ (Using the formulas)

$=(x^3)^2-(y^3)^2$

$=x^6-y^6$

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Exercise 4.6

1. Find the following product.

i) $(a-1)(a^2+a+1)$

ii) $(3-b)(9+3b+b^2)$

iii) $(8+b)(64-8b+b^2)$

iv) $(a+2)(a^2-2a+4)$

2. Find the following product.

i) $\left(2p+\frac{1}{2p}\right)\left(4p^2+\frac{1}{4p^2}-1\right)$

ii) $\left(\frac{3}{2}p-\frac{2}{3p}\right)\left(\frac{9}{4}p^2+\frac{4}{9p^2}+1\right)$

iii) $\left(3p-\frac{1}{3p}\right)\left(9p^2+\frac{1}{9p^2}+1\right)$

iv) $\left(5p+\frac{1}{5p}\right)\left(25p^2+\frac{1}{25p^2}-1\right)$

3. Find the following continued product.

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i) $(x^2-y^2)(x^2-xy+y^2)(x^2+xy+y^2)$

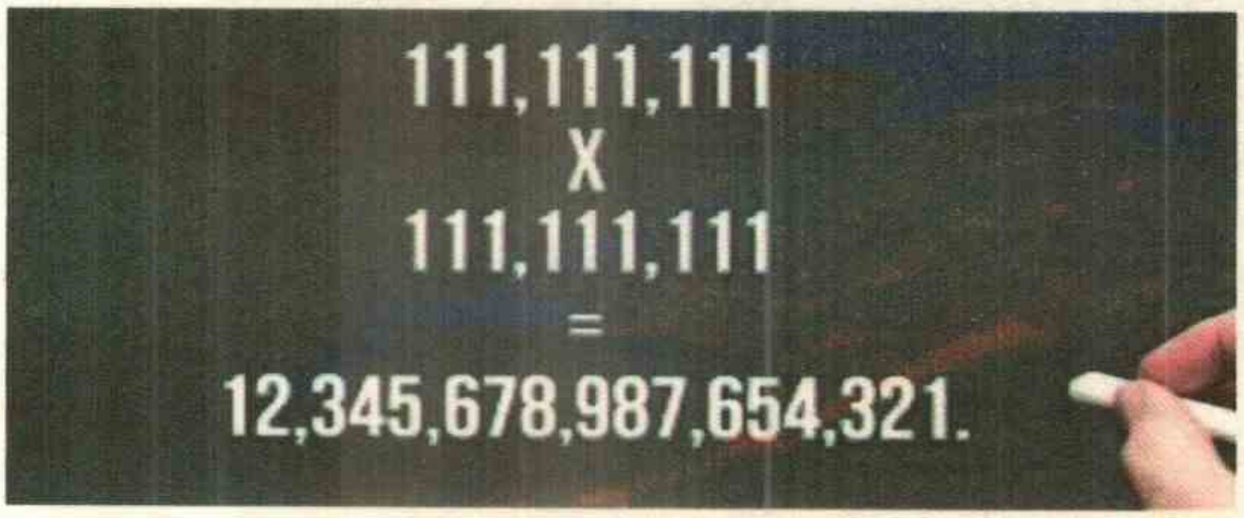
ii) $(x+y)(x-y)(x^2+y^2)(x^4+y^4)$

iii) $(2x-y)(2x+y)(4x^2-2xy+y^2)(4x^2-2xy+y^2)$

iv) $(x-2)(x+2)(x^2-2x+4)(x^2+2x+4)$

4. Find the product with the help of formula, $\left(\sqrt{x}-\sqrt{y}\right)\left(x+\sqrt{xy}+y\right)$.

5. Simplify with the help of formula, $\left(x^p+y^q\right)\left(x^{2p}-xy^{pq}+y^{2q}\right)$.



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4.3 SURDS AND THEIR APPLICATION

4.3.1 Surds

A number of the form $\sqrt[n]{a}$, where a is a positive rational number is called a surd.

A number, therefore, will be a surd if, and only if, (i) it is irrational and (ii) it is a root and (iii) a root of a rational number.

For example $\sqrt{3}$ and $\sqrt{5+\sqrt{3}}$ are both irrational numbers, but they are of different types. First number is a root of a rational number 3, whereas the second number is a root of an irrational number $5+\sqrt{3}$. The number of the first type is a surd. The number $\sqrt[3]{8}$ is not a surd. It has a form of a surd number; but its value 2 is rational. $\sqrt{-2}$, $\sqrt{-3}$ are not surds because -2 and -3 are negative.

4.3.2 Order of a Surd

In a surd $\sqrt[n]{a}$, n is called the order of the surd or the surd index and a is called the radicand.

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Thus if n = 2 then it is called a surd of order 2 i.e. $\sqrt{a}$ is a surd of order 2.

If n = 3 then it is called surd of order 3 and if n = 4 then it is called surd of order 4 and so on.

Surds of second, third and fourth orders are called quadratic, cubic and biquadratic surds respectively. For example $\sqrt{5}, \sqrt[3]{7}, \sqrt[4]{3}$ are quadratic, cubic and biquadratic surds respectively.

Surds of the same order are called equiradical surds. For example $\sqrt[3]{5}, \sqrt[3]{2}$ are equiradical surds.

4.3.3 Types of a Surd

Surds are also named w.r.t to the number of terms. Surd expressions involving one, two, and three terms are called monomial, binomial and trinomial surds respectively.

For example $\sqrt{3}$, $\sqrt{5}-1$, $2-\sqrt{7}+\sqrt{11}$ are monomial, binomial and trinomial surds respectively. In general expressions containing two or more surds are called compound surds. Thus $2-\sqrt{7}+\sqrt{11}$ is a compound surd while $\sqrt{3}$, $\sqrt{5}-1$ are not compound surd.

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If the coefficient of the surd is ± 1, the surd is called an entire surd; otherwise it is known as a mixed surd.

For example $\sqrt{5}, \sqrt[3]{3}, \sqrt[4]{7}$ are entire surds and $-4\sqrt[3]{5}, \frac{3}{4}\sqrt[3]{2}$ are mixed surds.

Surds which can be expressed as rational multiples of the same surd are called like or similar surds; and those which cannot be so expressed are called unlike or dissimilar. Thus $\sqrt{8}$, $\sqrt{18}$ are similar surds as they can be expressed as $\sqrt{8} = 2\sqrt{2}$ and $\sqrt{18} = 3\sqrt{2}$ and whereas $\sqrt{8}$, $\sqrt[3]{81}$, are unlike surds.

4.3.4 Conjugate Surds

Two binomial quadratic surds which differ only in the sign connecting their terms are called conjugate surds.

Thus $a\sqrt{x}+b\sqrt{y}$ and $a\sqrt{x}-b\sqrt{y}$ are conjugate surds. These are always rationalizing factors of each other.

i.e. $\left(a\sqrt{x}+b\sqrt{y}\right)\left(a\sqrt{x}-b\sqrt{y}\right)=a^2x-b^2y$

Thus $5+\sqrt{3}$ and $5-\sqrt{3}$, $2-\sqrt{5}$ and $2+\sqrt{5}$ are called conjugate surds of each other.

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4.4 RATIONALIZATION

The procedure of removing the surd quantity from the denominator of the surd fraction is very convenient and is known as the process of rationalizing the denominator or rationalization.

For purposes of calculating approximate values of expressions containing surds, much tedious work can be saved by first rationalizing the surd expression.

For example, if it is required to find the numerical value of $\frac{\sqrt{5}}{\sqrt{3}}$ .we have to find the value of $\sqrt{5}$ (i.e. 2.236), and the value of $\sqrt{3}$ (i.e. 1.732),and then obtain the quotient $\frac{2.236}{1.732}$, which is too tedious. Much of this labour can be saved by multiplying the numerator and denominator of $\frac{\sqrt{5}}{\sqrt{3}}$ by $\sqrt{3}$.

Thus $$\frac{\sqrt{5}}{\sqrt{3}}=\frac{\sqrt{5}}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}=\frac{\sqrt{15}}{3}=\frac{3.87298}{3}=1.2909$$

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Example 21

If $x = 2+\sqrt{3}$, find the values of $x+\frac{1}{x}$ and $x^2+\frac{1}{x^2}$.

Solution:

Since $x = 2+\sqrt{3}$, $\frac{1}{x} = \frac{1}{2+\sqrt{3}}$.

$$\frac{1}{x} = \frac{2-\sqrt{3}}{(2+\sqrt{3})(2-\sqrt{3})}$$

(Multiplying the denominator and numerator by $2-\sqrt{3}$ i.e. by the conjugate of $2+\sqrt{3}$)

$$\frac{1}{x} = \frac{2-\sqrt{3}}{4-3} = 2-\sqrt{3}$$

$$\therefore \quad x+\frac{1}{x} = (2+\sqrt{3})+(2-\sqrt{3}) = 4$$

Now, $\left(x+\frac{1}{x}\right)^2 = 16 \Rightarrow x^2+\frac{1}{x^2}+2 = 16$

$$\therefore \quad x^2+\frac{1}{x^2} = 14$$

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DID YOU KNOW?

There are just four numbers (after 1) which are the sums of the cubes of their digits:

$153 = 1^3 + 5^3 + 3^3$

$370 = 3^3 + 7^3 + 0^3$

$371 = 3^3 + 7^3 + 1^3$

$407 = 4^3 + 0^3 + 7^3$

Example 22

If $x = \sqrt{3}-\sqrt{2}$, find the values of $x-\frac{1}{x}$ and $x^2+\frac{1}{x^2}$.

Solution:

Since $x = \sqrt{3}-\sqrt{2}$, $\frac{1}{x} = \frac{1}{\sqrt{3}-\sqrt{2}}$.

$$\frac{1}{x} = \frac{\sqrt{3}+\sqrt{2}}{(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})} = \frac{\sqrt{3}+\sqrt{2}}{3-2} = \sqrt{3}+\sqrt{2}$$

$$\therefore \quad x-\frac{1}{x} = (\sqrt{3}-\sqrt{2})-(\sqrt{3}+\sqrt{2}) = -2\sqrt{2}$$

Now, $\left(x-\frac{1}{x}\right)^2 = (-2\sqrt{2})^2 \Rightarrow x^2+\frac{1}{x^2}-2 = 4\times 2 = 8$

$$\therefore \quad x^2+\frac{1}{x^2} = 10$$

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Exercise 4.7

1. State which of the following are surd quantities.

i) $\sqrt[3]{81}$ ii) $\sqrt{1+\sqrt{5}}$ iii) $\sqrt{\sqrt{5}}$

iv) $\sqrt[4]{32}$ v) π vi) $\sqrt{1+\pi^2}$

2. Express the following as the simplest possible surds:

i) $\sqrt{12}$ ii) $\sqrt{48}$ iii) $\sqrt{240}$

3. Simplify the following surds:

i) $(2-\sqrt{3})(3+\sqrt{5})$ ii) $(\sqrt{3}-4)(\sqrt{2}+1)$

iii) $(\sqrt{2}+\sqrt{3})(\sqrt{5}+\sqrt{2})$ iv) $(3-2\sqrt{3})(3+2\sqrt{3})$

4. Rationalize the denominator and simplify:

i) $\frac{1}{\sqrt{7}}$ ii) $\frac{3}{\sqrt{45}}$ iii) $\frac{1}{\sqrt{2}-1}$

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iv) $\frac{5}{2+\sqrt{5}}$ v) $\frac{1}{\sqrt{5}-2}+\frac{1}{\sqrt{5}+2}$

5. If $x=\sqrt{5}+2$, find the values of $x+\frac{1}{x}$ and $x^2+\frac{1}{x^2}$.

6. If $x=\sqrt{2}+\sqrt{3}$, find the values of $x-\frac{1}{x}$ and $x^2+\frac{1}{x^2}$.

7. If $x=5-2\sqrt{6}$, find the values of $x+\frac{1}{x}$ and $x^2+\frac{1}{x^2}$.

8. If $x=\frac{1}{\sqrt{2}-1}$, find the values of $x-\frac{1}{x}$ and $x^2+\frac{1}{x^2}$.

9. If $x=\sqrt{10}+3$, find the values of $x-\frac{1}{x}$ and $x^2+\frac{1}{x^2}$.

10. If $x=2-\sqrt{3}$, find value of $x^4+\frac{1}{x^4}$.

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REVIEW EXERCISE 4

1. Select the correct answer.

i) $mr^2+3mr^2-5mr^2=$

a) $-mr^2$ b) $-mr$ c) mr d) mr^2

ii) $(x^3y^2)(x^2y^3)=$

a) x^5y^5 b) x^5y^4 c) x^4y^5 d) x^4y^4

iii) $(4xy^4)^3=$

a) $64x^3y^8$ b) $64x^3y^8$ c) $64x^3y^{12}$ d) $64x^3y^7$

iv) $(7x+4y)-(3x-6y)=$

a) $3x$ b) $2x+10y$ c) $4x+y$ d) $4x+10y$

v) $(a+b)^2-(a-b)^2=$

a) $4ab$ b) $2(a^2+b^2)$

c) $a^2-4ab+2b^2$ d) a^4-b^4

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vi) $(a+b+c)^2=$

a) $a^2+b^2+c^2$

b) $a^2+b^2+c^2+2(a+b+c)$

c) $a^2+b^2+c^2+2(ab+bc+ca)$

d) $a+b+c+2(ab+bc+ca)$

vii) $a^3+b^3=$

a) $(a+b)^3-2ab(a+b)$ b) $(a+b)(a^2+ab+b^2)$

c) $(a-b)(a^2-ab+b^2)$ d) $(a+b)(a^2-ab+b^2)$

viii) Conjugate of $3-\sqrt{5}$ is

a) $-3-\sqrt{5}$ b) $-3+\sqrt{5}$

c) $3+\sqrt{5}$ d) $\sqrt{3}+\sqrt{5}$

NOTE FOR SALE

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ix) Which of the following expressions is equivalent to the expression $(m^2+4)^{-\frac{1}{2}}$.

a) $\frac{(m^2+4)}{2}$ b) $-\sqrt{m^2+4}$

c) $\frac{1}{\sqrt{m^2+4}}$ d) $\frac{1}{m+2}$

x) For which of the following expressions $a+b$ is not a factor?

a) a^2-b^2 b) a^2+b^2

c) a^3+b^3 d) a^4-b^4

2. Simplify: $\frac{12x^4y^5}{25a^3b^4} \cdot \frac{15a^5b^4}{16x^7y^2}$

3. Evaluate $\frac{2x-3}{x^2-x+1}$ for $x=2$.

4. Find the values of x^2+y^2 and xy, when $x+y=7$, $x-y=3$.

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5. Find the values of $a+b+c$, when $a^2+b^2+c^2=43$ and $ab+bc+ca=3$.

6. If $a+b+c=6$ and $a^2+b^2+c^2=24$, then find the value of $ab+bc+ca$.

7. If $2x-3y=8$ *and* $xy=2$, then find the value of $8x^3-27y^3$

8. If $x+\frac{1}{x}=8$, then find the value of $x^3+\frac{1}{x^3}$.

9. Find the product $\left(\frac{4}{5}x-\frac{5}{4x}\right)\left(\frac{16}{25}x^2+\frac{25}{16x^2}+1\right)$

Think

10. Simplify $\frac{2x^2}{x^4-16}-\frac{x}{x^2-4}+\frac{1}{x+2}$

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SUMMARY

- An algebraic expression is a statement that connects variables and constants by the help algebraic operations +, –, ×, ÷ .
- An expressions of the form $\frac{p(x)}{q(x)}$, where $p(x)$ & $q(x)$ are polynomials and $q(x) \neq 0$, is called a rational expression.
- A rational expression $\frac{p(x)}{q(x)}$ is said to be in lowest term if $p(x)$ & $q(x)$ are polynomials with integral coefficients and having no factor common.
- A number of the form $\sqrt[n]{a}$, where a is a positive rational number is called a surd..
- In a surd $\sqrt[n]{a}$, n is called the order of the surd or the surd index and a is called the radicand.
- Some important formulas

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$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a-b)^2 = a^2 - 2ab + b^2$$

$$a^2 - b^2 = (a+b)(a-b)$$

$$(a+b)^2 + (a-b)^2 = 2(a^2 + b^2)$$

$$(a+b)^2 - (a-b)^2 = 4ab$$

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$

$$(a+b)^3 = a^3 + 3ab(a+b) + b^3$$

$$(a-b)^3 = a^3 - 3ab(a-b) - b^3$$

$$\left(x + \frac{1}{x}\right)^3 = x^3 + \frac{1}{x^3} + 3\left(x + \frac{1}{x}\right)$$

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$





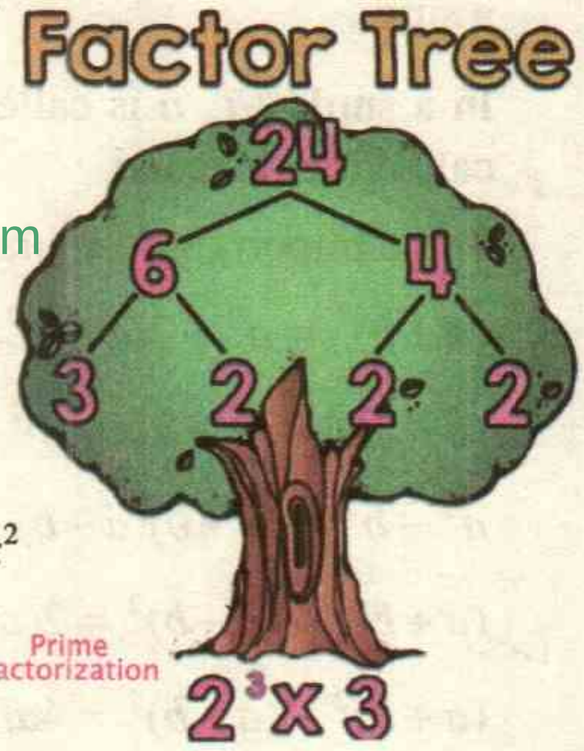
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UNIT 05

FACTORIZATION

After reading this unit, the students will be able to:

- Recall factorization of expressions of the following types.
  - $ka + kb + kc$,
  - $ac + ad + bc + bd$,
  - $a^2 \pm 2ab + b^2$,
  - $a^2 - b^2$,
  - $a^2 \pm 2ab + b^2 - c^2$.
- Faxtorize the expressions of the following types:
  - Type I: $a^4 + a^2b^2 + b^4$ or $a^4 + 4b^4$
  - Type II: $x^2 + px + q$.
  - Type III: $ax^2 + bx + c$
  - Type IV: $(ax^2 + bx + c)(ax^2 + bx + d) + k$
    $(x+a)(x+b)(x+c)(x+d) + k$
    $(x+a)(x+b)(x+c)(x+d) + kx^2$
  - Types V: $a^3 + 3a^2b + 3ab^2 + b^3$
    $a^3 - 3a^2b + 3ab^2 - b^3$
  - Type VI: $a^3 \pm b^3$
- State and prove Remainder theorem and explain through examples.
- Find remainder (without dividing) when a polynomial is divided by a linear polynomial.
- Define zeros of a polynomial.
- State and prove factor theorem.
- Use factor theorem to factorize a Cubic Pohynomial.



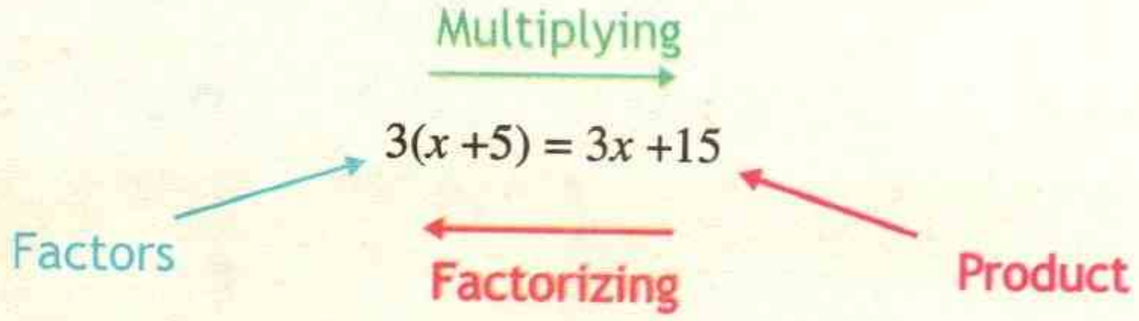
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Introduction

In Unit 4 we were given factors and asked to find a product. We are now going to reverse the process. We will be given a polynomial and asked to find its factors. This is called **factorization**.

In Unit 4, we have studied about algebraic expressions, polynomials and their multiplication to get a product. Before we start the new topic, we recall some previous definitions, some type of factorization for our logical warm-up and also for the reason that in Mathematics, each extended idea builds a new storey to old structure.



5.1 FACTORIZATION

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Writing an algebraic expression as the product of two or more algebraic expressions is called factorization of the algebraic expression. For example,

$5x+10x^2 = 5x(1+2x)$ (As $5x$ is a common factor to each term.)

This shows that $5x(1+2x)$ is the factorization of algebraic expression $5x+10x^2$,

whereas $5x$ and $(1+2x)$ are called factors of the given expression.

Recall the factorization of the following types.

5.1.1 Factorization of expressions of the type $ka+kb+kc$

Example 1

Factorize each of the following.

a) $15+10x-5x^2$ b) $12x^2y^2-20x^3y$

Solution:

a) $15+10x-5x^2 = 5\cdot3+5\cdot2x-5\cdot x^2 = 5(3+2x-x^2)$

b) $12x^2y^2-20x^3y = 4x^2y\cdot3y-4x^2y\cdot5x = 4x^2y(3y-5x)$

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5.1.2 Factorization of Expression of the Type $ac+ad+bc+bd$

Example 2

Factorize $a^2-ab-3a+3b$.

Solution:

$a^2-ab-3a+3b$

$=(a^2-ab^2)+(-3a+3b)$ (Grouping the terms)

$=a(a-b)-3(a-b)$ (Taking out a common from 1st group and -3 from 2nd group)

$=(a-b)(a-3)$ (As $(a-b)$ is a common factor)

5.1.3 Factorization of the Type $a^2 \pm 2ab+b^2$

Example 3

Factorize each of the following.

a) $x^2+8x+16$ b) $25y^2-30y+9$

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Solution:

a) $x^2+8x+16 = x^2+2\cdot x\cdot 4+4^2=(x+4)^2$

b) $25y^2-30y+9 =(25y)^2+2\cdot 5y\cdot 3+3^2=(5y-3)^2$

5.1.4 Factorization of the Type a^2-b^2

Example 4

Factorize each of the following.

a) x^2-16 b) $9a^2-25$ c) $6x^4-6y^4$

Solution:

a) $x^2-16= x^2-4^2=(x+4)(x-4)$

b) $9a^2-25=(3a)^2-52=(3a+5)(3a-5)$

c) $6x^4-6y^4=6(x^4-y^4)$

$=6(x^2+y^2)(x^2-y^2)$

$=6(x^2+y^2)(x+y)(x-y)$

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5.1.5 Factorization of the Type $a^2 \pm 2ab + b^2 - c^2$

Example 5

Factorize $a^2 + 4ab + 4b^2 - c^2$

Solution:

$$a^2 + 4ab + 4b^2 - c^2 = a^2 + 2(a)(2b) + (2b)^2 - c^2$$
$$= (a + 2b)^2 - (c)^2$$
$$= (a + 2b + c)(a + 2b - c)$$



Example 6

Factorize $a^2 - b^2 + 2b - 1$

Solution:

$$a^2 - b^2 + 2b - 1$$
$$= a^2 - (b^2 - 2b + 1)$$
$$= a^2 - \{(b)^2 - 2(b)(1) + (1)^2\}$$
$$= a^2 - (b-1)^2$$
$$= \{a + (b-1)\}\{a - (b-1)\}$$
$$= (a + b - 1)(a - b + 1)$$

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Exercise 5.1

Factorize the following expression:

1. $9s^3t + 15s^2t^3 - 3s^2t^2$
2. $10a^2b^3c^4 - 15a^3b^2c^2 + 30a^4b^3c^2$
3. $ax - a - x + 1$
4. $x^2 - 2y^3 - 2xy^2 + xy$
5. $4x^2 + 4 + \frac{1}{x^2}$
6. $4(x+y)^2 - 20(x+y)z + 25z^2$
7. $\frac{x^4}{y^4} - \frac{y^4}{x^4}$
8. $2x^2 - 288$
9. $1 + 2uv - u^2 - v^2$
10. $25a^2b^2 - 20abc + 4c^2 - 16d^2$

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5.1.6 Factorization of the Expression of the Type-I $a^4+a^2b^2+b^4$ or a^4+4b^4

This type includes those algebraic expressions which are neither perfect squares nor in the form of the difference of two squares. But the addition and subtraction of some suitable terms make it to convert in the form of a^2-b^2 and become factorizable. For further explanation consider the following examples.

Example 7

Factorize a^4+4b^4

Solution:

Since a^4+4b^4 is not a perfect square, therefore we add $2(a^2)(2b^2)=4a^2b^2$ then expression becomes perfect square, but we must subtract $4a^2b^2$, so that the value of the expression is not changed.

$$a^4+4b^4=a^4+4b^4+4a^2b^2-4a^2b^2$$
$$=(a^2)^2+(2b^2)^2+2(a^2)(2b^2)-(2ab)^2$$
$$=(a^2+2b^2)^2-(2ab)^2$$
$$=(a^2+2b^2+2ab)(a^2+2b^2-2ab)$$

$[\because\ a^2-b^2=(a+b)(a-b)]$

Example 8

Factorize $a^4+a^2b^2+b^4$.

Solution:

If we add a^2b^2, the given expression becomes a perfect square. Therefore we add and subtract a^2b^2.

$$a^4+a^2b^2+b^4=a^4+a^2b^2+b^4+a^2b^2-a^2b^2$$
$$a^4+a^2b^2+b^4=\{(a^2)^2+2(a^2)(b^2)+(b^2)^2\}-a^2b^2$$
$$=(a^2+b^2)^2-a^2b^2$$
$$=(a^2+b^2)^2-(ab)^2$$
$$=(a^2+b^2+ab)(a^2+b^2-ab)$$

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Example 9

Factorize $x^4 - 7x^2 + 1$.



Solution:

$$x^4 - 7x^2 + 1 = x^4 + 1 - 7x^2$$

Add and subtract $2x^2$

$$x^4 - 7x^2 + 1 = x^4 + 1 - 7x^2 + 2x^2 - 2x^2$$

$$= x^4 + 1 + 2x^2 - 7x^2 - 2x^2$$

$$= \left\{(x^2)^2 + (1)^2 + 2(x^2)(1)\right\} - 9x^2$$

$$= (x^2 + 1)^2 - (3x)^2$$

$$= (x^2 + 1 + 3x)(x^2 + 1 - 3x)$$

$$= (x^2 + 3x + 1)(x^2 - 3x + 1)$$

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Exercise 5.2

Factorize the following.

1. $x^4 + 64$
2. $4x^4 + 81$
3. $a^4 + a^2b^2 + b^4$
4. $x^4 + x^2 + 1$
5. $x^8 + x^4 + 1$
6. $x^4 + \frac{1}{x^4} - 7$
7. $81x^4 + \frac{1}{81x^4} - 14$
8. $4x^4 - 4x^2y^2 + 64y^4$
9. $16m^4 + 4m^2n^2 + n^4$
10. $4x^5y + 11x^3y^3 + 9xy^5$

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5.1.7 Factorization of Trinomials of the Type-II: x^2+bx+c

This type of trinomials can be factored into the product of two binomials. To factor a trinomial of the form x^2+bx+c, we look for binomial factors of the form $(x+p)(x+q)$, where $pq=c$ and $p+q=b$. That is, we look for two numbers p and q whose sum is the coefficient of the middle term of the polynomial, b, and whose product is the constant term, c.

Example 10

Factorize: $y^2-7y+12$

Solution:

$y^2-7y+12$

We look for two numbers whose product is 12 and whose sum is –7. Since the constant term, 12, is positive and the coefficient of the middle term, –7, is negative, we look for a factorization of 12 in which both factors are negative.

Pairs of Factors	Sum of Factors
–1, –12	–13
2, 6	–8
3, 4	–7

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The numbers we need are –3 and –4.

Thus we have,

$$y^2-7y+12=(y-3)(y-4)$$

We are find the factors of –90 with opposite the signs whose sum is 13.

Factors of 90	Paris whose sum is 13.
1, -1	2 | 90
2, -2	3 | 45
3, -3	3 | 15
5, -5	By inspection the factors are $2\times3\times3,5$
6, -6	*i.e.* 18,5
9, -9	
10, -10	
15, -15	
18, -18	
30, -30	
45, -45	
90, -90	

Example 11

Factorize $x^2+10x+21$.

Solution:

Here 3 and 7 are two numbers whose sum is 10 and product is 21.

Therefore

$$x^2+10x+21=(x+3)(x+7)$$

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Example 12

Find an expression for the perimeter of a rectangle whose area is given by $x^2+13x-90$.

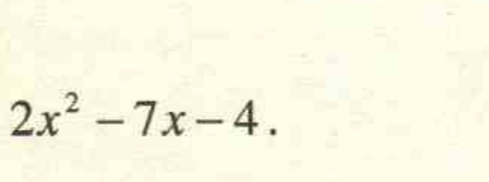
Solution:

Area $= x^2+13x-90$

To answer the problem, we factorize the trinomial.

$$x^2+13x-90=(x+18)(x-5)$$

Taking,



The perimeter p of the rectangle is

$$p=2(x+18)+2(x-5)$$

$$p=2x+36+2x-10=4x+26$$

5.1.8 Factorization of the Type - III: ax^2+bx+c

To factorize the expression ax^2+bx+c, remember the following steps.

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1. Find out the product ac where a is the co-efficient of x^2 and c is constant.
2. Then find two numbers b_1 and b_2 such that $b_1b_2=ac$ and $b_1+b_2=b$.

   Hence $ax^2+bx+c=ax^2+b_1x+b_2x+c$.

Example 13

Factorize $2x^2-7x-4$.

Solution:

Comparing $2x^2-7x-4$ with ax^2+bx+c, we have

$a=2, b=-7, \quad c=-4$

Therefore, $\quad ac=2(-4)=-8$

Now find two numbers whose sum is –7 and product is –8. The numbers are 1 and –8.

Thus,
$$\begin{aligned}2x^2-7x-4&=2x^2+x-8x-4\\&=(2x^2+x)-(8x+4)\\&=x(2x+1)-4(2x+1)\\&=(2x+1)(x-4)\end{aligned}$$

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Exercise 5.3

Factorize the following.

1. $x^2-7x+12$
2. x^2+x-12
3. $20-x-x^2$
4. $2y^2-7y+3$
5. $4x^2+8x+3$
6. $10y^2-3y-1$
7. $6x^3-15x^2-9x$
8. $2xy^2+8xy-24x$
9. $2+5t-12t^2$
10. $-16x^3y-20x^2y^2-6xy^3$
11. $(x+1)^2+3(x+1)+2$
12. $4x^8y^{10}-40x^5y^7+84x^2y^4$
13. Find an expression for the perimeter of a rectangle with area given by $x^2+24x-81$.

5.1.9 Factorization of the Type - IV $(ax^2+bx+c)(ax^2+bx+d)+k$

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Example 14

Factorize $(x^2+7x+10)(x^2+7x+12)+1$

Solution:

$(x^2+7x+10)(x^2+7x+12)+1$

Let $x^2+7x=y$, then we have

$(y+10)(y+12)+1$

$=y^2+12y+10y+120+1$

$=y^2+22y+121$

$=y^2+11y+11y+121$

$=y(y+11)+11(y+11)$

$=(y+11)(y+11)$

$=(y+11)^2$ Putting the value of y.

$\therefore (x^2+7x+10)(x^2+7x+12)+1=(x^2+7x+11)^2$

$=(x^2+7x+11)(x^2+7x+11)$

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Example 18

Factorize $8a^3+36a^2b+54ab^2+27b^3$

Solution:

$$8a^3+36a^2b+54ab^2+27b^3$$
$$=(2a)^3+3(2a)^2(3b)+3(2a)(3b)^2+(3b)^3$$
$$=(2a+3b)^3$$

Example 19

Factorize $27x^3-27x^2y+9xy^2-y^3$.

Solution:

$$27x^3-27x^2y+9xy^2-y^3$$
$$=(3x)^3-3(3x)^2(y)+3(3x)(y)^2-(y)^3$$
$$=(3x-y)^3$$

Exercise 5.4

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Factorize the following.

1. $(4x^2-16x+7)(4x^2-16x+15)+16$
2. $(9x^2+9x-4)(9x^2+9x-10)-72$
3. $(x+2)(x+4)(x+6)(x+8)-9$
4. $x(x+1)(x+2)(x+3)+1$
5. $(x+1)(x+2)(x+3)(x+6)-3x^2$
6. $64x^3-144x^2y+108xy^2-27y^3$
7. $\frac{a^3}{8}-\frac{1}{4}a^2b+\frac{1}{6}ab^2-\frac{b^3}{27}$
8. $\frac{x^3}{a^3}+\frac{3x}{a}+\frac{3a}{x}+\frac{a^3}{x^3}$
9. $27a^3+189a^2b+441ab^2+343b^3$
10. $8x^3-4x+\frac{2}{3x}-\frac{1}{27x^3}$



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5.1.13 Factorization of an expression of the Type-VI a^3+b^3 and a^3-b^3

We know that

$$a^3+b^3=(a+b)(a^2-ab+b^2)$$

and $$a^3-b^3=(a-b)(a^2+ab+b^2)$$

Now we use these formulas to factorize different expressions.

Example 20

Factorize $2x^3+16$

Solution:

$$2x^3+16=2(x^3+8)$$
$$=2[(x)^3+(2)^3]$$
$$=2(x+2)(x^2-2x+4)$$

Example 21

Factorize a^3-64b^3.

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Solution:

$$a^3-64b^3$$
$$=(a)^3-(4b)^3$$
$$=(a-4b)\{(a)^2+(a)(4b)+(4b)^2\}$$
$$=(a-4b)(a^2+4ab+16b^2)$$

Exercise 5.5

Factorize the following.

1. a^3-27
2. a^6+b^6
3. $24x^3+3$
4. $1-27r^3$
5. $2x^3-128$
6. $4x^5-256x^2$
7. $18(x-y)^3-144(a-b)^3$
8. x^9+1
9. $a^3+(c+d)^3$
10. $27x^3-y^3$

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5.2 REMAINDER THEOREM AND FACTOR THEOREM

5.2.1 Remainder Theorem

Statement:

When a polynomial $P(x)$ of degree $n \geq 1$ is divided by $x - r$, gives a constant R as remainder then $R = P(r)$.

Proof:

Let $P(x)$ be a polynomial of degree $n \geq 1$, divided by $x - r$, remainder R is obtained. i.e.

$$\text{Divisor} \overline{)\ \text{Dividend}}^{\ \text{Quotient}}$$

....

....

Remainder

∴ Dividend = Divisor × Quotient + Remainder

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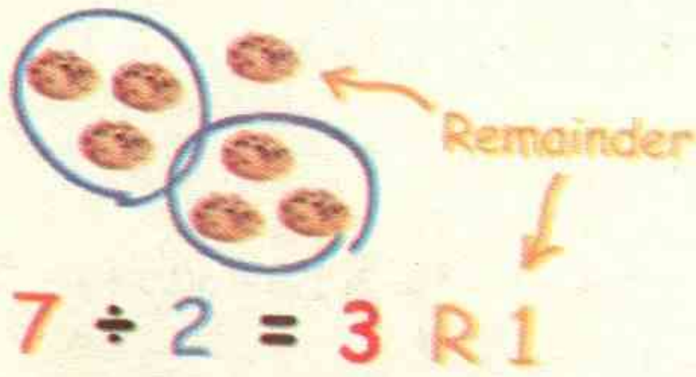
$P(x) = (x - r) \cdot Q(x) + R$, where $Q(x)$ is quotient.

When $x = r$, $\quad P(r) = (r - r)\,Q(r) + R$

$$P(r) = 0 + R$$

or $\quad R = P(r)$

which proves the result using remainder theorem. We can easily find the remainder only by replacing x of the polynomial by the number r.



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Example 22

Divide $P(x)=3x^3-7x^2+6x-3$ by $x-2$, and verify your answer by Remainder theorem.

Solution:

$$
\begin{array}{r|l}
 & 3x^2-x+4 \\
\hline
x-2 & 3x^3-7x^2+6x-3 \\
 & \pm 3x^3 \mp 6x^2 \\
\hline
 & -x^2+6x-3 \\
 & \mp x^2 \pm 2x \\
\hline
 & 4x-3 \\
 & \pm 4x \mp 8 \\
\hline
 & 5
\end{array}
$$

Divisor — Quotient — Remainder

We see that remainder = 5

Since $P(x)=3x^3-7x^2+6x-3$. Divisor = $x-2$

Therefore $r=2$

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or $P(r)=P(2)=3(2)^3-7(2)^2+6(2)-3$

$=24-28+12-3=36-31=5$

$\Rightarrow P(2)=5$ or $R=P(2)=5$

Which is same as we obtained by actual division.

Example 23

Without performing division, find the remainder when $2x^3-3x^2+x-2$ is divided by $x-3$.

Solution:

Here $P(x)=2x^3-3x^2+x-2$

and $x-r=x-3$

$\Rightarrow r=3$

$R=P(r)=P(3)$

$=2(3)^3-3(3)^2+3-2=2(27)-3(9)+1=54-27+1=28$

Hence remainder = 28.

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Example 24

For what value of k, $3x^3 + 9x^2 - (3k - 4)x + 2$ will be exactly divisible by $x - 1$.

Solution:

Since $P(x) = 3x^3 + 9x^2 - (3k - 4)x + 2$ is exactly divisible by $x - 1$ then reminder is 0.

i.e. $R = P(1) = 0$

or $3(1)^3 + 9(1)^2 - (3k - 4)(1) + 2 = 0$

$3 + 9 - 3k + 4 + 2 = 0$

$18 - 3k = 0$

$18 = 3k$

$\frac{18}{3} = k$

$6 = k$

or $k = 6$



5.2.2 Zero of a Polynomial

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Let $P(x)$ be a polynomial. A value of x for which $P(x) = 0$ is called zero of the polynomial.

Example 25

Find the zero of the polynomial $P(x) = x^2 - 4x + 3$.

Solution:

Let $x = 1$, then

$P(1) = 1^2 - 4(1) + 3$

$= 1 - 4 + 3$

$= 4 - 4 = 0$

Hence 1 is a zero of the polynomial $x^2 - 4x + 3$.

Similarly when $x = 3$, then

$P(3) = (3)^2 - 4(3) + 3$

$= 9 - 12 + 3$

$= 12 - 12 = 0$

Therefore 3 is another zero of the polynomial.

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5.2.3 The Factor Theorem

Let $P(x)$ be a polynomial. A linear polynomial $x-r$ is a factor of $P(x)$ if and only if r is a zero of the polynomial $P(x)$ i.e. $P(r)=0$.

Proof:

Since $P(x)$ is a polynomial and let $x-r$ be a factor of $P(x)$. It means that when $P(x)$ is divided by $x-r$, then remainder will be zero.

We can write

$$P(x)=(x-r)Q(x)+0$$

or $$P(x)=(x-r)Q(x)$$

Put $x=r$

$$\therefore \quad P(r)=(r-r)Q(r)$$

$$\Rightarrow \quad =0\cdot Q(r)=0$$

$$\Rightarrow \quad P(r)=0$$

If $y+3$ is a factor of $3y^2+ky+6$ Find the value of k

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It means that $x=r$ is a zero of the polynomial $P(x)$.

Conversely, let $x=r$ be a zero of the polynomial $P(x)$. Then by the definition of zero of a polynomial $P(r)=0$. But by remainder theorem

$$P(r)=R$$

$$\Rightarrow \quad R=0 \qquad \because P(r)=0$$

Therefore, $$P(x)=(x-r)Q(x)+R$$

$$\Rightarrow \quad P(x)=(x-r)Q(x)+0$$

$$\Rightarrow \quad P(x)=(x-r)Q(x)$$

$$\Rightarrow \quad x-r \text{ is a factor of } P(x).$$

This completes the proof of factor theorem.



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5.3 FACTORIZATION OF A CUBIC POLYNOMIAL

Now we shall use the factor theorem to factorize cubic polynomial.

Example 26

Using factor theorem, prove that $x-3$ is a factor of x^3-x^2-5x-3 and find the other factor.

Solution:

$P(x)=x^3-x^2-5x-3$

Put $x=3$

$P(3)=(3)^3-(3)^2-5(3)-3$

$=27-9-15-3$

$=27-27$

$P(3)=0$

Therefore by factor theorem $x-3$ is a factor of $P(x)$. To find the remaining factors divide the $P(x)$ by $x-3$ as under.

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$$
\begin{array}{r|l}
x-3 & x^3-x^2-5x-3 \\
 & \pm x^3 \mp 3x^2 \\
\hline
 & 2x^2-5x-3 \\
 & \pm 2x^2 \mp 6x \\
\hline
 & x-3 \\
 & \pm x \mp 3 \\
\hline
 & 0
\end{array}
\qquad x^2+2x+1
$$



Here $Q(x)=x^2+2x+1$ and $R=0$.

Since $P(x)=(x-r)Q(x)+R$

$x^3-x^2-5x-3=(x-3)(x^2+2x+1)+0$

$=(x-3)(x^2+2\cdot 1\cdot x+1^2)$

$=(x-3)(x+1)^2$

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Example 27

Prove that $(x+2)$ is a factor of x^3-7x-6 and hence find the other factors.

Solution:

$P(x) = x^3-7x-6$

Put $x=-2$

$P(-2) = (-2)^3-7(-2)-6$

$= -8+14-6$

$= 14-14$

$= 0$

Since $P(-2)=0$, therefore by factor theorem $x+2$ is factor of $P(x)$. To find other factors, divide $P(x)$ by $x+2$ as under

$$
\begin{array}{r|l}
x+2 & x^2-2x-3 \\
\hline
x^3-7x-3 & \\
\pm x^3 \mp 2x^2 & \\
\hline
-2x^2-7x-6 & \\
\mp 2x^2 \mp 4x & \\
\hline
-3x-6 & \\
\mp 3x \mp 6 & \\
\hline
0 &
\end{array}
$$

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Remember

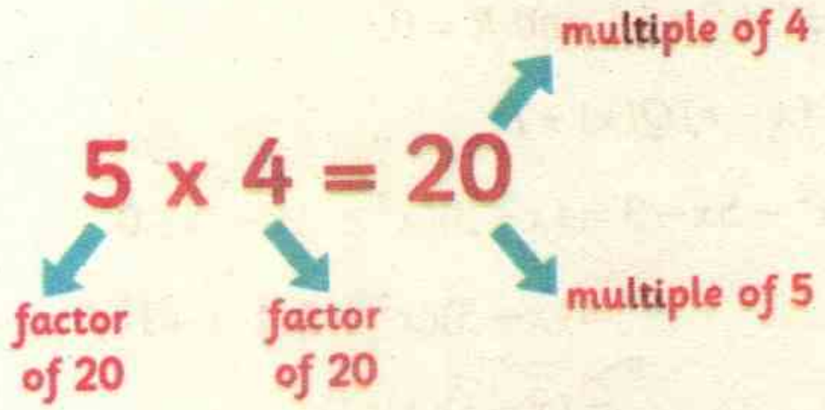
To factorize a cubic polynomial by Factor theorem, it is necessary that one of the factor or one of the zero of the polynomial is known.

Hence $x^3-7x-6 = (x+2)(x^2-2x-3)$

$= (x+2)(x^2-3x+x-3)$

$= (x+2)[x(x-3)+1(x-3)]$

$= (x+2)(x-3)(x+1)$



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Exercise 5.6

1. By using remainder theorem find the remainder of the following polynomials when:

i. $x^3+6x^2+8x-11$ is divided by $x-1$.

ii. $2x^3+4x^2+7x-5$ is divided by $x+3$.

iii. $3x^3+x-200$ is divided by $x-4$.

2. Without performing division find the value of a, when $2x^3-ax^2-2ax+3x+2$ is exactly divisible by $x+1$.

3. Without performing division find the value of 'b' when x^3-4x^2+bx-2 is exactly divisible by $x-1$.

4. Using Factor theorem, factorize the following polynomials.

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i. x^3-2x^2-5x+6

ii. x^3+x^2-4x-4

iii. x^3-7x+6

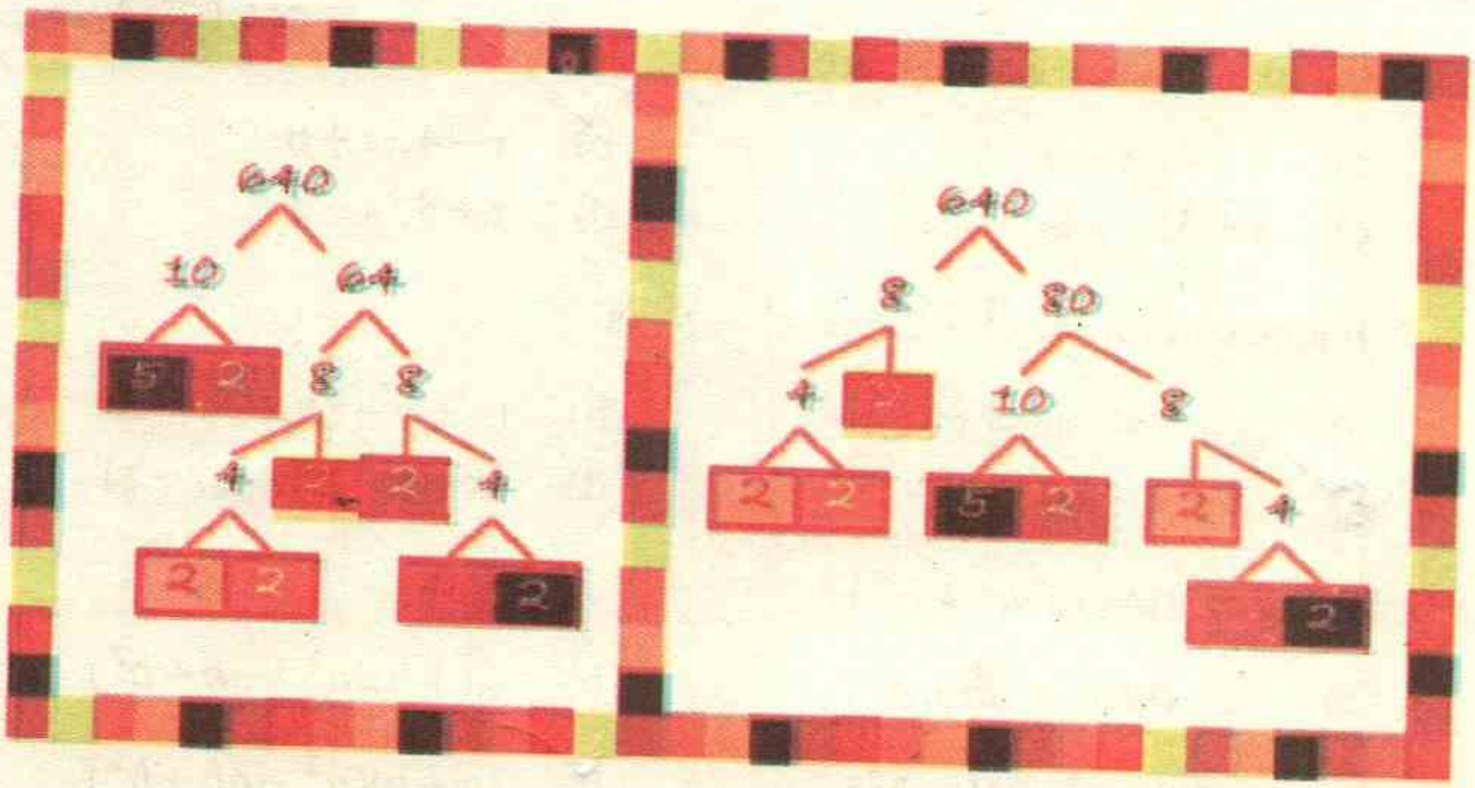
iv. $x^3-9x^2+23x-15$

v. $x^3-4x^2-3x+18$

vi. $x^3+2x^2-19x-20$

vii. $x^3-x^2-14x+24$

viii. x^3-6x^2+32



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REVIEW EXERCISE 5

1. True and false questions.

Read the following sentences carefully and encircle 'T' in case of true and 'F' in case of false statement.

(i) $x^2 - 11x + 30 = (x-6)(x-5)$ — T — F

(ii) $x^3 + 27 = (x+3)(x^2 - 3x + 9)$ — T — F

(iii) $x^3 - 8 = (x-2)(x^2 + 2x + 4)$ — T — F

(iv) $x^4 - y^4 = (x-y)(x+y)(x+y)^2$ — T — F

(v) $x^6 + y^6 = (x^3 + y^3)(x^3 - y^3)$ — T — F

(vi) $x^5 + y^5 = (x+y)^5$ — T — F

2. Complete the following sentences.

(i) $9x^2 - 16y^4 = (3x - 4y^2)$ _____

(ii) $x^3 - 27y^3 = (x - 3y)$ _____

(iii) $x^2 - 5x + 6 = (x-2)$ _____

(iv) $a^2 + b^2 = (a-b)^2$ _____

(v) $8b^3 + c^3 = (2b + c)$ _____



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3. Choose the correct answer.

i) Factors of $x^2 + 2x - 24$ are

a) $x+4, x-6$ b) $x-4, x+6$

c) $x+3, x-8$ d) $x+8, x-3$

ii) Factorization of $x^2 + 2xy + y^2 - z^2$ is

a) $(x-y+z)(x-y-z)$ b) $(x+y-z)(x-y-z)$

c) $(x+y+z)(x+y-z)$ d) $(x+y+z)(x-y-z)$

iii) Factorization of $a^3 + b^3$ is

a) $(a-b)(a^2 + ab + b^2)$ b) $(a+b)(a^2 + ab + b^2)$

c) $(a+b)(a^2 + ab - b^2)$ d) $(a+b)(a^2 - ab + b^2)$

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iv) Factors of $8y^3 - z^3$ are

a) $2y - z, 4y^2 + 2yz + z^2$
b) $2y - z, 2y - z, 2y - z$
c) $2y - z, 4y^2 - 2yz + z^2$
d) $2y - z, 4y^2 - 2yz + z^2$

v) In simplified form $\frac{1}{a+b} + \frac{b}{a^2 - b^2} =$

a) $\frac{b+1}{a^2 - b^2}$
b) $\frac{a}{a^2 - b^2}$
c) $\frac{b}{a^2 - b^2}$
d) $\frac{b+a}{a^2 - b^2}$

vi) Factorization of $a^2 - b^2 + 10b - 25$ is

a) $(a - b + 5)(a - b + 5)$
b) $(a + b - 5)(a - b + 5)$
c) $(a + b - 5)(a + b - 5)$
d) $(a + b - 5)(a - b - 5)$

vii) Completely factorize $16x^4 - 81$.

a) $(2x - 3)(2x + 3)(4x^2 + 9)$
b) $(4x^2 - 9)(4x^2 + 9)$
c) $(2x - 3)(2x + 3)(2x - 3)$
d) $(4x^2 - 9)(2x - 3)(2x + 3)$

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4. **Factorize the following.**

i) $3x^3 - 3x^2 - 18x$
ii) $64y^3 + 27$
iii) $x^6 - y^6$
iv) $343a^3 - 125b^3 + 5b - 7a$

5. Factorize $(x+1)(x+2)(x+3)(x+4) - 120$.

6. **Using Factor theorem, factorize the following polynomials.**

i) $x^3 - 17x + 26$
ii) $x^3 - 39x^2 - 124x - 84$

7. Factorize $a^4 + a^2b^2 + b^4$.

8. By using long division method, find the quotient and the remainder when $(2x^3 + 4x^2 - 6)$ is divided by $(x + 3)$.

9. Find the remainder when the polynomial $x^3 + 2x^2 - 3x + 1$ is divided by
a) $x - 1$,
b) $x + 2$.

Think

10. By using remainder theorem find the remainder of the following polynomials when: $x^{72} - 7x^{48} + 1$ is divided by $x + 1$.

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SUMMARY

- Writing an algebraic expression as the product of two or more algebraic expressions is called factorization of the expression.
- When a polynomial $P(x)$ of degree $n \geq 1$ is divided by $x-r$, The remainder $R = P(r)$.
- Let $P(x)$ be a polynomial. A value of x for which $P(x) = 0$ is called zero of the polynomial.
- Let $P(x)$ be a polynomial. A linear polynomial $x-r$ is a factor of $P(x)$ if and only if r is a zero of the polynomial $P(x)$ i.e. $P(r) = 0$.
-

Quotient

Divisor) **Dividend**

....

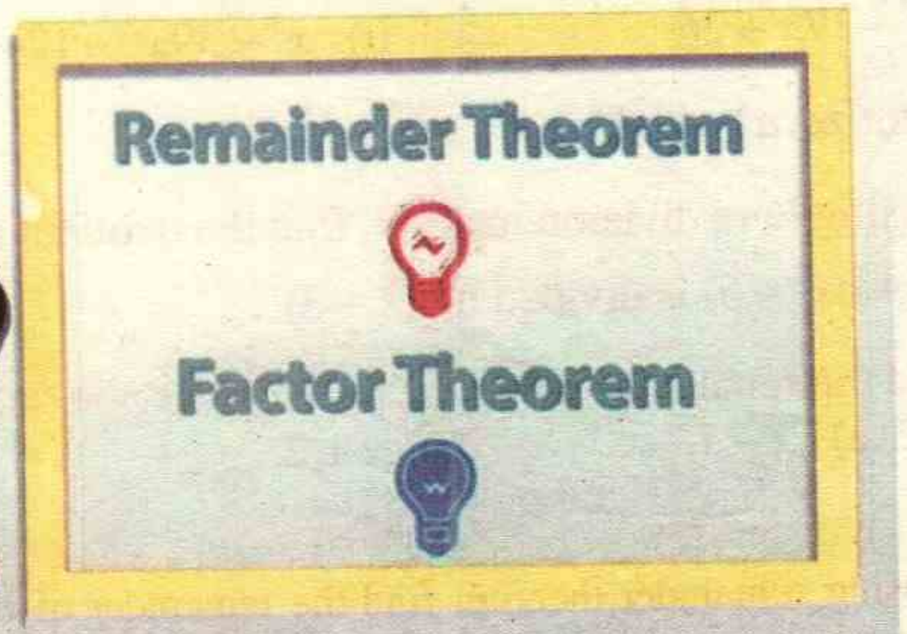
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Remainder

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- Dividend = Divisor × Quotient + Remainder





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UNIT 06

ALGEBRAIC MANIPULATION

After reading this unit, the students will be able to:

- Find highest common factor and least common multiple of algebraic expressions.
- Use factor or division method to determine highest common factor and least common multiple.
- Know the relationship between HCF and LCM.
- Solve real life problems related to HCF and LCM.
- Use highest common factor and least common multiple to reduce fractional expressions involving + − × ÷.
- Find square root of algebraic expression by factorization and division.

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Introduction

In the previous classes we have studied the concepts of H.C.F, L.C.M and square root of numbers. In this unit we shall study these terms for algebraic expressions, different methods for finding them, and their use in simplification of algebraic fractions.

6.1 HIGHEST COMMON FACTOR AND LEAST COMMON MULTIPLE

6.1.1 Highest Common Factor (H.C.F)

The highest number of factors common to all given expressions or polynomials is called Highest Common Factor (H.C.F).

In other words H.C.F of two or more polynomials is a polynomial of the highest degree, which divides exactly the given polynomials.

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There are two methods for finding H.C.F.

a) H.C.F by Factorization.

b) H.C.F by Division.

We shall explain these methods one by one through examples.

a) H.C.F by Factorization

Example 1

Find H.C.F of $x^2 - y^2$, $x^2 - xy$

Solution:

$$x^2 - y^2 = (x+y)(x-y)$$

$$x^2 - xy = x(x-y)$$

We see that $x - y$ is a common factor.

$\therefore$ H.C.F $= x - y$

Example 2

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Find H.C.F of $ax^2 + 5ax + 6a$, $ax^3 + 9ax^2 + 14ax$ and $15a(x^2 - 4)$.

Solution:

$$ax^2 + 5ax + 6a = a(x^2 + 5x + 6)$$
$$= a(x^2 + 3x + 2x + 6)$$
$$= a[x(x+3) + 2(x+3)]$$
$$= a(x+3)(x+2)$$

$$ax^3 + 9ax^2 + 14ax = ax(x^2 + 9x + 14)$$
$$= ax(x^2 + 7x + 2x + 14)$$
$$= ax[x(x+7) + 2(x+7)]$$
$$= ax(x+7)(x+2)$$

$$15a(x^2 - 4) = 15a(x-2)(x+2)$$

In the above factorization we see that $a(x+2)$ is common to given three expressions, it also divides exactly the given expressions.

$\therefore$ H.C.F $= a(x+2)$

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b) **H.C.F by Division**

Following steps are used in finding H.C.F by division method.

- Write the expression or polynomial in descending order with respect to variable.
- Divide higher degree polynomial by the polynomial of lower degree.
- If degrees are same then any one can be divided by the other.
- First divisor is divided by the remainder.
- Repeat the process till the remainder is zero.
- Last divisor is the required H.C.F of the given polynomials.

Note

To find the H.C.F of three polynomials, first find H.C.F of any two of them, then find H.C.F of this H.C.F and the third polynomial.

Example 3

Find H.C.F of $2x^3+7x^2+4x-4$ and $2x^3+9x^2+11x+2$.

Solution:

Let $P(x) = 2x^3+7x^2+4x-4$ and $Q(x) = 2x^3+9x^2+11x+2$

$$
\begin{array}{l}
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 1 \\
2x^3+7x^2+4x-4 \;\overline{)\; 2x^3+9x^2+11x+2} \\
\qquad \pm 2x^3 \pm 7x^2 \pm 4x \mp 4 \\
\hline
2x^2+7x+6 \;)\; 2x^3+7x^2+4x-4 \;(\; x \\
\qquad \pm 2x^2 \pm 7x^2 \pm 6x \\
\hline
\qquad -2 \;|\; -2x-4 \qquad\qquad \text{Dividing by } -2 \\
\hline
x+2 \;)\; 2x^2+7x+6 \;(\; 2x+3 \\
\qquad \pm 2x^2 \pm 4x \\
\hline
\qquad\qquad 3x+6 \\
\qquad\qquad \pm 3x \pm 6 \\
\hline
\qquad\qquad\qquad \times
\end{array}
$$

Hence H.C.F of $P(x)$ and $Q(x) = x+2$

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Example 4

Find H.C.F of $x^3-6x^2+11x-6$, $3x^3-5x^2+6x-4$ and $3x^3+5x^2-6x-2$.

Solution:

Let $P(x) = x^3-6x^2+11x-6$

$Q(x) = 3x^3-5x^2+6x-4$

and $R(x) = 3x^3+5x^2-6x-2$

First we shall find the H.C.F of Q(x) and R(x).

$3x^3-5x^2+6x-4 \overline{)3x^3+5x^2-6x-2 (1}$

$\pm 3x^3 \mp 5x^2 \pm 6x \mp 4$

$2 \mid 10x^2-12x+2$ Dividing by 2

$5x^2-6x+1 \overline{)3x^3-5x^2+6x-4 (3x-7}$

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$\times 5$ Multiplying by 5

$15x^3-25x^2+30x-20$

$\pm 15x^3 \mp 18x^2 \pm 3x$

$-7x^2+27x-20$

$\times 5$ Multiplying by 5

$-35x^2+135x-100$

$\mp 35x^2 \mp 42x \mp 7$

$93 \mid 93x-93$ Dividing by 93

$x-1 \overline{)5x^2-6x+1 (5x-1}$

$\pm 5x^2 \mp 5x$

$-x+1$

$\mp x \pm 1$

$\times$

$\therefore$ H.C.F of $Q(x)$ and $R(x) = x-1$

Now we shall find the H.C.F of $x-1$ and $P(x)$.

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$$
\begin{array}{r|l}
 & x^2-5x+6 \\
\hline
x-1 & x^3-6x^2+11-6 \\
 & \pm x^3 \mp x^2 \\
\hline
 & -5x^2+11x \\
 & \mp 5x^2 \pm 5x \\
\hline
 & 6x-6 \\
 & \pm 6x \mp 6 \\
\hline
 & 0
\end{array}
$$

Hence the required H.C.F of $x^3-6x^2+11x-6$, $3x^3-5x^2+6x-4$ and $3x^3+5x^2-6x-2$ is $x-1$.

6.1.2 Least Common Multiple (L.C.M)

The Least Common Multiple (L.C.M) of two or more polynomials is the polynomial of least degree which is divisible by the given polynomials.

There are two methods of finding L.C.M.

(a) L.C.M by factorization **(b) L.C.M by formula**

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(a) L.C.M by factorization

To find L.C.M by factorization we use the following formula.

L.C.M = Common factors × Non common factors

Example 5

Find L.C.M of x^2+4x+4 and x^2+5x+6.

Solution:

$$x^2+4x+4 = (x+2)^2$$
$$= (x+2)(x+2)$$
$$x^2+5x+6 = x^2+3x+2x+6$$
$$= x(x+3)+2(x+3)$$
$$= (x+3)(x+2)$$

L.C.M = Common factors × Non-common factors.

$L.C.M = (x+2)\times(x+2)(x+3)$

$L.C.M = (x+2)^2(x+3)$

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Example 6

Find L.C.M of x^2-4x+3, x^2-3x+2 and x^2-5x+6.

Solution:

$$x^2-4x+3 = x^2-3x-x+3$$
$$= x(x-3)-1(x-3)$$
$$= (x-3)(x-1) \text{ -------- (1)}$$

$$x^2-3x+2 = x^2-2x-x+2$$
$$= x(x-2)-1(x-2)$$
$$= (x-2)(x-1) \text{ -------- (2)}$$

$$x^2-5x+6 = x^2-3x-2x+6$$
$$= x(x-3)-2(x-3)$$
$$= (x-3)(x-2) \text{ -------- (3)}$$

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In expressions (1) and (2) $x-1$ is a common factor.

In expressions (2) and (3) $x-2$ is a common factor.

In expressions (1) and (3) $x-3$ is a common factor.

Therefore ,

$L.C.M$ = Common factors × Non-common factors.

$$= (x-1)(x-2)(x-3)\times 1$$
$$= (x-1)(x-2)(x-3)$$

6.1.3 Relationship between H.C.F and L.C.M

Relationship between H.C.F and L.C.M is shown by the following formula which we have discussed in section 6.1.2.

Product of two polynomials = H.C.F×L.C.M

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Example 7

Find L.C.M of $x^3-6x^2+11x-6$ and x^3-4x+3.

Solution:

First we shall find H.C.F of given polynomials.

$$x^3-4x+3\overline{)x^3-6x^2+11x-6}\,(1$$
$$\pm x^3 \quad \mp 4x \pm 3$$
$$-3\,|\,-6x^2+15x-9 \qquad \text{Dividing by } -3$$
$$2x^2-5x+3\overline{)x^3-4x+3}\,(x+5$$



$$\times 2 \qquad \text{Multiplying by 2}$$
$$2x^3-8x+6$$
$$\pm 2x^3 \pm 3x \quad \mp 5x^2$$
$$5x^2-11x+6$$

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$$\times 2 \qquad \text{Multiplying by 2}$$
$$10x^2-22x+12$$
$$\pm 10x^2 \mp 25x \pm 15$$
$$3\,|\,3x-3 \qquad \text{Dividing by 3}$$
$$x-1\overline{)2x^2-5x+3}\,(2x-3$$
$$\pm 2x^2 \mp 2x$$
$$-3x+3$$
$$\mp 3x \pm 3$$
$$\times$$

Hence H.C.F of $x^3-6x^2+11x-6$ and x^3-4x+3 is $x-1$.

$$\therefore L.C.M = \frac{Product\ of\ two\ Polynomials}{H.C.F}$$
$$= \frac{(x^3-4x+3)(x^3-6x^2+11x-6)}{x-1}$$

Divide $x^3-6x^2+11x-6$ by $x-1$.

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Theorem

If A and B are given polynomials and their H.C.F and L.C.M are represented by H and L respectively then

$$A \times B = H \times L$$

Proof:

Since H (H.C.F) is common factor of polynomials A and B, then it divides exactly A and B.

Let $\frac{A}{H} = a$ --------- (1)

and $\frac{B}{H} = b$ ---------- (2)

Clearly a and b have no common factors.

Equation (1) and (2) can be written as

$A = Ha$ -------- (3)

$B = Hb$ ---------(4)

Then L.C.M = common factor × non-common factor

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Or $L = H \times a \times b$

Multiply both sides by H.

$H \times L = H\,(H \times a \times b)$

Or $H \times L = H \times a \times H \times b$

Or $H \times L = (Ha) \times (Hb)$

Or $H \times L = A \times B$ using equation (3) and (4)

Or

$$A \times B = H \times L$$

b) **L.C.M by formula**

L.C.M of two polynomials can be found out by using the following formula.

$$L = \frac{A \times B}{H}$$

$$L.C.M = \frac{\text{Product of two polynomials}}{H.C.F}$$

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$$x-1\overline{)x^3-6x^2+11x-6}\,(x^2-5x+6$$

$$\pm x^3 \mp x^2$$

$$-5x^2+11x$$

$$\mp 5x^2 \pm 5x$$

$$6x-6$$

$$\pm 6x \mp 6$$

$$0$$

Therefore L.C.M = $(x^3-4x+3)(x^2-5x+6)$

Example 8

Find H.C.F and L.C.M of $3x^3-2x^2-3x+2$ and $6x^3-7x^2-x+2$.

Solution:

Let $A = 3x^3-2x^2-3x+2$

and $B = 6x^3-7x^2-x+2$

$$3x^2-2x^2-3x+2\overline{)6x^3-7x^2-x+2}\,(2$$

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$$\pm 6x^3 \mp 4x^2 \mp 6x \pm 4$$

$$-1 \mid -3x^2+5x-2 \qquad \text{(Dividing by } -1\text{)}$$

$$3x^2-5x+2\overline{)3x^3-2x^2-3x+2}\,(x+1$$

$$\pm 3x^3 \mp 5x^2 \pm 2x$$

$$3x^2-5x+2$$

$$\pm 3x^2 \mp 5x \pm 2$$

$$0$$

$\therefore$ H.C.F of A and B $= 3x^2-5x+2$.

To find L.C.M we have the formula

$$L = \frac{A \times B}{H}$$

$$= \frac{(3x^3-2x^2-3x+2)(6x^3-7x^2-x+2)}{(3x^2-5x+2)} \quad (x+1)$$

L.C.M = $(x+1)(6x^3-7x^2-x+2)$

Note

In the above step we divided $3x^3-2x^2-3x+2$ by $3x^2-5x+2$ and got the quotient as $x+1$.

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Example 9

If H.C.F and L.C.M of two polynomials are $x-3$ and $x^3-9x^2+26x-24$ respectively. Find the second polynomial when one polynomial is x^2-5x+6.

Solution:

H.C.F $=x-3$, L.C.M $=x^3-9x^2+26x-24$

$A=x^2-5x+6$, $B=?$

We know that, $A\times B=H\times L$

$$\Rightarrow \quad B=\frac{H\times L}{A}$$

$$=\frac{(x-3)(x^3-9x^2+26x-24)}{x^2-5x+6}$$

$$x^2-5x+6\overline{)\,x^3-9x^2+26x-24\,(\,x-4}$$

$$\pm x^3 \mp 5x^2 \pm 6x$$

$$-4x^2+20x-24$$

$$\mp 4x^2 \pm 20x \mp 24$$

$$0$$

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Thus $B=(x-3)(x-4)$

$=x^2-7x+12$

Hence second polynomial is $x^2-7x+12$.



Example 10

If H.C.F and L.C.M of two polynomials are $x-1$ and x^3+4x^2+x-6 respectively. Find the polynomials of degree 2.

Solution:

$H=x-1$, $L=x^3+4x^2+x-6$, $A=?, B=?$

Consider $L=x^3+4x^2+x-6$

$L=(x-1)(x+2)(x+3)$

$L=$ Common factors $\times$ Non-common factors.

Clearly $x-1$ is common factor of A and B, and $(x+2)$ and $(x+3)$ are non-common factors. Therefore $A=(x-1)(x+2)=x^2+x-2$

$B=(x-1)(x+3)=x^2+2x-3$

Which are the required polynomials of degree 2.

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6.1.4 Application of H.C.F and L.C.M

Example 11

The sum of two numbers is 120 and their H.C.F is 12. Find the numbers.

Solution:

Let the number be $12x$ and $12y$, where x, y are numbers prime to each other.

Then $12x+12y=120$

i.e., $x+y=10$

Thus we have to find two numbers whose sum is 10. The possible such pairs of numbers are (1,9), (2,8), 3,7), (4,6), (5,5).

The pairs of numbers which are prime to each other are (1,9) and (3,7).

Thus the required numbers are:

1×12, 9×12; 3×12, 7×12

i.e., 12, 108 and 36, 84.

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Exercise 6.1

1. Find H.C.F of the following expressions by factorization method.

i) $(x+6)^2$ and x^2-36.

ii) x^4-y^4 and $x^4+2x^2y^2+y^4$.

iii) $x-3, x^2-9, (x-3)^2$

iv) $2^3 3^2 (x+y)^3 (x+2y)^2$, $2^3 3^2 (x-y)^2 (x+2y)^3$, $3^2 (x-y)^2 (x+2y)$

v) $2x^4-2y^4, 6x^2+12xy+6y^2, 9x^3+9y^3$

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2. Find H.C.F by division method.

i) x^2-x-6 and x^2-2x-3.

ii) y^3-3y+2 and y^3-5y^2+7y-3.

iii) $2x^5-4x^4-6x$ and $x^5+x^4-3x^3-3x^2$.

iv) $2x^3+10x^2+5x+25$ and x^3+5x^2-x-5.

3. Find L.C.M by factorization.

i) $x+y,\ x^2-y^2$

ii) $x^3-y^3, x-y$

iii) $x^5-x,\ x^5-x^2$ and x^5-x^3

iv) $2^3 3^2 (x-y)^3 (x+2y)^2,\ 2^3 3^2 (x-y)^2 (x+2y)^3$ and $3^2 (x-y)^2 (x+2y)$

4. Find H.C.F and L.C.M of the following expressions.

i) $x^3-2x^2-13x-10$ and $x^3-x^2-10x-8$.

ii) $2x^4-2x^3+x^2+3x-6$ and $4x^4-2x^3+3x-9$.

iii) a^4-a^3-a+1 and a^4+a^2+1.

iv) $1-x^2-x^4+x^5$ and $1+2x^2+x^3-x^4-x^5$.

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5. H.C.F and L.C.M of two polynomials are $x-2$ and x^3+3x^2-6x-8 respectively. If one polynomial is x^2+2x-8, find the second polynomial.

6. If product of two polynomials is $x^4+5x^3-6x^2-2x-28$ and their H.C.F is $x-2$. Find their L.C.M.

7. H.C.F and L.C.M of two polynomials are $x+5$ and $2x^3+11x^2+2x-15$ respectively. Find polynomials of degree 2.

8. If product of two polynomials is $x^4+6x^3-3x^2-56x-48$ and their L.C.M is $x^3+2x^2-11x-12$. Find their H.C.F.

9. Waqar wishes to distribute 128 bananas and also 176 apples equally among a certain number of children. Find the highest number of children who can get the fruit in this way.

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6.2 BASIC OPERATIONS ON ALGEBRAIC FRACTIONS

An algebraic fraction is the quotient of two algebraic expressions. Examples of these are;

i) $\frac{3x^2+4a}{x+1}$

ii) $\frac{4x^3+3x^2+8}{x^2+9}$

iii) $\frac{\sqrt{x^3-1}+4x}{\sqrt{x+7}}$

iv) $\frac{\frac{1}{x^2}+\sqrt{3x}}{\frac{2}{x+5}}$

We shall now carry out the operations of addition, subtraction, product and division on algebric fractions in order to reduce them by giving some examples . We assume all fractions are defined.

Example 12

Simplify $\frac{x+y}{3x+2y}+\frac{x-y}{3x+2y}$.

Solution:

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$$\frac{x+y}{3x+2y}+\frac{x-y}{3x+2y}$$

$$=\frac{(x+y)+(x-y)}{(3x+2y)} \quad \because L.C.M = 3x+2y$$

$$=\frac{x+\cancel{y}+x-\cancel{y}}{3x+2y}=\frac{2x}{3x+2y}$$



Example 13

Simplify $\frac{x-y}{x+y}-\frac{x^2-2y^2}{x^2-y^2}$.

Solution:

$$\frac{x-y}{x+y}-\frac{x^2-2y^2}{x^2-y^2}=\frac{x-y}{x+y}-\frac{x^2-2y^2}{(x+y)(x-y)}$$

$$=\frac{(x-y)(x-y)-(x^2-2y^2)}{(x+y)(x-y)}=\frac{x^2-2xy+y^2-x^2+2y^2)}{(x+y)(x-y)}=\frac{3y^2-2xy}{x^2-y^2}$$

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Example 14

Simplify $\frac{x^2-xy+y^2}{x^3+y^3}+\frac{x^2+xy+y^2}{x^3-y^3}-\frac{1}{x^2-y^2}$.

Solution:

$$\frac{x^2-xy+y^2}{x^3+y^3}+\frac{x^2+xy+y^2}{x^3-y^3}-\frac{1}{x^2-y^2}$$

$$=\frac{\left(x^2-xy+y^2\right)}{(x+y)\left(x^2-xy+y^2\right)}+\frac{\left(x^2+xy+y^2\right)}{(x-y)\left(x^2+xy+y^2\right)}-\frac{1}{(x+y)(x-y)}$$

$$=\frac{1}{x+y}+\frac{1}{x-y}-\frac{1}{(x+y)(x-y)}=\frac{(x-y)+(x+y)-1}{(x+y)(x-y)}$$

$$=\frac{x-y+x+y-1}{x^2-y^2}=\frac{2x-1}{x^2-y^2}$$

Example 15

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Simplify $\frac{y}{y^2-y-2}-\frac{1}{y^2+5y-14}-\frac{2}{y^2+8y+7}$

Solution:

$$\frac{y}{y^2-y-2}-\frac{1}{y^2+5y-14}-\frac{2}{y^2+8y+7}$$

$$=\frac{y}{(y+1)(y-2)}-\frac{1}{(y-2)(y+7)}-\frac{2}{(y+1)(y+7)}$$

$$=\frac{y(y+7)-(y+1)-2(y-2)}{(y+1)(y-2)(y+7)}$$

$$=\frac{y^2+7y-y-1-2y+4}{(y+1)(y-2)(y+7)}=\frac{y^2+4y+3}{(y+1)(y-2)(y+7)}$$

$$=\frac{(y+1)(y+3)}{(y+1)(y-2)(y+7)}=\frac{y+3}{(y-2)(y+7)}$$

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Example 16

Simplify $\frac{x+4}{x-3}\times\frac{x^2-9}{x^2-x-2}$.

Solution:

$$\frac{x+4}{x-3}\times\frac{x^2-9}{x^2-x-2}=\frac{x+4}{x-3}\times\frac{(x+3)(x-3)}{(x-2)(x+1)}$$ (Factorization)

$$=\frac{(x+4)(x+3)}{(x-2)(x+1)}$$ (Eliminating common factors)

Example 17

Multiply $\frac{x^2-2x}{2x^2+5x+3}$ *by* $\frac{2x^2-3x-9}{x^2-9}$ and write the answer in simplified form.

Solution:

The work may be indicated as follows:

$$\frac{x^2-2x}{2x^2+5x+3}\cdot\frac{2x^2-3x-9}{x^2-9}=\frac{x(x-2)(x-3)(2x+3)}{(x+1)(2x+3)(x+3)(x-3)}$$

$$=\frac{x(x-2)}{(x+1)(x+3)}=\frac{x^2-2x}{x^2+4x+3}$$

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Example 18

$$\left(\frac{x^3-y^3}{y^3}\times\frac{y}{x-y}\right)\div\frac{x^2+xy+y^2}{y^2}$$

Solution:

$$\left(\frac{x^3-y^3}{y^3}\times\frac{y}{x-y}\right)\div\frac{x^2+xy+y^2}{y^2}=\frac{(x-y)(x^2+xy+y^2)y}{y^3(x-y)}\cdot\frac{y^2}{(x^2+xy+y^2)}=1$$

Example 19

$$\left(\frac{3}{x-2}-\frac{1}{x+1}\right)\div\frac{x+4}{x-2}$$

Solution:

$$\left(\frac{3}{x-2}-\frac{1}{x+1}\right)\div\frac{x+4}{x-2}$$

$$=\frac{3(x+1)-(x-2)}{(x-2)(x+1)}\cdot\frac{x-2}{x+4}=\frac{(2x+5)(x-2)}{(x-2)(x+1)(x+4)}=\frac{(2x+5)}{(x+1)(x+4)}$$

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Exercise 6.2

1. Simplify:

i) $\frac{x}{x+y}+\frac{2y}{x+y}$

ii) $\frac{x+y}{3x+2y}+\frac{x-y}{3x+2y}$

iii) $\frac{3}{y-2}-\frac{2}{y+2}-\frac{y}{y^2-4}$

iv) $\frac{x-y}{x+y}-\frac{x^2-2y^2}{x^2-y^2}$

v) $\frac{x}{2x^2+3xy+y^2}-\frac{x-y}{y^2-4x^2}+\frac{y}{2x^2+xy-y^2}$

vi) $\frac{a}{3x-y}+\frac{a}{3x+y}-\frac{6ax}{9x^2-y^2}$

vii) $\frac{y}{x-y}+\frac{y}{x+y}+\frac{2xy}{x^2+y^2}+\frac{4x^3y}{x^4+y^4}$

viii) $\frac{1}{a^2+7a+10}+\frac{1}{a^2+10a+16}$

ix) $\frac{1}{a-b}+\frac{1}{a+b}+\frac{2a}{a^2+b^2}+\frac{4a^3}{a^4+b^4}$

x) $\frac{x^2-xy+y^2}{x^3+y^3}+\frac{x^2+xy+y^2}{x^3-y^3}-\frac{1}{x^2-y^2}$

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2. Simplify:

i) $\frac{x^2-25}{5-x}$

ii) $\frac{x^2+5x+4}{4y^3}\times\frac{2y^2}{x^2+3x+2}$

iii) $\frac{x^2-5x+4}{x^2-3x-4}\div\frac{x^3-4x^2+x-4}{2x-1}$

iv) $\frac{a(a+b)}{a^3-b^3}\times\frac{a^2+ab+b^2}{a^2-b^2}$

v) $\frac{7}{x^2-4}\div\frac{xy}{x+2}$

vi) $\frac{a^3-b^3}{a^4-b^4}\div\frac{a^2+ab+b^2}{a^2+b^2}$

vii) $\frac{2x}{3x-12}\div\frac{x^2-2x}{x^2-6x+8}$

viii) $\frac{a^4-8a}{2a^2+5a-3}\times\frac{2a-1}{a^2+2a+4}\div\frac{a^2-2a}{a+3}$

ix) $\frac{9-x^2}{x^4+6x^3}\div\frac{x^3-2x^2-3x}{x^2+7x+6}$

x) $\frac{ax+ab+cx+bc}{a^2-x^2}\times\frac{x^2-2ax+a^2}{x^2+(b+a)x+ab}$

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6.3 SQUARE ROOT OF ALGEBRAIC EXPRESSIONS

Square root of an algebraic expression can be found out by the following two methods.

a) By Factorization *b)* By Division

a) Square root by factorization

In this method make the expression a perfect square then find square root. To explain this method, consider the following examples.

Example 20

Find the square root of $x^2+ax+\frac{1}{4}a^2$ by factorization.

Solution:

We are to find $\sqrt{x^2+ax+\frac{1}{4}a^2}$

$$x^2+ax+\frac{1}{4}a^2=(x)^2+2\cdot\frac{1}{2}a\cdot x+(\frac{1}{2}a)^2$$

$$=(x+\frac{1}{2}a)^2$$

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$$\therefore \pm\sqrt{x^2+ax+\frac{1}{4}a^2}=x+\frac{1}{2}a$$

Highest Common Factor (HCF)

20: 1, 2, 4, 5, 10, 20
28: 1, 2, 4, 7, 14, 28

Least Common Multiple (LCM)

9: 9, 18, 27, 36, 45, ...
12: 12, 24, 36, 48, ...

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Example 21

Find square root of $x^2+\frac{1}{x^2}-10\left(x+\frac{1}{x}\right)+27$

Solution:

$$x^2+\frac{1}{x^2}-10\left(x+\frac{1}{x}\right)+27=x^2+\frac{1}{x^2}-10\left(x+\frac{1}{x}\right)+25+2$$

$$=x^2+\frac{1}{x^2}+2-10\left(x+\frac{1}{x}\right)+25 \quad =\left(x+\frac{1}{x}\right)^2-2\cdot5\left(x+\frac{1}{x}\right)+(5)^2$$

$$=\left(x+\frac{1}{x}\right)^2-2\left(x+\frac{1}{x}\right)\cdot5+(5)^2$$

$$=\left(x+\frac{1}{x}-5\right)^2 \qquad \because a^2-2ab+b^2=(a-b)^2$$

Therefore, $\pm\sqrt{x^2+\frac{1}{x^2}-10\left(x+\frac{1}{x}\right)+27}=\left(x+\frac{1}{x}-5\right)$

b) Square root by division

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To find square root by division method, study the following examples.

Example 22 Find the square root of $16x^4-24x^3+25x^2-12x+4$.

Solution:

$$\begin{array}{r|l} & 4x^2-3x+2 \\ \hline 4x^2 & 16x^4-24x^3+25x^2-12x+4 \\ & \pm16x^4 \\ \hline 8x^2-3x & -24x^3+25x^2-12x+4 \\ & \mp24x^3\pm9x^2 \\ \hline 8x^2-6x+2 & +16x^2-12x+4 \\ & \pm16x^2\mp12x\pm4 \\ \hline & 0 \end{array}$$

$\sqrt{16x^4}=4x^2$

2nd element of square root $=\frac{-24x^3}{8x^2}=-3x$

3rd element of square root $=\frac{16x^2}{8x^2}=2$

Double of the first elements of square root

Double of the first two elements of square root

Therefore $\sqrt{16x^4-24x^3+25x^2-12x+4}=\pm(4x^2-3x+2)$.

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Rules for finding square root by division method

In previous example we have followed the following steps:

1. Write the expression in descending order.
2. Take square root of first element which is $4x^2$.
3. Subtract the square of $4x^2$ from the given expression and get remainder $-24x^3+25x^2-12x+4$.
4. Find the second element of square root by dividing the first term of remainder by twice of the first term of square root which is $-3x$.
5. Multiply the second element with the sum of twice the first term of square root and second element i.e. $-3x(8x^2-3x)=-24x^3+9x^2$ and subtract from the remainder.
6. Find the third element of square root which is 2.
7. Multiply third element with sum of twice the first two terms and third term. i.e. $2(8x^2-6x+2)=16x^2-12x+4$, then subtract from remainder, we get zero as remainder.

The above rules are quite general and will be used in every problem for finding square root by division method.

Example 23

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Find the square root of $\frac{x^4}{4}+4x^2+\frac{ax^2}{3}+\frac{a^2}{9}-2x^3-\frac{4ax}{3}$.

Solution:

Write the expression in descending order.

$$\frac{x^2}{2}-2x+\frac{a}{3}$$

$\frac{x^2}{2}$	$\frac{x^4}{4}-2x^3+4x^2+\frac{ax^2}{3}-\frac{4ax}{3}+\frac{a^2}{9}$
	$\pm\frac{x^4}{4}$
x^2-2x	$-2x^3+4x^2$
	$\mp 2x^3 \pm 4x^2$
$x^2-4x+\frac{a}{3}$	$\frac{ax^2}{3}-\frac{4ax}{3}+\frac{a^2}{9}$
	$\pm\frac{ax^2}{3}\mp\frac{4ax}{3}\pm\frac{a^2}{9}$

$\therefore$ Required square root $= \pm\left(\frac{x^2}{2}-2x+\frac{a}{3}\right)$

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Example 24

(i) What should be added to

(ii) What should be subtracted from

iii) For what value of x

The expression $9x^4 - 12x^3 + 10x^2 - 3x - 3$ to make the perfect square.

Solution:

$$
\begin{array}{r|l}
 & 3x^2 - 2x + 1 \\
\hline
3x^2 & 9x^4 - 12x^3 + 10x^2 - 3x - 3 \\
 & \pm 9x^4 \\
\hline
6x^2 - 2x & -12x^3 + 10x^2 \\
 & \pm 12x^3 \pm 4x^2 \\
\hline
6x^2 - 4x + 1 & +6x^2 - 3x - 3 \\
 & \pm 6x^2 \pm 3x \pm 1 \\
\hline
 & -x - 4
\end{array}
$$

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To make the given expression *a* perfect square, remainder should be zero. Hence

(i) $-x + 4$ should be added.

(ii) $x - 4$ should be subtracted.

(iii) $x - 4 = 0 \Rightarrow x = 4$

DID YOU KNOW?

We still use the base-60 number,system developed by the ancient Babylonians in order to tell time. This is why we have 60 seconds in 1 minute and 60 minutes in 1 hour.

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Exercise 6.3

1. **Find the square root by factorization method.**

i) x^2+4x+4

ii) $(x+y)^2+6(x+y)+9$

iii) $x^2y^2-8xy+16$

iv) $x^2+\frac{1}{x^2}-8\left(x+\frac{1}{x}\right)+18$

v) $x(x+1)(x+2)(x+3)+1$

vi) $\left(x^2+\frac{1}{x^2}\right)^2-5\left(x-\frac{1}{x}\right)+\frac{9}{4}$

vii) $\left(x^2+\frac{1}{x^2}\right)^2-4\left(x+\frac{1}{x}\right)^2+12$

viii) $\frac{4x^6-12x^3y^3+9y^6}{9x^4+24x^2y^2+16y^4}$

2. **Find the square root of the following by division method.**

i) $4x^4-4x^3+13x^2-6x+9$

ii) $x^4+x^3-\frac{31}{4}x^2-4x+16$

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iii) $x^2-2x+1+2xy-2y+y^2$

iv) $\left(x^2-\frac{1}{x^2}\right)^2-12\left(x^2-\frac{1}{x^2}\right)+36$

3. For what value of k the expression $4x^4+32x^2+96+\frac{128}{x^2}+\frac{k}{x^4}$ will become a perfect square.

4. (i) What should be added to

(ii) What should be subtracted from

(iii) For what value of x the expression $4x^4-12x^3+17x^2-13x+6$ so that it becomes a perfect square.

5. What should be subtracted and added to the expression $x^4-4x^3+10x+7$. so that the expression is made perfect square.

6. Find the values of l & m for which the following expressions will become perfect squares.

(i) $x^4+4x^3+16x^2+1x+m$ (ii) $49x^4-70x^3+109x^2+1x-m$.

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REVIEW EXERCISE 6

1. Select the correct answer.

i) H.C.F of $a^3 - 8b^3$ and $a^2 - 4ab + 4b^2$ is

a) $a - 2b$ b) $a^2 + 2ab + b^2$ c) $a + 2b$ d) $(a + 2b)^2$

ii) L.C.M of $(2x+3y)^5$ and $(2x+3y)^3$ is

a) $2x+3y$ b) $(2x+3y)^3$ c) $(2x+3y)^2$ d) $(2x+3y)^5$

iii) H.C.F of $a^3 - b^3$ and $a^2 + ab + b^2$ is

a) $a+b$ b) $a^2 + ab + b^2$ c) $a-b$ d) $(a-b)^2$

iv) L.C.M of $(a-b)^4$ and $(a-b)^3$ is

a) $(a-b)$ b) $(a-b)^3$ c) $(a-b)^4$ d) $(a-b)^7$

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v) Reduce to lowest terms: $\dfrac{10(x+3)(x-2)}{15(x-2)}$

a) $\dfrac{2(x+3)}{3}$ b) $2x$ c) $\dfrac{10(x+3)}{15}$ d) $2(x+1)$

vi) Simplified form of $\dfrac{b}{25a^2 - b^2} - \dfrac{1}{5a-b}$ is

a) $\dfrac{5a}{25a^2 - b^2}$ b) $\dfrac{+5a}{5a-b}$ c) $\dfrac{-5a}{5a+b}$ d) $\dfrac{-5a}{25a^2 - b^2}$

vii) $\dfrac{5}{x^2 - x - 2} + \dfrac{3}{x^2 + 4x + 3} =$

a) $\dfrac{8x+21}{(x-1)(x+2)(x+3)}$ b) $\dfrac{8x-3}{(x+1)(x-2)(x+3)}$

c) $\dfrac{8x+6}{(x-1)(x+2)(x+3)}$ d) $\dfrac{8x+9}{(x-1)(x+2)(x+3)}$

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viii) $\dfrac{x^2-2x-3}{3x^2+x-2}$

a) $\dfrac{x-3}{3x-2}$ b) $\dfrac{-2x-3}{3x-2}$

c) $\dfrac{-2x-2}{x+1}$ d) $\dfrac{x-3}{3x+2}$

ix) L.C.M =

a) $\dfrac{H.C.F}{A\times B}$ b) $\dfrac{A\times B}{H.C.F}$

c) $\dfrac{A}{H.C.F}$ d) $\dfrac{B}{H.C.F}$

x) L.C.M of a^2-a+1 and a^3+1 is

a) $a+1$ b) a^2-a+1

c) a^3+1 d) a^2+a+1

2. Simplify the following. perfect24u.ocm

i) $\dfrac{5}{2s+4}-\dfrac{3}{s^2+3s+2}+\dfrac{s}{s^2-s-2}$

ii) $\dfrac{a}{(c-a)(a-b)}+\dfrac{b}{(a-b)(b-c)}+\dfrac{c}{(b-c)(c-a)}$

iii) $\dfrac{x^2-4}{xy^2}\cdot\dfrac{2xy}{x^2-4x+4}$

iv) $\dfrac{a^3-b^3}{a^4-b^4}\div\dfrac{a^2+ab+b^2}{a^2+b^2}$

3. Find L.C.M of $x^3-6x^2+11x-6$ and x^3-4x+3.

4. Find the square root of:

i) $4x^2-12x+9$.

ii) $x^4+4x^2+6x^2+4x+1$

Think

5. Simplify $\dfrac{x^3-y^3}{x^3+z^3}\times\dfrac{x^2+xy+xz+yz}{x^4+x^2y^2+y^4}\times\dfrac{x^3+y^3}{x^2-y^2}$

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SUMMARY

- The highest number of factors common to all given expressions or polynomials is called Highest Common Factor (H.C.F).
- The smallest number of factors which contain all the factors of two or more given algebraic expressions is called Least Common Multiple (L.C.M)..
- $L.C.M = Common\ factors \times Non\ common\ factors$

  $Dividend = Divisor \times Quotient + Remainder$
- $L.C.M = \dfrac{Product\ of\ two\ polynomials}{H.C.F\ of\ two\ ploynomials}$

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- Product of two polynomials = H.C.F×L.C.M

DID YOU KNOW?

1 divided by 998001 gives a complete sequence from 000 to 999 in order.

1/998001 = 0.000001002003004005006007008009010011012013014015016017018019020
021022023024025026027028029030031032033034035036037038039040041042043044045046
047048049050051052053054055056057058059060061062063064065066067068069070071072
073074075076077078079080081082083084085086087088089090091092093094095096097098
099100101102103104105106107108109110111112113114115116117118119120121122123124
125126127128129130131132133134135136137138139140141142143144145146147148149150
151152153154155156157158159160161162163164165166167168169170171172173174175176
177178179180181182183184185186187188189190191192193194195196197198199200201202
203204205206207208209210211212213214215216217218219220221222223224225226227228
229230231232233234235236237238239240241242243244245246247248249250251252253254
255256257258259260261262263264265266267268269270271272273274275276277278279280

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UNIT 07

LINEAR EQUATIONS AND INEQUALITIES

After reading this unit, the students will be able to:

- Recall linear equation in one variable.
- Solve linear equation with rational coefficients.
- Reduce equations, involving radicals, to simple linear form and find their solutions.
- Define absolute value.
- Solve the equation, involving absolute value, in one variable.
- Define inequalities ($>$, $<$) and ($\geq$, $\leq$) .
- Recognize properties of inequalities (i.e. trichotomy, transitive, additive and multiplicative).
- Solve linear inequalities with rational coefficients.

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7.1 LINEAR EQUATIONS

7.1.1 Introduction

You have already studied in previous classes that an equation of the form

$$ax+b=0 \text{ where } a\neq 0 \text{ and } a\in Q$$

is general form of linear equation in one variable with rational coefficient.

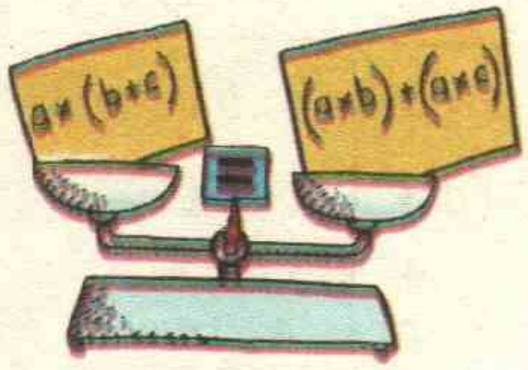
Example 1

(i) $2x+3=0$

(ii) $\frac{5}{2}y-4=0$

(iii) $5x-15=2x+3$

(iv) $\frac{1}{3}x-\frac{2}{3}=4$



are linear equations in one variable with rational coefficient.

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7.1.2 Solution of Linear Equations with rational coeffeciants

To solve a linear equation we will follow the following rules:

1. Transfer all terms containing variable on one side and constant terms on the other side of the equation.
2. Simplify both sides of the equation.
3. Divide both sides of the equation by the rational coefficient of the variable.
4. Value of variable which is single numerical value is obtained.
5. Verify this value by putting it in the original equation.
6. If L.H.S = R.H.S of the equation, then this value of variable is the required solution of the equation.

Example 2

Solve: $2x + 3 = 1 - 6(x - 1)$.

Solution:

We begin by using the distributive property to remove the parentheses.

$2x + 3 = 1 - 6(x - 1)$

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$2x+3 = 1-6x+6$	Using the distributive property
$2x + 3 = 7-6x$	Combining like terms
$2x + 6x + 3 = 7-6x +6x$	Using the addition principle to add 6x on both sides
$8x +3 -3 = 7- 3$	Using the addition principle to add–3, or subtract 3, on both sides
$x = 4/8$	Using the multiplication principle to multiply by 1/8, or divide by 8, on both sides
$x = 1/2$	Simplifying

We check the result in the original equation.

Check:

$2x + = 1 - 6\,(x - 1)$	
$2 \cdot \frac{1}{2} + 3$?	$1 - 6\,(\frac{1}{2} - 1)$
$1 + 3$	$1 - 6\,(-\frac{1}{2})$
4	$1 + 3$
4	4 True

The solution is $\frac{1}{2}$.

Note

An equation remains balanced when

- Equal numbers are added to each side.
- Equal numbers are subtracted from each side.
- Each side is multiplied by equal numbers

Each side is divided by equal numbers except 0.

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Example 3

Solve $3x+\frac{x}{5}-5=\frac{1}{5}+5x$.

Solution:

$$3x+\frac{x}{5}-5=\frac{1}{5}+5x$$

or $3x+\frac{x}{5}-5x=\frac{1}{5}+5$ (shifting variable to one side & constants to other)

or $\frac{x}{5}-2x=\frac{1}{5}+5$ or $\frac{x-10x}{5}=\frac{1+25}{5}$ or $\frac{-9x}{5}=\frac{26}{5}$

or $-5\left(\frac{-9x}{5}\right)=(-5)\cdot\frac{26}{5}$ or $9x=-26$

or $x=\frac{-26}{9}$ (dividing by 9)

Note

Substitute the answer back into the equation to check if the equation remains balanced.
LHS =RHS

Example 4

Age of mother is 13 times the age of her daughter. It will be only five times after four years. Find their present ages.

Solution:

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Let present age of daughter = x years.
and present age of mother = $13x$ years.
After four years,
Age of daughter = $(x + 4)$ years
and age of mother = $(13x + 4)$ years
According to given condition
Age of mother = 5 (Age of daughter)

or $13x+4=5(x+4)$

$\Rightarrow 13x+4=5x+20$

$\Rightarrow 13x-5x+4=5x-5x+20$ (Subtracting $5x$ from both sides)

$\Rightarrow 8x+4=20$

$\Rightarrow 8x+4-4=20-4$ (Subtracting 4 from both sides)

$\Rightarrow 8x=16$

$\Rightarrow \frac{8x}{8}=\frac{16}{8}$

$\Rightarrow x=2$

Hence present age of mother = $13\times 2 = 26$ years
and present age of daughter = 2 years.

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Example 5

A number consist of two digits. The sum of digits is 8. If digits are interchanged, then new number becomes 36 less than the original number. Find the number.

Solution:

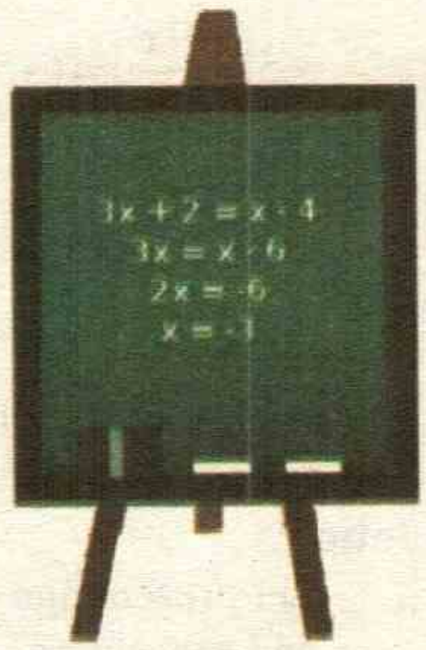
Let the digit in Ones place = x

and the digit in Tens place = $8 - x$ ($\because$ sum of digits = 8)

then original number = $10(8-x)+x$

$= 80 - 10x + x$

$= 80 - 9x$



When digits are interchanged,

then new number $= 10x + (8-x)$

$= 10x + 8 - x$

$= 9x + 8$

According to the given condition

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New number = (Original number) – 36

or $9x + 8 = (80 - 9x) - 36$

$\Rightarrow 9x + 8 = 80 - 9x - 36$

$\Rightarrow 9x + 8 = 44 - 9x$

$\Rightarrow 9x + 9x + 8 = 44 - 9x + 9x$ (Adding $9x$ to both sides)

$\Rightarrow 18x + 8 = 44$

$\Rightarrow 18x + 8 - 8 = 44 - 8$ (Subtracting 8 from both sides)

$\Rightarrow 18x = 36$

$\Rightarrow \frac{18x}{18} = \frac{36}{18}$ (Dividing by 18)

$\Rightarrow x = 2$

Hence digit in Ones place = 2

and digit in Tens place $= 8 - 2 = 6$

$\therefore$ The required number $= 62$.

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Exercise 7.1

1. Find the solution sets of the following equations and verify the answers:

i) $5x+8=23$ ii) $\frac{3}{5}x-\frac{2}{3}=2$ iii) $6x-5=2x+9$

iv) $\frac{2}{x-1}=\frac{1}{x-2}$ v) $\frac{1}{7x+13}=\frac{2}{9}$ vi) $10(x-4)=4(2x-1)+5$

2. Awais thought of a number, added 3 with it. Then he doubled the sum, he got 40. What was the original number?

3. The sum of two numbers is –4 and their difference is 6. What are the numbers?

4. The sum of three consecutive odd integers is 81. Find the numbers.

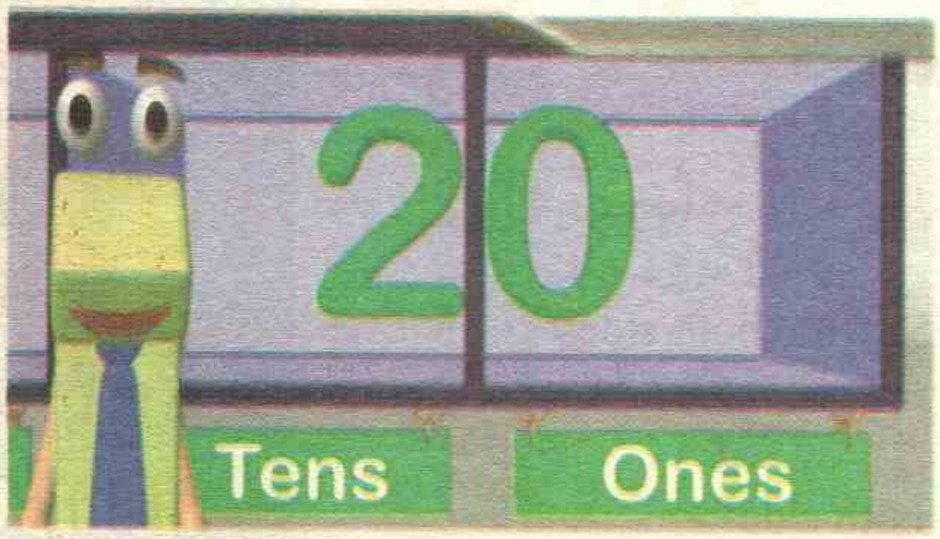
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5. A man is 41 years old and his son is 9 years old. In how many years will the father be three times as old as the son?

6. The tens digit of a certain two-digit number exceeds the units digit by 4 and is 1 less than twice the ones digit. Find the two-digit number.

7. The sum of two digits is 10. If the places of the digits are changed then the new number is decreased by 18. Find the number.

8. If breadth of a room is one fourth of its length and the perimeter of the room is 20m. Find length and breadth of the room.



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7.1.3 Solution of Equations Involving Radicals

Radical equations are equations in which the unknown is included in a radicand.

Thus, $\sqrt[2]{x}+5=9$ is a radical equation?

but $2x+\sqrt{5}=9$ is not a radical equation,

It should be remembered that in this unit radical equations will be considered as positive numbers and these numbers will have only positive square root.

Therefore equation like $\sqrt{x+6}=-11$ has no solution because this equation is not true for any value of x. On the other hand $\sqrt{x+6}=11$ has solution set consisting of a real number.

Solution of Radical Equations:

Example 6

Solve $\sqrt{2x}+5=9$.

Solution:

$\sqrt{2x}=9-5$ Reasoning

i.e. $\sqrt{2x}=4$ Isolate the term containing the radical:

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$2x=16$ Square both sides

$x=8$ Divide both sides by 2

Check for $x=8$:

$\sqrt{2(8)}+5=9 \Rightarrow \sqrt{16}+5=9 \Rightarrow 4+5=9$

$9=9 \quad \therefore x=8$ verified.

Example 7

Solve $\sqrt{3x-2}=\sqrt{5x+4}$

Solution:

$\sqrt{3x-2}=\sqrt{5x+4}$

Squaring both sides

$\left(\sqrt{3x-2}\right)^2=\left(\sqrt{5x+4}\right)^2$

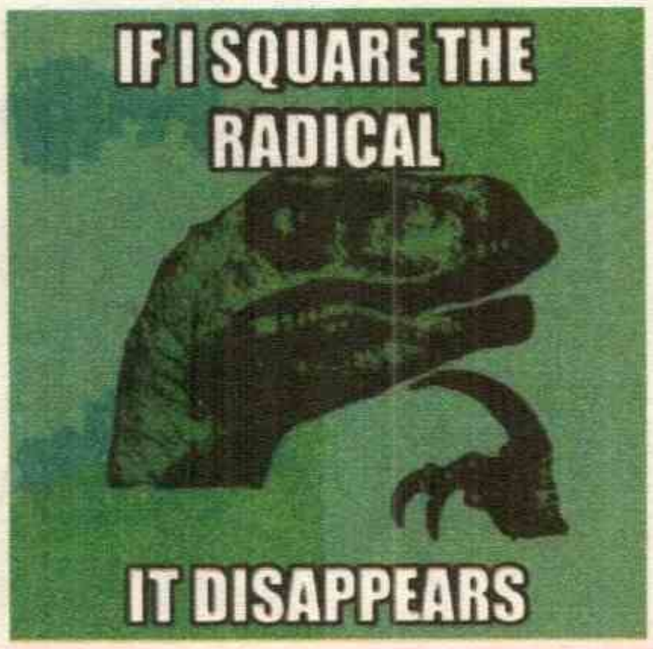
$\Rightarrow \quad 3x-2=5x+4$

Subtracting $5x$ from both sides

$3x-5x-2=5x-5x+4$

$\Rightarrow -2x-2=4$

Adding 2 to both sides



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$$-2x-2+2=4+2$$
$$-2x=6$$

Dividing both sides by -2

$$\frac{-2x}{-2}=\frac{6}{-2}$$
$$\Rightarrow \quad x=-3$$

To verify put $x=-3$ in given equation

$$\sqrt{3(-3)-2}=\sqrt{5(-3)+4}$$
$$\Rightarrow \sqrt{-9-2}=\sqrt{-15+4}$$
$$\Rightarrow \quad \sqrt{-11}=\sqrt{-11}$$

L.H.S = R.H.S

Hence $x=-3$ satisfies the given equation.

Therefore solution set = $\{-3\}$.

Note

When solving problems that involves radical sign, square both sides of the equation to obtain the value of unknown.

Example 8

Find the solution set of the equation $\sqrt{3x+2}+6=2$.

Solution:

$$\sqrt{3x+2}+6=2$$
$$\Rightarrow \quad \sqrt{3x+2}+6-6=2-6 \qquad \text{(Subtracting 6 from both sides)}$$
$$\Rightarrow \quad \sqrt{3x+2}=-4$$
$$\left(\sqrt{3x+2}\right)^2=(-4)^2 \qquad \text{(Squaring both sides)}$$
$$\Rightarrow \quad 3x+2=16$$
$$\Rightarrow 3x+2-2=16-2$$
$$\Rightarrow \quad 3x=14$$
$$\Rightarrow \quad x=\frac{14}{3}$$

To verify it, put $x=\frac{14}{3}$ in given equation.

$$\sqrt{3\left(\frac{14}{3}\right)+2}+6=2 \quad \Rightarrow \sqrt{14+2}+6=2$$
$$\Rightarrow \sqrt{16}+6=2 \quad \Rightarrow 4+6=2$$
$$\Rightarrow 10=2$$

Which is not possible.

Thus the given equation has no solution.

i.e. Solution set = $\{\ \}$.

Note

When we take square of both sides of a radical equation, sometimes, we get roots, which do not satisfy the given equation. Such roots are called extraneous roots. Therefore it is necessary to check the correctness of the roots by verification.

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Exercise 7.2

Solve the following radical equations:

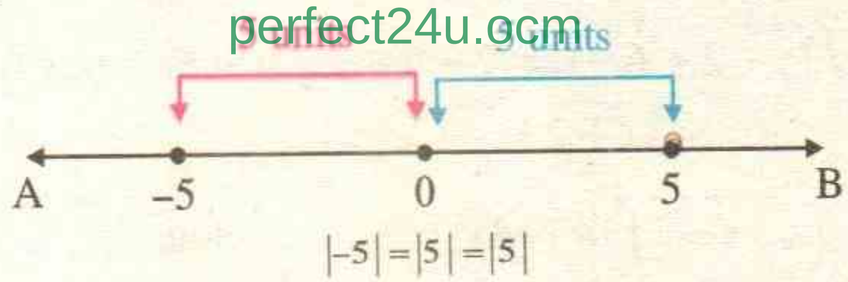
1. $2\sqrt{a}-3=7$
2. $8+3\sqrt{b}=20$
3. $7-\sqrt{2b}=3$
4. $8\sqrt{r}-5=\sqrt{r}+9$
5. $20-3\sqrt{t}=\sqrt{t}-4$
6. $2\sqrt{5x}-3=7$
7. $\sqrt{2x-7}+8=11$
8. $22=17+\sqrt{40-3y}$

7.2 EQUATIONS INVOLVING ABSOLUTE VALUE

7.2.1 Absolute Value

On the real number line, the distance of a number x and is denoted by $|x|$. For example $|5|=5$ and $|-5|=5$ because both 5 and -5 are 5 units from 0, as is obvious form the figure

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Similarly $|0|=0$. Notice that $|x|$ can never be negative, that is $|x|\geq 0$.

If x is positive or zero that $|x|$ is simply x itself, so we can omit the vertical bars and write $|x|=x$. On the other hand, consider the absolute value of a negative number like $x=-5$. $|x|=|-5|=5=x$

Thus if x is negative, then $|x|$ is the positive number $-x$. The minus sign indicates that we have changed the sign of x. Thus aside from its geometrical interpretation, absolute value can be defined as follows.

Definition:

The absolute of a real number x, written $|x|$, is defined as

$$|x|=\begin{cases} x & \text{if } x\geq 0 \\ -x & \text{if } x<0 \end{cases}$$

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7.2.2 Solution of Equations Involving Absolute Value in one variable

To solve equations involving absolute value in one variable we have to consider both the possible values of the variable, for example if $|x| = 2$ then $x = 2$ and $x = -2$ because $|2| = |-2| = 2$.

To explain this concept we consider the following examples.

Example 9

Solve $|x-1| = 7$

Solution:

Given equation is $|x-1| = 7$, then there are two possibilities;

(i) $x-1 = 7 \Rightarrow x-1+1 = 7+1 \Rightarrow x = 8$ and

(ii) $x-1 = -7 \Rightarrow x-1+1 = -7+1 \Rightarrow x = -6$

Therefore solution set = $\{8, -6\}$.

Example 10

Find the solution set of $|3x-5|+7 = 11$

Solution:

$|3x-5|+7 = 11$ perfect24u.ocm

$|3x-5|+7-7 = 11-7$ Subtracting 7 from both sides

$|3x-5| = 4$

The two possibilities of equations are

$3x-5 = 4$ and $3x-5 = -4$

$\Rightarrow 3x-5+5 = 4+5$ and $3x-5+5 = -4+5$

$\Rightarrow 3x = 9$ and $3x = 1$

$\Rightarrow \frac{3x}{3} = \frac{9}{3}$ and $\frac{3x}{3} = \frac{1}{3}$ Dividing both sides by 3

$\Rightarrow x = 3$ and $x = \frac{1}{3}$

Therefore the solution set = $\left\{3, \frac{1}{3}\right\}$

Solution Sets

$3x = 4, x \in R$

Solution Set = { 4/3 }

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Exercise 7.3

Solve for x.

1. $|x+3|=5$
2. $|-5x+1|=6$
3. $\left|\frac{3}{4}x-8\right|=1$
4. $|x-4|=3$
5. $|3x+4|=2$
6. $|2x-9|=0$
7. $\left|\frac{3x-2}{5}\right|=7$
8. $4|5x-2|+3=11$
9. $\frac{2}{5}|4x-3|-9=-1$

7.3 LINEAR INEQUALITIES

Inequality is a statement that one quantity is less than, greater than, less than or equal to and greater than or equal to another mathematical quantity is called linear inequality and the relations are shown by inequality symbols as under;

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'$<$' means less than
'$>$' means greater than
'$\leq$' means less than or equal to
'$\geq$' means greater than or equal to

Thus if a and b are two real numbers we can compare their positions on the real line by using inequality symbols. For example;

$a < b$ means a is less than b, and on number line a lies to the left of b.

$a >$ b means a is greater than b, and a lies to the right of b on number line.

$a \leq b$ means a is less than or equal to b and a coincides with b or lies to the left of b on number line.

$a \geq b$ means a is greater than or equal to b and a coincides with b or lies to the right of b.

Linear Inequality in one variable

A linear inequality in one variable say x is an inequality in one of the following forms or can be converted into the following form:

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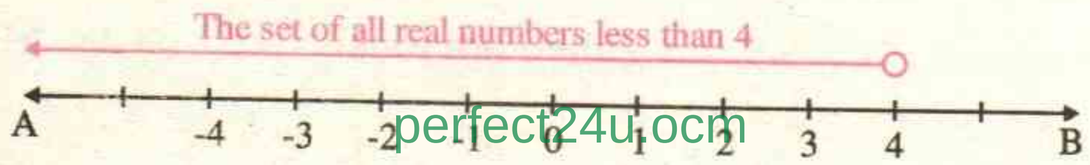
$$\left.\begin{array}{l} ax+b>c \\ ax+b\geq c \\ ax+b<c \\ ax+b\leq c \end{array}\right\}$$

where a, b, c are any real numbers with $a\neq 0$. x is also a real variable.

Thus $3x+5>4$ is a linear inequality in one variable. $2n-\frac{1}{6}\leq 2$ is also a linear-inequality in one variable. By the solution of an inequality in one variable, we mean the value of the variable that makes the inequality a true statement. The set of all possible values of the variable which makes the inequality a true statement is called the solution set of the inequality,

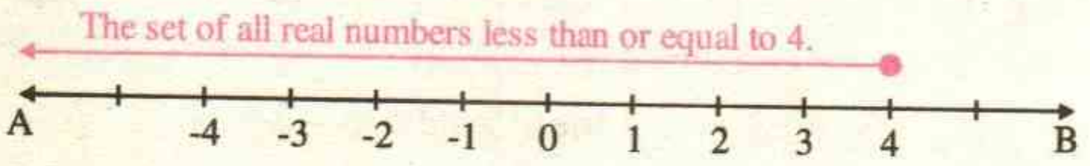
It is always convenient to represent the solution of an inequality with the help of real number line.

For example, suppose $x<4$ is a simple inequality and it means all real numbers which are less than 4. Geometrically we can say that all those real numbers which are lying to the left of 4. Here we are to exclude 4. This is done by using a hollow circle around 4 as shown in the figure.



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But if we have $x\leq 4$ then 4 is included in our discussion, so in this case we will use "thick", filled or "solid" circle as shown below in the figure.



Example 11

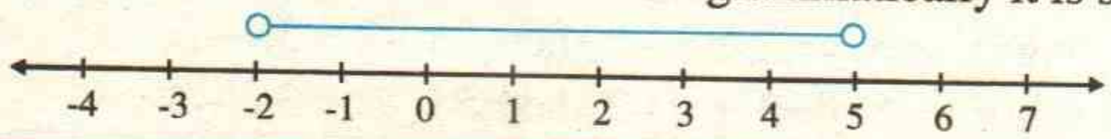
Show $-2<x<5$ on a number line.

Solution:

$-2<x<5$ means the set of those real numbers which are greater than -2 but less than 5.

or $-2<x<5$ means the set of all those real numbers which are between -2 and 5, obviously -2 and 5 are nor included.

or geometrically $-2<x<5$ means the set of all those real numbers are lying to the right of -2 but to the left of 5. Diagrammatically it is shown below.



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7.3.1 Properties of Inequalities

The following are some properties of inequalities.

(i) Trichotomy Property:

If x and y are any two real numbers then one and only one of the following statements is true

a) $x > y$ *b)* $x < y$ *c)* $x = y$

(ii) Transitive Property:

If x, y and z are any three real numbers then

a) If $x < y$ and $y < z \Rightarrow x < z$

e.g if $x = 1$, $y = 5$, $z = 6$,then $1<5$, $5<6$ and $1<6$ is obviously true

b) If $x > y$ and $y > z \Rightarrow x > z$

if $x = 1$, $y = 0$, $z = -4$, then $1 > 0$, $0 > -4$ and $1 > -4$

(iii) Additive Property:

If x, y and z are any three real numbers then

a) $x < y \Rightarrow x + z < y + z$

e.g if $x = 3$, $y = 5$, $z = 2$ then obviously $3<5$, $3+2 < 5+2$

$x < y \Rightarrow x - z < y - z$

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e.g if $x = 3$, $y = 5$, $z = 2$ then $3 - 2 < 5 - 2$



b) $x > y \Rightarrow x + z > y + z$

e.g if $x = 5$, $y = 3$, $z = 2$ then $5 + 2 > 3 + 2$

$x > y \Rightarrow x - z > y - z$

e.g if $x = 5$, $y = 3$, $z = 2$ then $5 - 2 > 3 - 2$

(iv) Multiplicative Property:

If x, y and z are any three real numbers then

a) (i) $x < y \Rightarrow xz < yz$
(ii) $x > y \Rightarrow xz > yz$
} when z is positive number.

e.g if $x = 5$, $y = 6$, $z = 2$ then obviously $5 < 6$, $5 \cdot 4 < 6 \cdot 4$ *or* $20 < 24$

and if $x = 5$, $y = 3$, $z = 2$ then obviously $5 > 3$, $5 \cdot 2 > 3 \cdot 2$ *or* $10 > 6$.

b) (i) $x < y \Rightarrow xz > yz$
(ii) $x > y \Rightarrow xz < yz$
} when z is negative number.

e.g if $x = 3$, $y = 5$, $z = -2$ then $3x - 2 > 5x - 2$

e.g if $x = 5$, $y = 3$, $z = -2$ then $5x - 2 < 3x - 2$

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Example 12

Write the names of properties used in the following statements.

(i) $21 < 31 \Rightarrow 31 < 41$ (ii) $15 > 8 \Rightarrow 22 > 15$

(iii) $10 < 20 \Rightarrow 30 < 60$ (iv) $-12 > -15 \Rightarrow 24 < 30$

(v) If $x > 4$ and $4 > z$ then $x > z$

Solution:

(i) $21 < 31 \Rightarrow 21+10 < 31+10$ (Additive Property)

Hence $21 < 31 \Rightarrow 31 < 41$

(ii) $15 > 8 \Rightarrow 15+7 > 8+7$ (Additive Property)

Hence $15 > 8 \Rightarrow 22 > 15$

(iii) $10 < 20 \Rightarrow 10 \times 3 < 20 \times 3$ (Multiplicative Property)

Hence $10 < 20 \Rightarrow 30 < 60$

(iv) $-12 > -15 \Rightarrow -12 \times -2 < -15 \times -2$ (Multiplicative Property)

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Hence $-12 > -15 \Rightarrow 24 < 30$

(v) $x > 4$ and $4 > z \Rightarrow x > z$ (Transitive Property)

7.4 SOLUTION OF LINEAR INEQUALITIES WITH RATIONAL COEFFICIENT

Linear inequalities are solved in almost the same way as linear equations

PRINCIPLES OF INEQUALITIES:

(i) if $a > b$, then

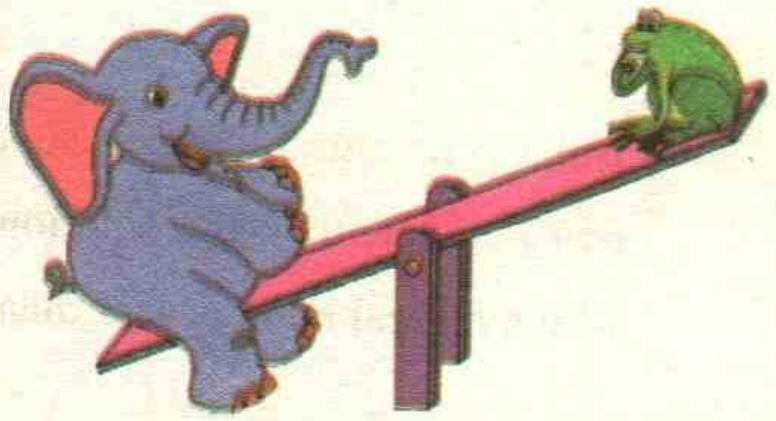
$a+c < b+c$, *and* $a-c < b-c$, *and* $a-b > 0$.

(ii) if $a > b$ *and* $k > o$, then

$ka > kb$ and $\frac{a}{k} > \frac{b}{k}$.

(iii) if $a > b$ and $k < 0$, then

$ka < kb$ and $\frac{a}{k} < \frac{b}{k}$.



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7.4.1 Application of inequality in Real Life

Luggage Weights:

You are checking a bag at an airport. Bags can weigh no more than 50 Kgs. Your bag weighs 16.8 kg. Find the possible weight w (in Kg)that you can add to the bag.

Example 13

Solution:

Weight of bag + weight you can add ≤ weight limit

$$16.8+W \le 50$$

$$5016.8+W-16.8 \le 50-16.8$$

$$W \le 33.2$$

You can add no more than 33.2 kgs

Example 14

Solve the inequality $2(\frac{x}{4}+1)<\frac{3}{2}$ where (i) x is a natural number, (ii) x is a real number.

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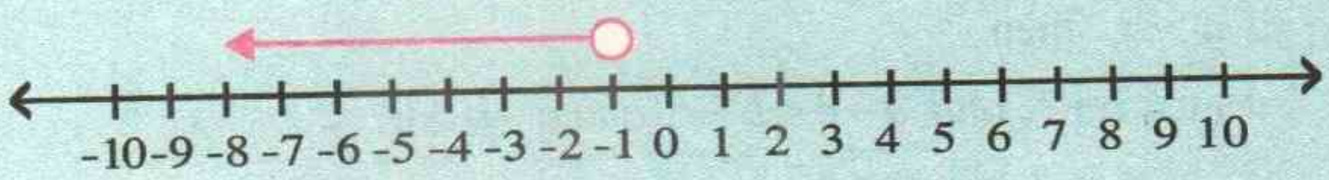
Solution:

$$2(\frac{x}{4}+1)<\frac{3}{2}$$ Original inequality

$$2(x+4)<6$$ Multiply both sides by 4

$$x+4<3$$ Divide both sides by 3

$$x<-1$$ Subtract 4 from both sides.



But x is a natural number, so above in-equality has no solution. Bur x is a real number, then the solution of above in-equality is shown in diagram which all real number less than -1.

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Example 15

Solve the inequality $x-\frac{5}{7}\leq\frac{15+2x}{7}$ where (i) x is natural number, (ii) x is a real number.

Solution:

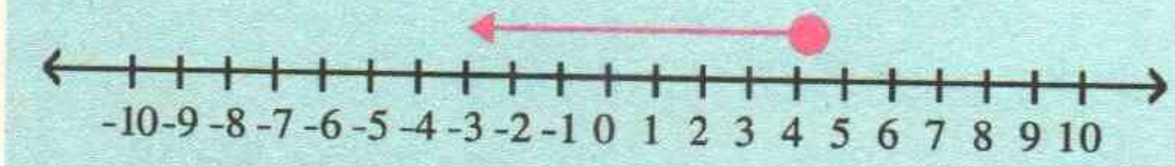
$x-\frac{5}{7}\leq\frac{15+2x}{7}$ Original inequality

$7x-5\leq 15+2x$ Multiply by 7

$5x-5\leq 15$ Subtract $2x$ from both sides.

$5x\leq 20$ Add 5 on both sides.

$x\leq 4$ Divide both sides by 5.



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If x is a natural number then the solution set is $\{1,2,3,4\}$. But if x is real number then the solution set is show in the above diagram.

Example 16

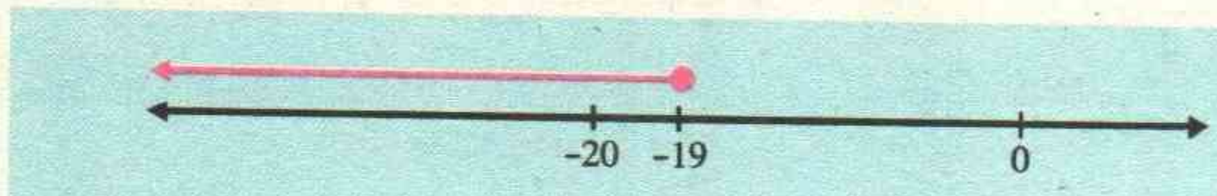
Solve the in-equality $\frac{x+3}{2}\leq\frac{x-5}{3}$ where $x\in R$.

Solution:

$\frac{x+3}{2}\leq\frac{x-5}{3}\Rightarrow\frac{x+3}{2}-\frac{x-5}{3}\leq 0$ or $\frac{3(x+3)-2(x-5)}{6}\leq 0$

$\Rightarrow\frac{3x+9-2x+10}{6}\leq 0\Rightarrow\frac{x+19}{6}\leq 0\Rightarrow 6\cdot\frac{x+19}{6}\leq 6\cdot 0$

$\Rightarrow x\leq -19$. The solution is shown in the diagram.



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Exercise 7.4

1. **Show the following inequalities on number line.**

i) $x > 0$ ii) $x < 0$ iii) $\frac{x-3}{2} \leq -1$

iv) $x \leq -5$ v) $x \geq -3$ vi) $\frac{3x-2}{6} > \frac{5}{2}$

vii) $-5 \leq x \leq 6$ viii) $3 \geq x \geq -2$ ix) $0 < \frac{x}{4} - 1 < \frac{1}{2}$

x) $0 < \frac{x+3}{2} < \frac{3}{2}$

2. **Find the solution sets of the following inequalities:**

i) $7 - 2x \geq 1$, $x \in N$ ii) $5x + 4 < 34$, $x \in N$

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iii) $\frac{8x+1}{2} < 2x - 1.5$, $x \in R$ iv) $(4x+3) \geq 23$, $x \in \{1,2,3,4,5,6\}$

v) $5x + 1 \geq 13 - x$, $x \in \{-2,-1,0,1,2,3,4,5\}$

vi) $\frac{2x+6}{2} \leq \frac{x-9}{3}$, $x \in R$ vii) $\frac{x-1}{3} \leq \frac{1-x}{2}$, $x \in Z$

3. **Solve the following inequalities and plot the solution on the number line.**

i) $\frac{x}{12} \leq \frac{1}{4}$ ii) $x + 7 \geq 2$ iii) $3(x-2) > 15$

iv) $\frac{1}{2} > \frac{x}{4} > -2$ v) $2.5 \leq \frac{x}{2} + 1 \leq 4.5$ vi) $-2 \leq x < 2$

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REVIEW EXERCISE 7

1. Select the correct answer.

i) Solve for $x: \frac{1}{2}|x-6|-4=-1$.

a) $x=12$ *b)* $x=8$ and $x=4$

c) $x=12$ and $x=0$ d) there is no solution

ii) Solve for $x: |3x-1|=2$.

a) $x=1$ b) $x=1$ and $x=-\frac{1}{3}$

c) $x=1$ and $x=-1$ d) $x=1$ and $x=\frac{1}{3}$

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iii) The solution set of $\sqrt{5x+3}+2=4$ is

a) $\left\{\frac{1}{5}\right\}$ b) $\left\{-\frac{1}{5}\right\}$ c) $\{2\}$ d) $\{1\}$

iv) Solve for x, $\sqrt{x}=-10$

a) $\{-10\}$ b) $\{\ \}$ c) $\{100\}$ d) $\{10\}$

v) $\sqrt{2x+1}-5=4$ is a

a) Linear equation b) Radical equation

c) Cubic equation d) Quadratic equation

vi) What is the solution for $|x-7|=1$?

a) $x=8$ *b)* $x=6$ *and* $x=8$ *c)* $x=8$ *and* $x=-8$ *d)* $x=6$

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vii) The solution set of $\sqrt{5x+3}+2=4$ is

a) $\left\{\frac{1}{5}\right\}$ b) $\left\{-\frac{1}{5}\right\}$ c) $\{2\}$ d) $\{1\}$

viii) The solution set of $\left|\frac{5x}{3}\right|=5$ is

a) $\{3\}$ b) $\{5,-5\}$ c) $\{4,-4\}$ d) $\{-4\}$

ix) Which one is the solution set of $|-x|=0$

a) $\{-1\}$ b) $\{1\}$ c) $\{\ \}$ d) $\{0\}$

x) Solve for $x: \frac{x+2}{x-2}>0$.

a) $(-2,\infty)$ *b*) $(-2,2)$

c) $(-2,\infty)\cup(2,\infty)$ *d*) $(-\infty,-2)\cup(2,\infty)$

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2. Solve the following equations for x .

i) $5(3x+1)=2(x-4)$ ii) $\frac{x-8}{3}+\frac{x-3}{2}=0$

iii) $\sqrt{2(5x-1)}=\sqrt{2x+14}$ iv) $|2x+7|=9$

3. Solve the following inequalities and graph the solution on the number line.

i) $-1<\frac{x-3}{2}<0$ ii) $-1<\frac{x-4}{5}<0$ iii) $7<-3x+1\leq 13$

Think

4. A father is 4 times older than his son. In 20 years he will be twice as old as his son. What ages have they now?

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SUMMARY

- An equation is a statement that two expressions are equal.
- To solve an equation in one variable is to find all the values of the variable that make the equation true. Each of these numbers is a solution of the equation.
- The set of all solutions of an equation is its solution set.
- Equations that have the same solution set are called equivalent .
- Radical equations are equations in which the unknown is included in a radicand.
- Absolute value of any number x is denoted by $|x|$ and defined as

$$|x| = x \quad \text{if} \quad x > 0$$
$$= 0 \quad \text{if} \quad x = 0$$
$$= -x \quad \text{if} \quad x < 0$$

- The absolute value of a number is always non-negative.
- Inequality is a statement that one quantity is less or greater than another quantity.
- An inequality has as infinite number of real solutions whereas equation has finite number of solutions.

If x and y are any two real numbers then one and only one statement is true

a) $x > y$ *b)* $x < y$ *c)* $x = y$

- If x, y and z are any three real numbers then

a) If $x < y$ and $y < z \Rightarrow x < z$

- If x, y and z are any three real numbers then

a) $x < y \Rightarrow x + z < y + z$ *b)* $x > y \Rightarrow x + z > y + z$

- If x, y and z are any three real numbers then

a) (i) $x < y \Rightarrow xz < yz$
(ii) $x > y \Rightarrow xz > yz$
} when z is positive number.

b) (i) $x < y \Rightarrow xz > yz$
(ii) $x > y \Rightarrow xz < yz$
} when z is negative number.

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UNIT 08

LINEAR GRAPHS AND THEIR APPLICATIONS

After reading this unit, the students will be able to:

- Identify pair of real numbers as an ordered pair.
- Recognize an ordered pair through different examples;
- Describe rectangular or Cartesian plane consisting of two number lines intersecting at right angles at the point O.
- Identify origin (O) and coordinate axes (horizontal and vertical axes or x-axis and y-axis) in the rectangular plane.
- Locate an ordered pair (a, b) as a point in the rectangular plane and recognize:
  - a as the x-coordinate (or abscissa),
  - b as the y-coordinate (or ordinate).
- Draw different geometrical shapes (e.g., line segment, triangle and rectangle etc) by joining a set of given points.
- Construct a table for pairs of values satisfying a linear equation in two variables.
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- Plot the pairs of points to obtain the graph of a given expression.
- Choose an appropriate scale to draw a graph.
- Draw the graph of
  - an equation of the form $y = c$.
  - an equation of the form $x = a$.
  - an equation of the form $y = mx$.
  - an equation of the form $y = mx + c$
- Draw a graph from a given table of (discrete) values.
- Solve appropriate real life problems.
- Interpret conversion graph as a linear graph relating to two quantities which are in direct proportion.
- Read a given graph to know one quantity corresponding to another.
- Read the graph for conversions of the form:
  - miles and kilometers,
  - acres and hectares,
  - degrees Celsius and degrees Fahrenheit,
  - Pakistani currency and another currency, etc.
- Solve simultaneous linear equations in two variables using graphical method.

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Introduction:

In the previous class we learned a technique for solving first-degree equations containing a single variable, which are also known as linear equations. Recall that the solution set for a linear equation is a real number.

In this unit we will learn how to graphically represent an ordered pair, which consists of two real numbers. Just as a real number is associated with a point on a number line, and ordered pair is associated with a point in 2-dimentional plane called the Cartesian Coordinate System or Rectangular Coordinate System. Also we will learn the conversion graphs of different quantities and the solution set of two linear equations in two variables by graphical method.

8.1 IDENTIFYING PAIR OF REAL NUMBERS AS AN ORDERED PAIR

8.1.1 Cartesian Plane and Linear Graphs

a) Identification of pair of real numbers

Let $A = \{1, 2\}$ and $B = \{3, 4\}$ are any two sets. The cartesian product of A and B denoted by $A \times B$ is given as;

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$A \times B = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$

which shows that each element of $A \times B$ is an ordered pair. If we take the cartesian product of a real number set to itself then it is defined as

$$R \times R = \{(a, b) \mid a, b \in R\}$$

Each element of $R \times R$ is an ordered pair (a, b) where both a and b are real numbers. $(\sqrt{2}, \sqrt{3}), (5, -7), (2,3)$ and $(-1, 4)$ etc. are the examples of ordered pairs.

b) Recognizing ordered pairs of real numbers through examples

Example 1

Let a rectangular box is filled with balls of equal size as shown in the figure.



Then the pair (3, 4) represents the position of the ball which is in the third row and fourth column in the box.

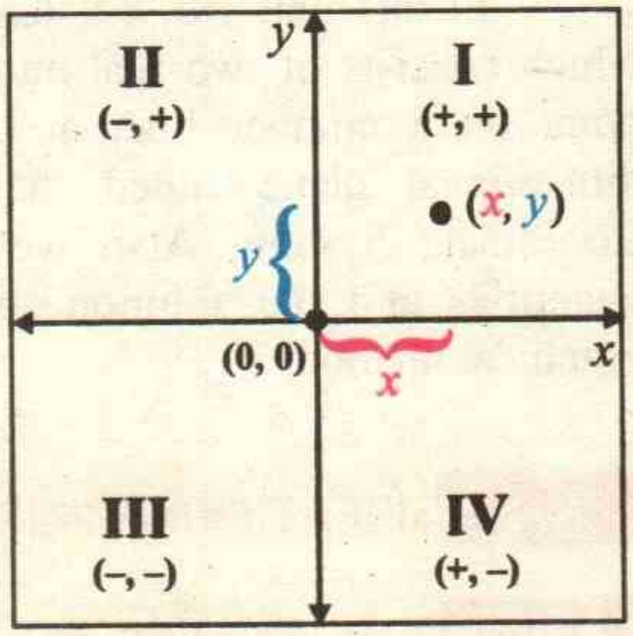
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Example 2

If a farmer has planted trees in a garden at equal distance. Then (3, 5) represents the tree located in 3rd row and 5th column in the garden.

c) Cartesian Plane

A rectangular coordinate system is formed by two perpendicular number lines, one horizontal and one vertical, that intersect at the zero point of each line. The point of intersection O(0,0) is called the origin. The two lines are called coordinate axes, or simply axes. The axes determine a plane, called rectangular or Cartesian plane.



The two axes divide the plane into four regions called quadrants. The quadrants are numbered counterclockwise from I to IV.

Each point in the plane can be identified by a pair of numbers called an ordered pair.

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The first number of the pair measures a horizontal distance and is called the abscissa.

The second number of the pair measures avertical distance and is called the ordinate.

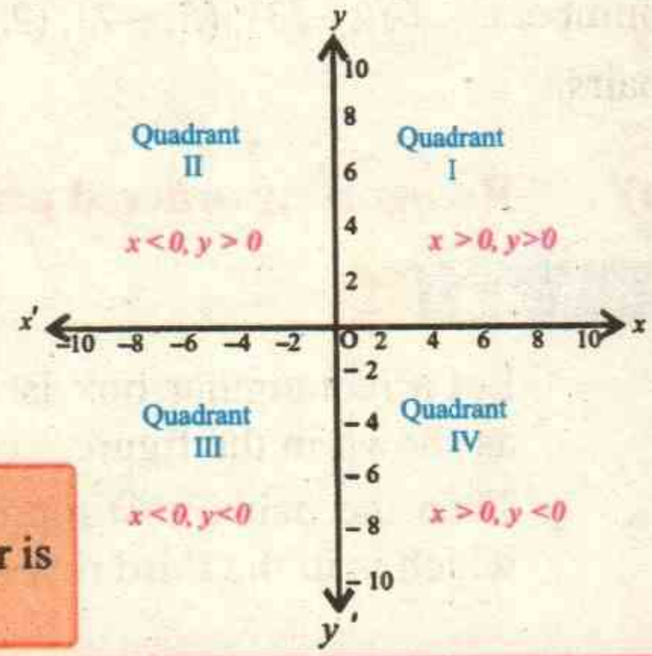
The coordinates of a point are the numbers in the ordered pair associated with the point. The abscissa is also called the first coordinate of the ordered pair or x-coordinate, and the ordinate is also called the second coordinate of the ordered pair or y-coordinate.

In Quadrant I $x > 0$, $y > 0$

In Quadrant II $x < 0$, $y > 0$

In Quadrant III $x < 0$, $y < 0$

In Quadrant IV $x > 0$, $y < 0$



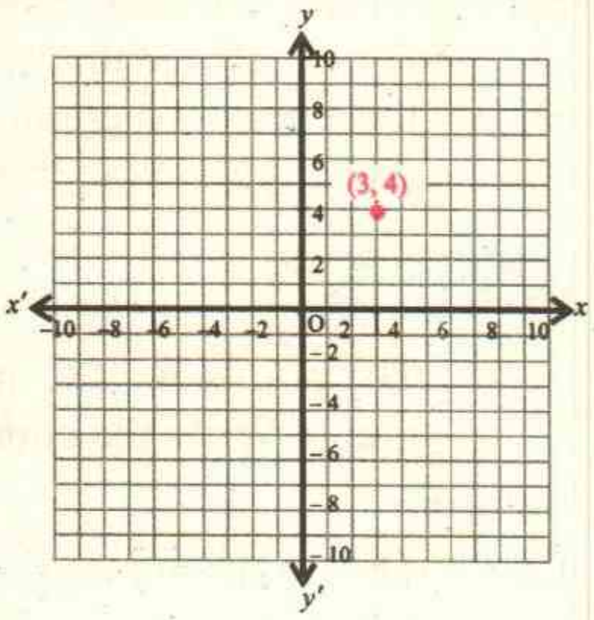
Tit-Bit

The **order** of the coordinates of an ordered pair is important.

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d) Locating a point

Locating or graphing, or plotting, an ordered pair in the plane means placing a dot at the location given by the ordered pair. The graph of an ordered pair is the dot drawn at the coordinates of the point in the plane. The point whose coordinates are (3, 4) is graphed in the figure at the right.



Example 3

Determine the x-coordinate and y-coordinate of the following points. Mention the quadrant in which each point lies. Also plot the points.

i) (3, 1) ii) (– 2, – 4) iii) (4, 0) iv) (1, – 3)

Solution:

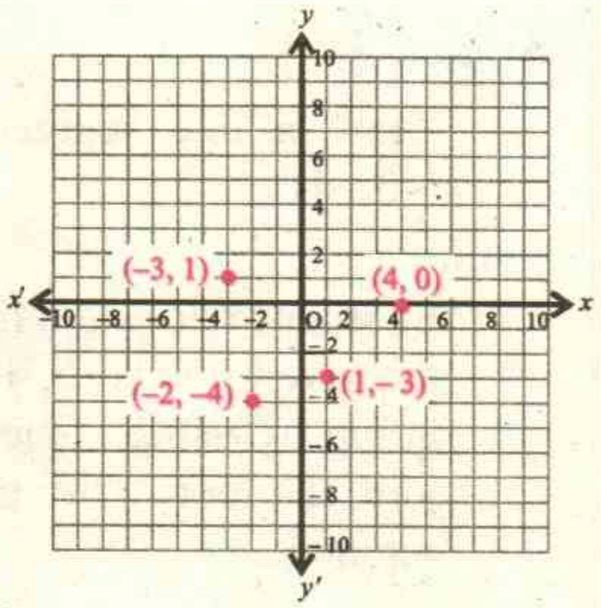
i) In the point (–3, 1), the x-coordinate is 3 and y-coordinate is 1. The point (–3, 1) lies in the quadrant II.

ii) In the point (–2,–4),the x-coordinate is – 2 and y-coordinate is –4. The point (–2,–4), lies in quadrant III.

iii) In the point (4, 0), the x-coordinate is 4 and y-coordinate is 0. The point (4, 0) lies on x-axis.

iv) In the point (1, –3), the x-coordinate is 1 and y-coordinate is –3. The point (1, –3), lies in the quadrant IV.

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Scale: 1 small square = 1 unit along x-axis.
Scale: 1 small square = 1 unit along y-axis.

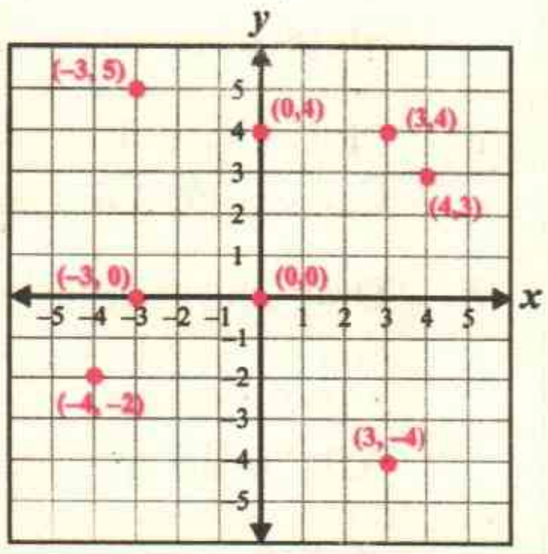
Example 4

Graph and label the points (-3,5), (4,3), (3,4), (-4,-2), (3,-4), (0,4), (-3,0), and (0,0).

Solution:

Scale: *1 small square = 1 unit along x-axis and y-axis.*

To graph or plot (-3, 5), we note that the x- coordinate, –3, tells us to move from the origin 3 units to the left of the y-axis. Then

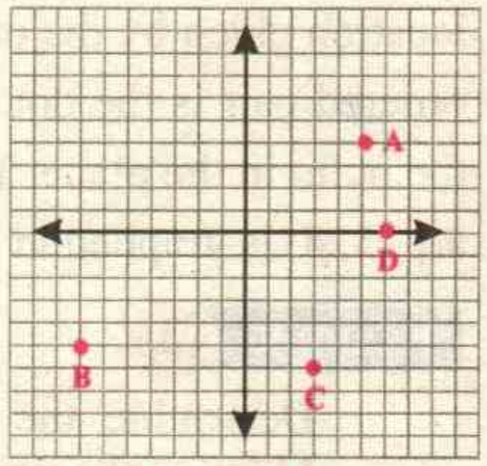


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the origin 3 units to the left of the y-axis. Then we move 5 units up from the x-axis. To graph the other points, we proceed in a similar manner (See the graph). Note that the point (4,3) is different from the point (3,4).

Example 5

The points A, B,C and D are shown in the graph. Write the coordinates of the points.



Solution:

The points are A(5, 4), B(-6, -5), C(3, –6), D(6, 0).

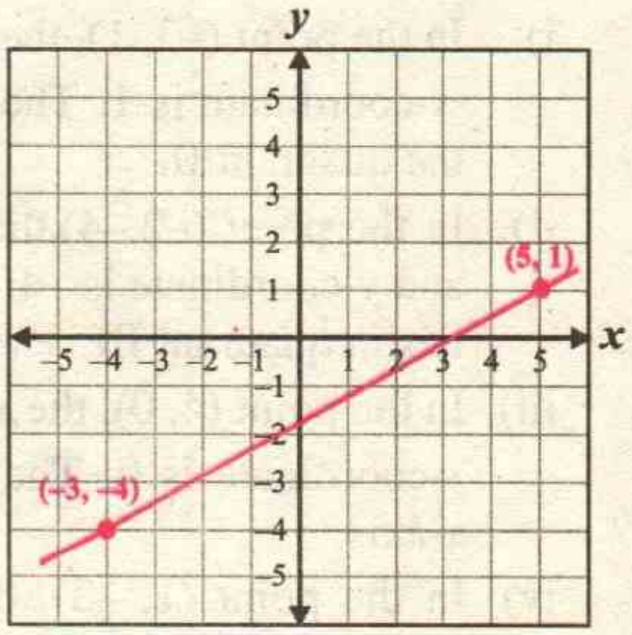
e) To draw line segment, Triangle, Rectangle, Square and Parallelogram by joining the set of given points:

Example 6

Draw a line segment joining (5,1) and (–3,–4)

Solution:

In order to draw the line segment, first plot the given points (5, 1) and (–3, –4) on the graph by using some suitable scale. Then join the points to get the required line segment.



Example 7

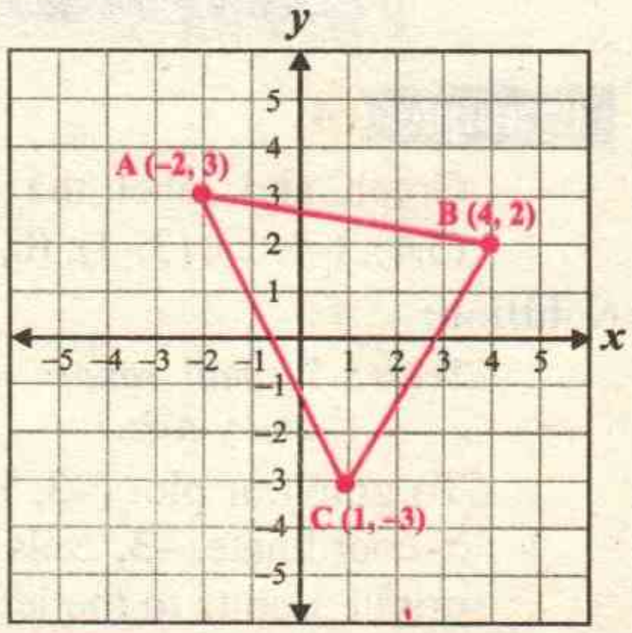
Draw a triangle ABC by joining the points A (–2, 3), B(4, 2), C(1, –3).

Solution:

Steps of construction:

(a) Plot the points A(–2, 3), B(4, 2) and C(1, –3) on the xy-plane.

(b) Draw the straight line through the points A and B, B and C, A and C to get the sides AB, BC and AC of the triangle ABC. Thus ABC is the triangle.



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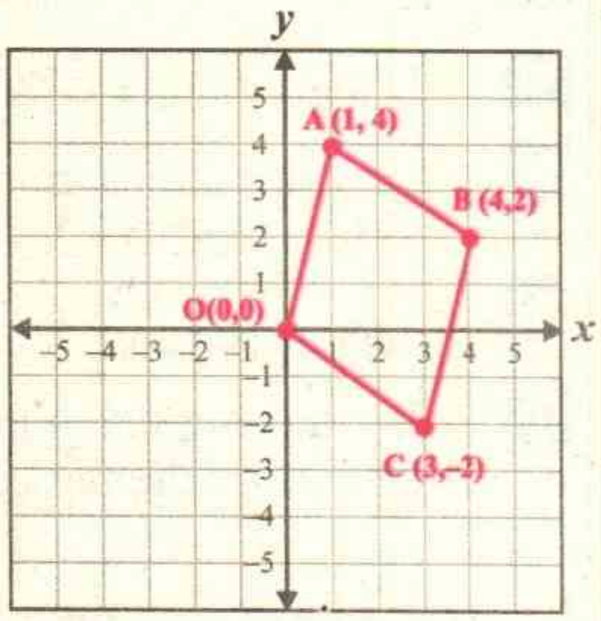
Example 8

Draw a parallelogram by joining the points O(0, 0), A(1, 4), B(4, 2) and C(3, –2).

Solution:

(a) Plot the points O(0, 0), A(1, 4), B(4, 2) and C(3, –2) on the *xy*-plane.

(b) Draw the straight line through O and A, A and B, B and C and O and C to get the sides OA, AB, BC, and OC respectively of the parallelogram or joined all the points by drawing straight line. OABC is a Parallelogram.



Example 9

Draw a rectangle by joining the points P(2,3), Q(2,0), S(–2,0) and R(–2,3).

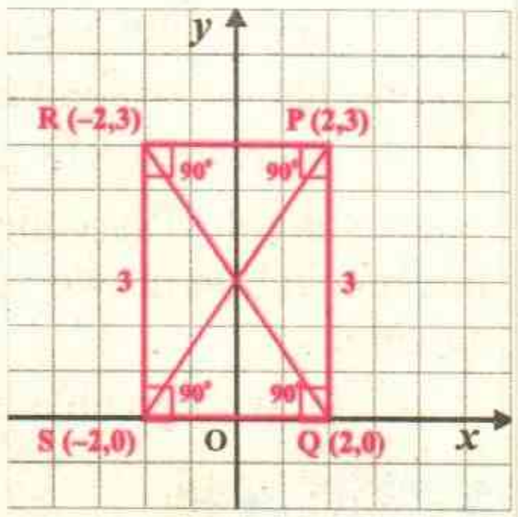
Solution:

Plot the points P(2,3), Q(2,0), S(–2,0) and R(–2,3). Joining the points P, Q, S and R, we get a rectangle PQRS.

Along *y*-axis,

2(length of square) = 1 unit

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Exercise 8.1

1. Determine the *x* and *y* coordinates of the following points.

i) A(–7, 5) ii) B(0, 7) iii) C(–3, 8)

iv) D(–3, –3) v) E(10, 12)

2. Mention the quadrant in which each of the following point lies.

i) $A(-1, \sqrt{2})$ ii) B(–3, – 2) iii) C(5, 5)

iv) D(3, – 5) v) $E(-\sqrt{5}, \sqrt{7})$

3. Plot the points A, B, C and D on the *xy*-plane:

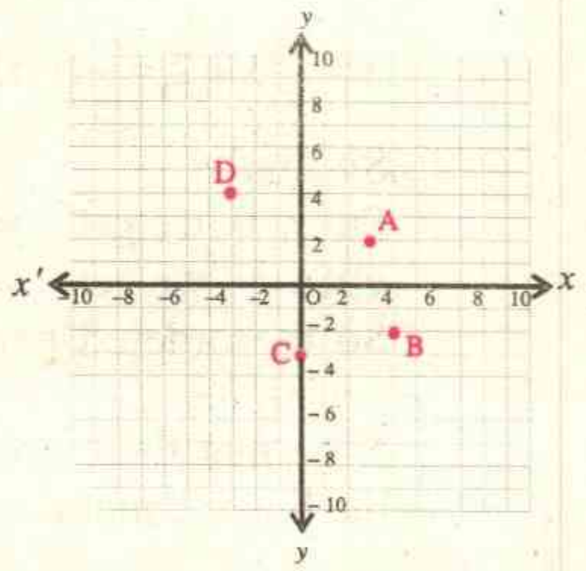
i) A(3, 1), B(2,4), C(–5,6), D(3,–3)

ii) A(–1, 0) , B(0, – 1) , C(2, – 2), D(3. 3)

iii) A(4, 4) , B(0, 0) , C(8, – 6) , D(– 7, 5)

4. Plot the points associated with the ordered pairs A(3, 5), B(4, 3), C(5, –5), D(– 4, –5), E(– 4,4) and F(0, 5).

5. Write the coordinates of the points A, B, C and D in the given graph.



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6. Draw a line segment by joining the points (5, 7) and (–7, 9).

7. Draw a triangle ABC by joining the points A(5,7), B(8, –3) and C(9,4).

8. Draw a parallelogram OABC by joining the points O(0, 0), A(3, –4), B(1,–7) and C(– 2, – 3).

9. Join the points O (0,0), A (5,0), B (5,5) and C (0,5) to draw a square.

10. By drawing the graph, show that the points A(0, 1), B(1, 2), C(2, 1) and D(1, 0), are the vertices of a rectangle.

f) A Linear Equation in Two Variables:

Let's begin to examine the connections among equations, functions, and graphs that are straight lines.

An equation of the form $ax+by=c$ (where a, b, and c are constants and x, y are variables) is called a linear equation in two variables. The solution of a linear equation in two variables is a set of ordered pairs satisfying the given linear equation.

How to graph a linear function, we explain it by the help of an example.

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Example 10

Graph: $3x - 5y = -10$

Solution:

We first solve for y:

$3x - 5y = -10$

$-5y = -3x - 10$

$y = \frac{3}{5}x + 2$

Put $x = -5$, we get

$y = \frac{3}{5}x + 2 = \frac{3}{5}(-5) + 2 = -3 + 2 = -1$

Similarly,

Put $x=0$, we get $y=2$.

Put $x=5$, we get $y=5$.

So the ordered pairs (–5, –1), (0, 2), (5, 5) satisfy the given linear equation.

Note that there are infinite number of ordered pairs that are solutions to the linear equation e.g. (10, 8), (-10 ,-4) etc. (continued)

Note

To see whether an ordered pair is a solution to a linear equation in two variables or not, just substitute the values of x and y in the equation. If the resulting statement is true, then the ordered pair is the solution to the linear equation otherwise not.

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g) Construction of the table for ordered pair of values satisfying a linear equation in two variables:

Frequently a linear equation involve the variables x and y. To construct a table for pairs of values satisfying the equation, the variables x and y are written in the top row. Some values for x and y are chosen to satisfy the given linear equation. The values of x are written below the variable x in first column and the corresponding values of y are written in the second column below the variable y. Then the ordered pairs are formed from the corresponding values of x and y. For example the following table lists a few points in our above example.

x	y	(x, y)
5	1	(5, 1)
0	2	(0, 2)
5	5	(5, 5)

h) Drawing the graph of a linear equation:

The graph of a linear equation is a straight line, which contain the set of points representing the ordered pairs of the solution set. Since a straight line is determined by two points, so we need to find at least two ordered pairs that are solutions to the linear equation in order to draw the graph. We summarize this as follows:

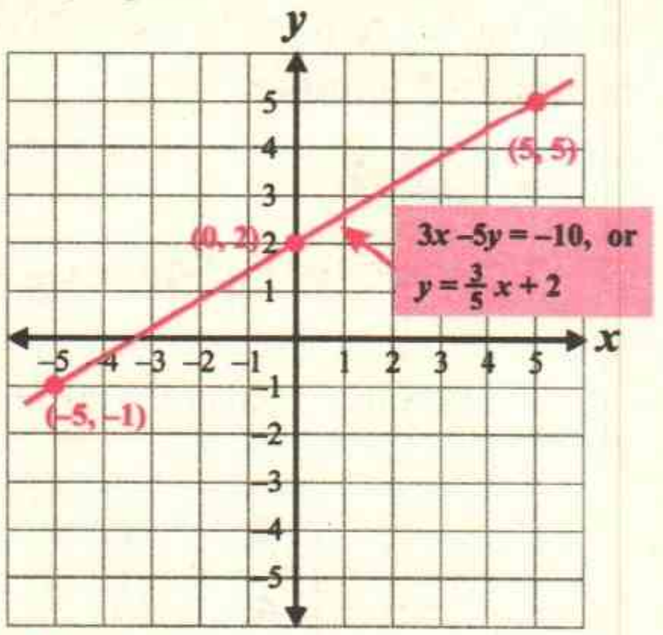
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To draw graph of a linear equation:

i) Find at least two ordered pairs (points) that are solutions to the linear equation.

ii) Plot the points that correspond to these ordered pairs in the xy-plane.

iii) Draw a straight line through those points.



i) Drawing the graph of the equation of the form $y = c$:

The graph of the equation $y = c$ is a horizontal line (i.e. the line which is parallel to x-axis) and at a distance of 'c' units from each point on x-axis. 'c' is the constant value of y which can be any real number. In any point on the line $y = c$, the y-coordinate be 'c' and the x-coordinate can be any real number.

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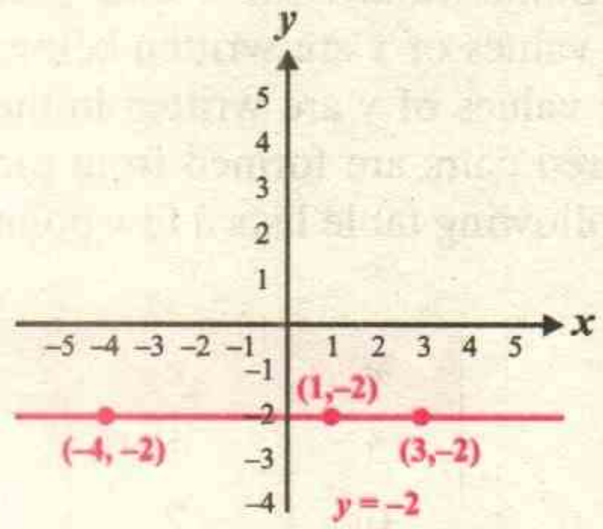
Example 11

Draw the graph of $y = -2$.

Solution:

To draw the graph of $y = -2$, we first construct a table showing a few values of x and y.

x	y
– 4	–2
–2	–2
0	–2
2	–2
3	–2



Plot the points (–4, 2), (–2, –2), (0, –2), (2, –2), (3, –2) in the xy-plane.
By joining the points we get the graph of $y = -2$.
In the graph of y = –2 the distance of the line is 2 units below the x-axis.

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j) Drawing the graph of the equation of the form $x = a$:

The graph of the equation $x = a$ is a vertical line and at a distance of 'a' units from each point on y-axis. The x-coordinate of every point on the line $x = a$ must be 'a', where 'a' is a fixed constant real number while the y-coordinate can be any real number.

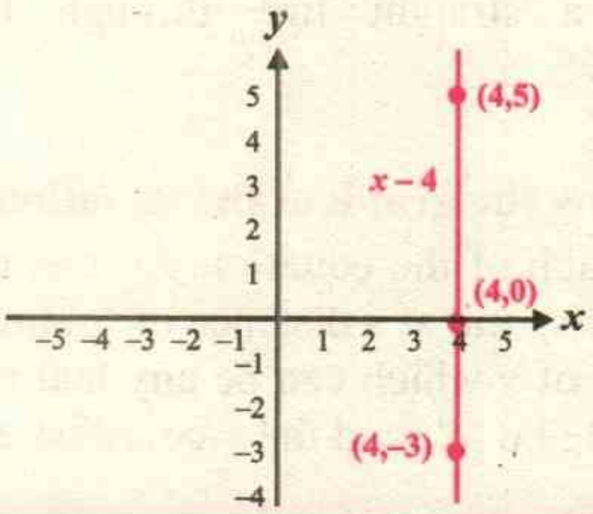
Example 12

Graph the equation $x = 4$.

Solution:

We is construct a table for some values of x and y.

x	y
4	–3
4	0
4	1
4	3
4	5



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We plot the points (4, –3), (4, 0) and (4, 3) on the xy-plane.
The line passing through the points (4, –3), (4, 0) and (4, –3) is the graph of $x = 4$.
The distance of line from y-axis is 4 units.

k) Drawing the graph of the equation of the form $y = m\,x$

In the equation y = mx, y-coordinate of every point is m times the x-coordinate where 'm' is a constant real number called the slope of the line. We assume some values of x and calculate for y, keeping m as constant number.

Example 13

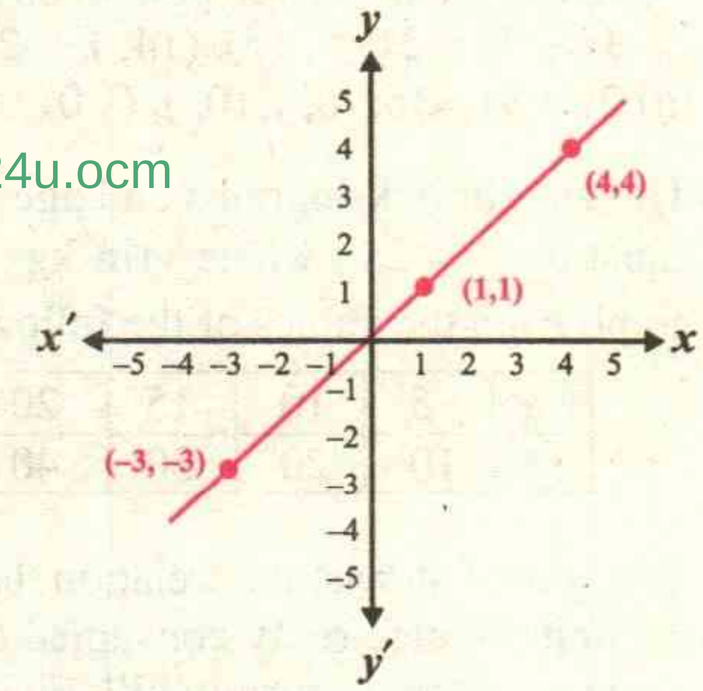
Graph the equation $y = x$.

Solution:

Here $m = 1$, let $x = -3$ then $y = -3$.
$x = 0$ then $y = 0$.
$x = 1$ then $y = 1$.
$x = 2$ then $y = 2$.
$x = 4$ then $y = 4$.

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x	y
–3	–3
0	0
1	1
2	2
4	4



Graph of line $y = x$

Now we plot the points (–3, –3), (1, 1), (4, 4) as shown in the graph. Thus the line joining the given points is the graph of $y = x$.

l) Drawing the graph of the equation of the form $y = mx + c$

In the graph of equation $y = mx + c$, 'm' and 'c' are fixed constant numbers. To draw the graph, some values of x are chosen and calculated for y by using the given equation. The solutions are listed in a table and then the points are

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Exercise 8.2

1. Determine whether or not each of the following ordered pairs are solutions to the linear equations given:

(a) (6, 1) , $x - 5y = 1$ (b) (5, –10) , $2x + y = 6$

(c) (0, 4) , $x - y = 2$ (d) (-3, 4) , $x + 3y = 2$

2. Which of the following points are on the line $3x + 2y - 6 = 0$:
(1, 1), (4,–3), (3, 0), (2, 0), (0, 2), (0, 3), (–2, 6)?

3. Construct a table for four pair of values satisfying the equation $x - y = 4$.

4. Draw the graphs of the equations

a) $y - 2x = 6$; b) $y = 1 - x$; c) $y = 2$; d) $y = x$;

5. Complete each ordered pair so that it satisfies the given equation.

i) $3x - 7y = 21$; (, 15), (14,), (–2,) ii) $5y + 6x = 30$; (–5,), (, –6), (, 4)

iii) $2y + 9x = 36$; (6,), (0,), (, 0) iv) $4x + 7y = 56$; (, 2), (, 0), (0,)

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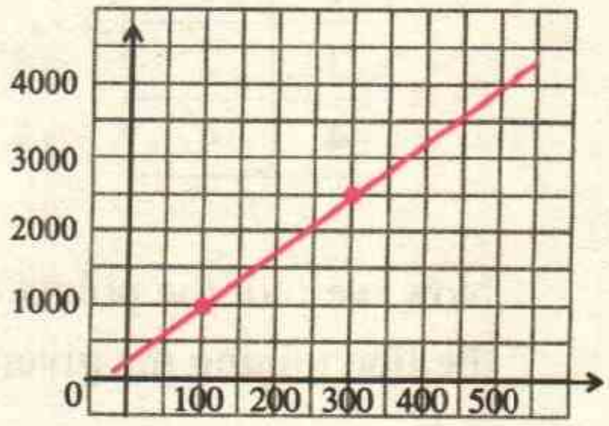
6. The weight in kilograms and age in years of a person is expressed by the equation $y = 2x$, where y (in kg) and x (in years). Draw the Age-Weight graph from the values of the following table.

x	5	10	15	20	25	30
y	10	20	30	40	50	60

7. The graph shows the relation between the units of electricity consumed and the total cost of an electricity bill. Find

(i) The cost (C) of the bill if 300 units (n) are consumed.

(ii) The number of units used when the bill is Rs.1500.



8. Draw the graph from the following table by using a suitable scale.

x	0	–1	5	7	–4
y	4	3	6	8	–5

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8.2 CONVERSION OF GRAPHS

If two quantities in a relation are either both increasing or decreasing, then the graph of the relation will be the straight line showing the levels of both quantities indicated by different coordinate axes.

8.2.1 Conversion graph of Miles into Kilometers

Let we discuss the graph of Miles-Kilometers, both are the units of distance. Let the variable 'M' denotes the distance in miles indicated along y-axis and 'K' denotes the distance in kilometers along x-axis. The relation is represented by the linear equation $M = f(K)$.

First we construct a table in which some values of Miles are converted in Km as shown below in the table.

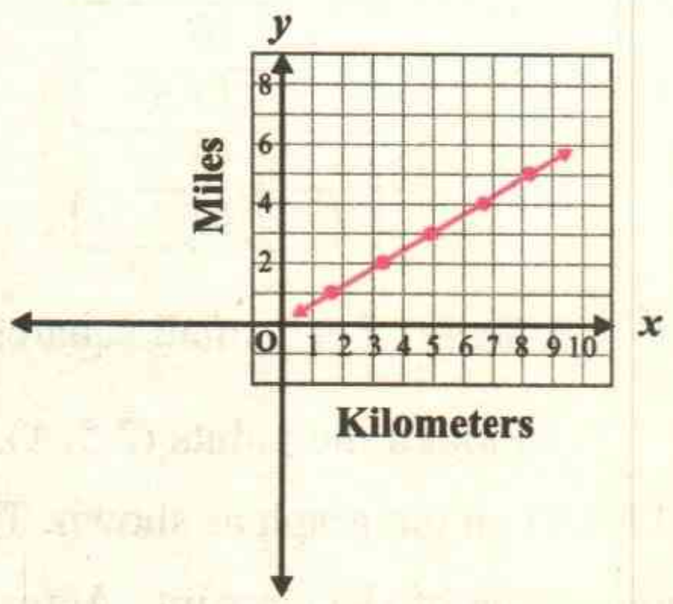
M	Km
1	1.60
2	3.20
3	4.80
4	6.40
5	8.00

1M = 1.60 Km

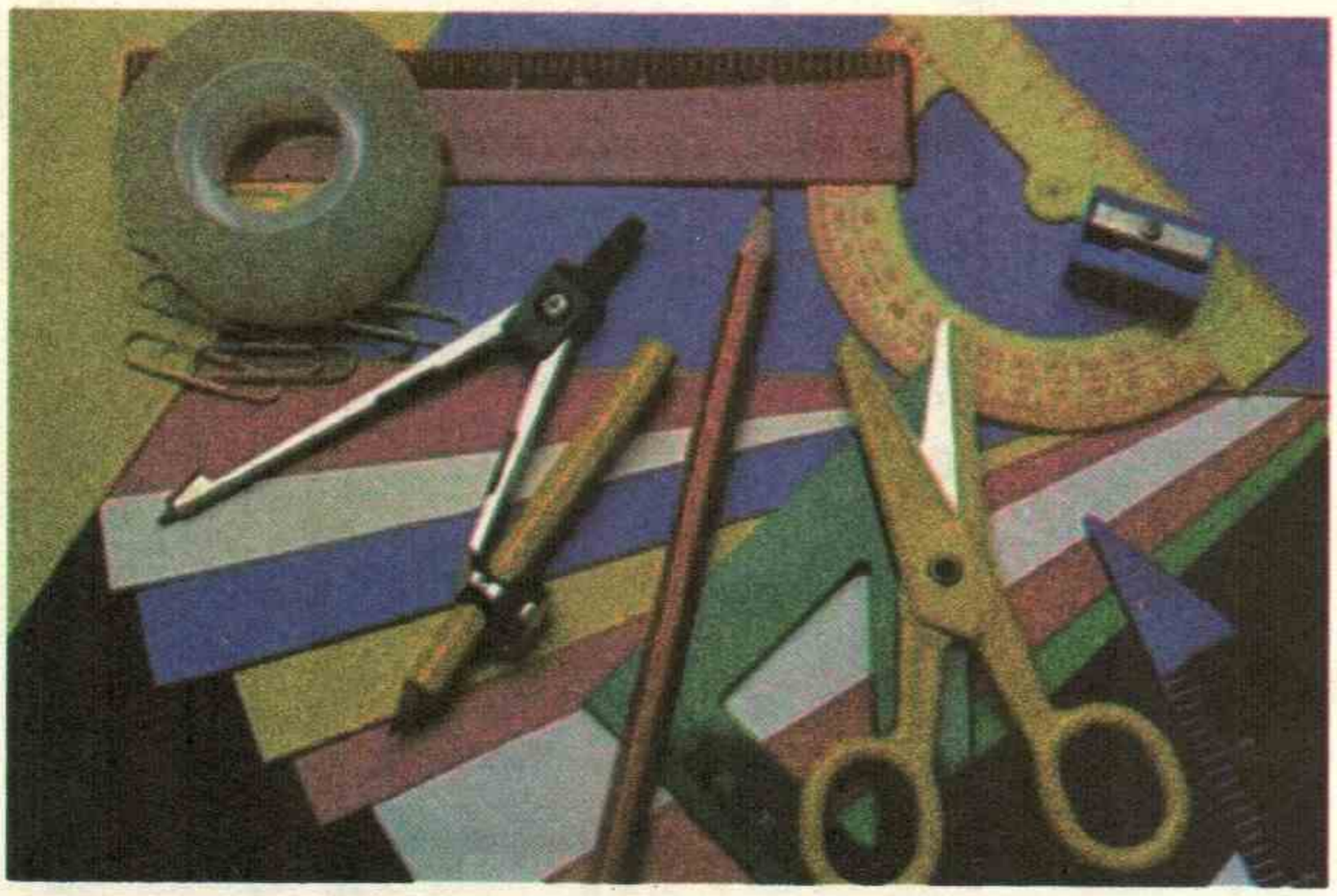
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Scale: 1 mile = 1 unit along y-axis.

1Km = 1 unit along x-axis.



Miles – Kilometers Graph



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8.2.2 Conversion of Hectares into Acres

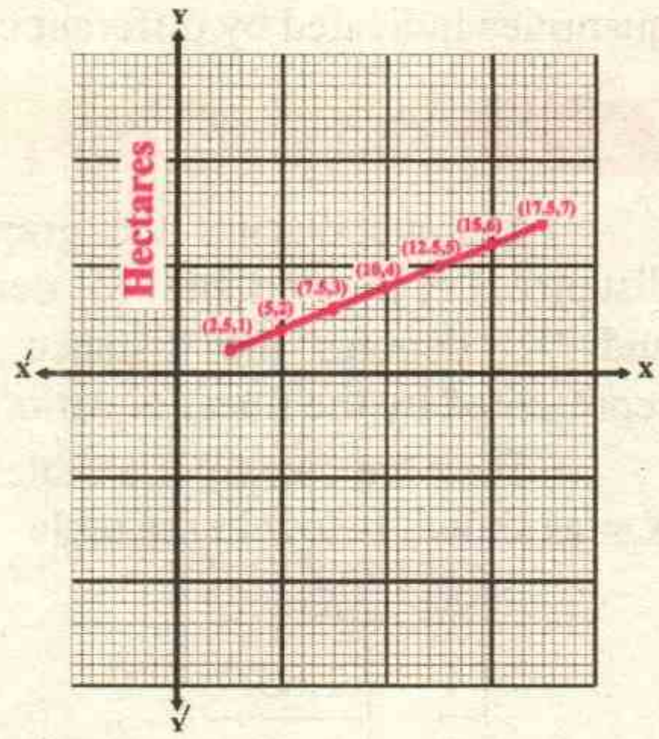
Acres-Hectares Graph:

Let 'H' denotes Hectare and 'A' denotes Acre.

$$1\ H = 2.5A$$

First we construct a table showing the converted values of Hectare into Acre.

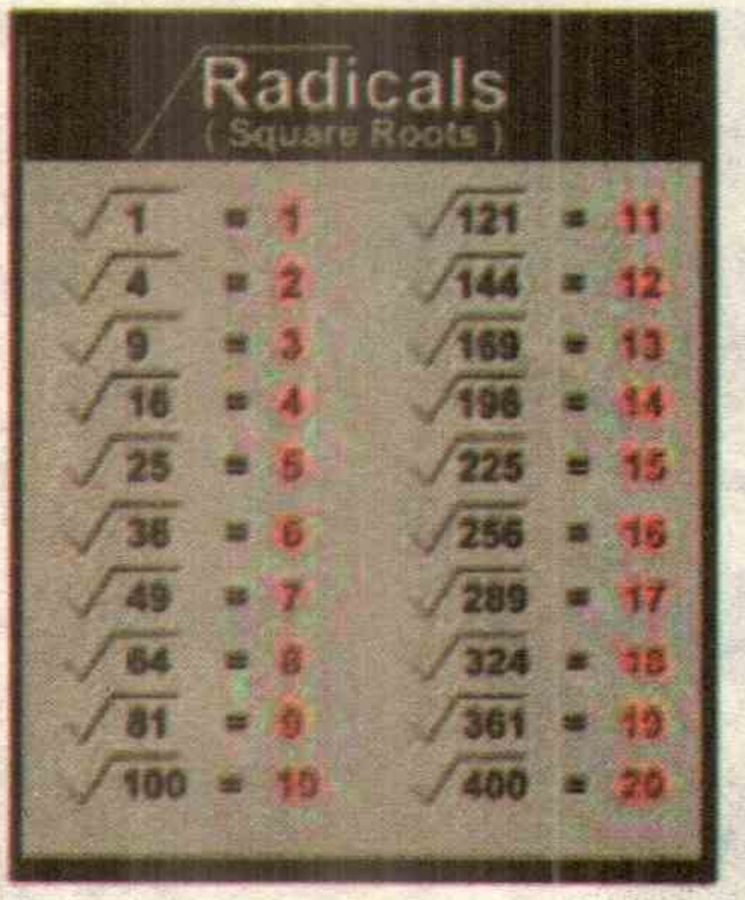
H	A
1	2.5
2	5
3	7.5
4	10
5	12.5
6	15
7	17.5



Scale : Let 2 small squares = 1 unit.

Plotted the points (2.5, 1), (5, 2), (7.5, 3), (10, 4), (12.5, 5), (15, 6), (17.5,7) on the graph as shown. The line joined all the points is the graph of conversion of Hectares into Acres.

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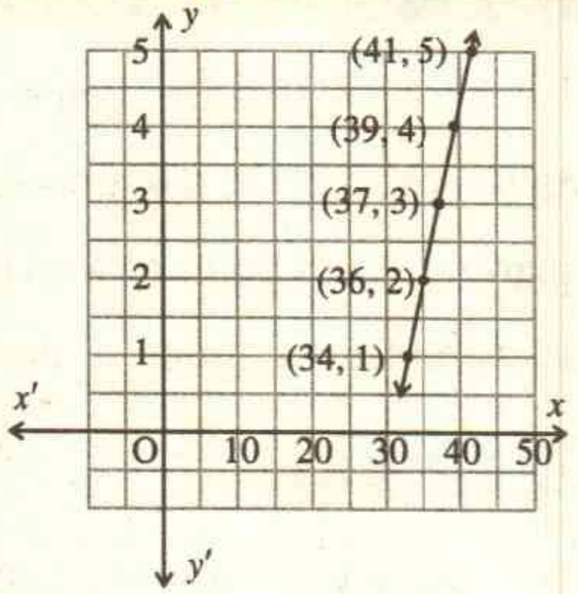
8.2.3 Conversion graph of Degrees Celsius into Degrees Fahrenheit

Let 'C' represents Celsius and 'F' represents Fahrenheit temperature. To draw the graph, a table is constructed showing the converted values of C° into F° temperatures.

For conversion the formula used as;

$$F° = \frac{9}{5}C° + 32$$

The values of F° have been rounded off.



Scale: 1C° = 1unit along *y*-axis.
10F° = 1unit along *x*-axis.

Plotted the points in the graph and joined then by the straight line which is the required graph.

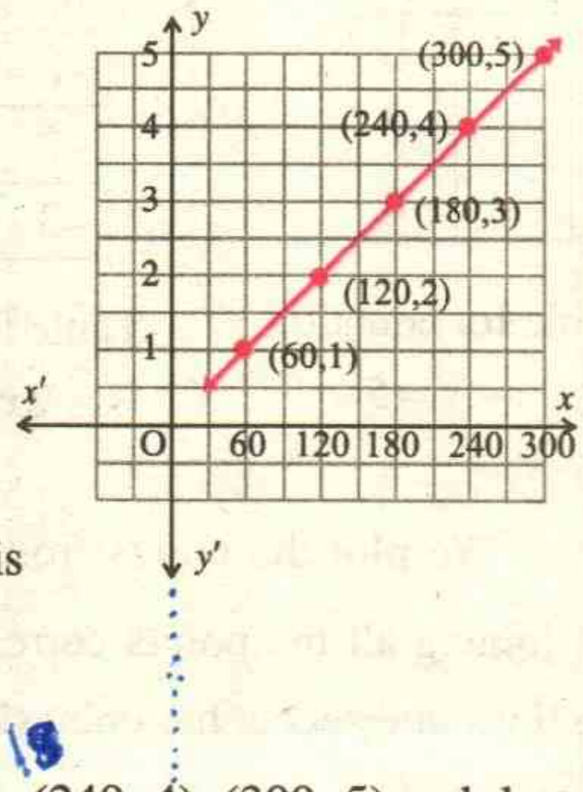
C°	F°
1	33.8 = 34
2	35.6 = 36
3	37.4 = 37
4	39.2 = 39
5	41

8.2.4 Conversion Graph of Pakistani Currency & another Currency

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Let us draw the conversion graph of Pakistani currency and US Dollars. If the exchange rate is US $1 = Rs.60, then draw the graph by using the converted values given in the table.

US $	PKR
1	60
2	120
3	180
4	240
5	300



In the graph the US $ is shown along *y*-axis and PKR (Pakistani Rupees) along *x*-axis.

Scale: Let US 1$ = 2 unit along *y*-axis
and Rs.60 = 2 unit along *x*-axis.

Plotted the points (60, 1), (120, 2), (180, 3), (240, 4), (300, 5) and draw a line passing through the points, which is the required graph.

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8.3 GRAPHIC SOLUTION OF EQUATION IN TWO VARIABLE

Recall that the graph of a linear equation is a line that contains all the ordered pairs in the solution set of the equation. When we graph a system of linear equations, each point at which the graphs intersect is a solution of both equations and therefore a solution of the system of equations.

Example 14

Solve the following system of equations graphically,

$x - y = 5$, $2x + y = 1$.

Solution:

Tables are constructed showing the values of x and y satisfying both equations.

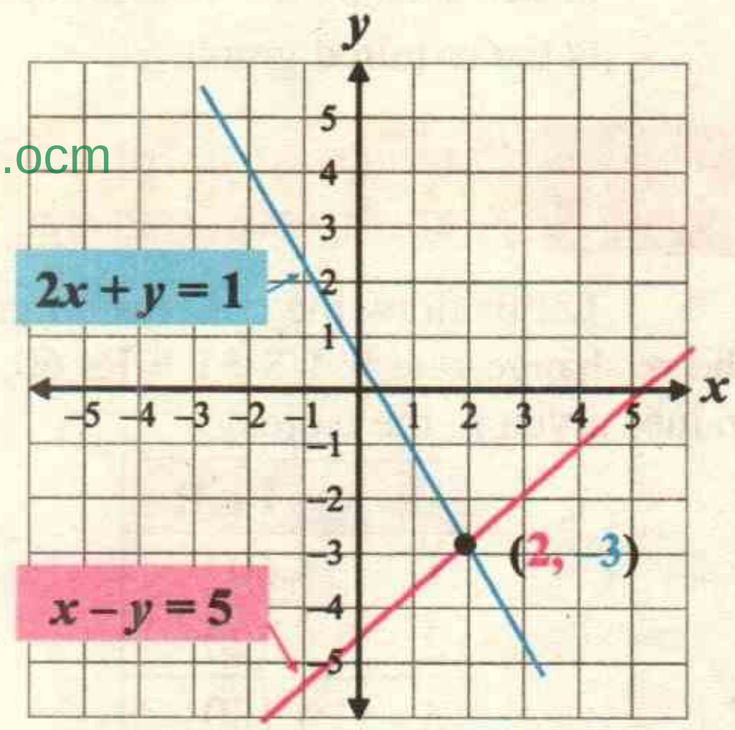
x	y
2	–3
1	–4
0	–5
-1	–6
–2	–7
–3	–8

Table for equation $x - y = 5$

x	y
2	–3
1	–1
0	1
-1	3
–2	5
–3	–7

Table for equation $2x + y = 1$

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We plot the points from both tables on the graph and then draw straight line joining all the points corresponding to the given equations. In the graph the two lines intersect at the point (2, –3), which is the solution of the two equations.

So the solution set $= \{(2, -3)\}$.

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Exercise 8.3

1. Using the conversion formula 1 mile = 1.6 km. Draw the conversion graph of Miles-Kilometres if distance in miles are given as;

   1M, 3M, 4M, 5M (M is used for mile).

2. Draw the Miles-Kilometres graph if distance in kilometres are given as 1km, 2km, 3km, 4km and 5km.

3. Given that I Hectare = 2.5 Acres (approximately), draw a conversion graph of Hectare-Acre from the given values of Hectares 2H, 4H, 8H.

4. Convert the following temperatures given in Celsius Degrees into Fahrenheit and then draw its graph.

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   Degree Celsius = 0C°, 2C°, 3C°, Conversion formula is $F° = \frac{9}{5}C° + 32$.

5. Draw the following conversion graphs.

   i) PKR (Pakistani Rupees) — US $ (1$, 3$, 5$)

   ii) PKR (Pakistani Rupees) — GB£ (1£, 2£, 3£)

   $ is used for US Dollar and £ is for Pound sterling.

   Where 1$ = Rs.60 (approx) , IGB£ = Rs.100 (approx)

6. Solve the following simultaneous equations by using graphical method.

   (i) $2x + y = 3$, $x - y = 0$ (ii) $y = 2x + 2$, $y = x - 1$

   (iii) $x + 4y = 5$, $2x + 3y = 0$ (iv) $3x + 5y = 2$, $3x + 5y = 8$

   (v) $3x - 2y = 13$, $2x + 3y = 13$

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REVIEW EXERCISE 8

1. Select the correct answer.

i) The point (5, –2) is located in
a) Quadrant I b) Quadrant II
c) Quadrant III d) Quadrant IV

ii) The two coordinate axes intersect at an angle of
a) 30° b) 60° c) 90° d) 45°

iii) The point (–3, 8) is located in
a) Quadrant I b) Quadrant II
c) Quadrant III d) Quadrant IV

iv) The lines represented by the equations $x+y=1$ and $x+y=4$ are
a) Parallel b) Inclined c) Intersecting d) Perpendicular

v) The point (–6, –6) is located in
a) Quadrant I b) Quadrant II
c) Quadrant III d) Quadrant IV

vi) The line $x=a$ where a is a real number is parallel to
a) y-axis b) x-axis
c) Both x-axis and y-axis d) Neither x-axis nor y-axis

vii) The point (2, 11) is located in
a) Quadrant I b) Quadrant II
c) Quadrant III d) Quadrant IV

viii) The solution set of the lines $y = 2$ and $y = 3$ is
a) {(5, 3)} b) {(4, 0)} c) {(0, 0)} d) { }

ix) The line $y = 5$ is parallel to
a) y-axis b) x-axis
c) Both x-axis and y-axis d) Neither x-axis nor y-axis

x) Solution of the simultaneous equations $y = x+1, y = 2x-2$
a) $x=2, y=4$ b) $x=3, y=4$
c) $x=4, y=3$ d) $x=5, y=6$

2. Determine the x-coordinate and y-coordinate of the following points. Also mention the quadrant in which each point lies.

i) (2, 3) ii) (– 4, 5) iii) (4, 0)

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3. Draw a triangle ABC by joining the points A (–2, 3), B(4, 2), C(1, –3).

4. Draw a parallelogram by joining the points O(0,0), A(1,4), B(4,2) and C(3,–2).

5. By joining the points A(1,7), B(4,2), C(-1,-1) D(-4, 4) draw a square.

6. Draw a rectangle by joining the points, A(–6,5), B(5,–5), C(5,8) and D(–6, –8).

7. Draw the graph of the equation $x + y = 4$.

8. Draw the graph of $y = 3$.

9. Draw a graph from the following table by using a suitable scale.

x	–3	4	3	–4	3	0	–3	0
y	5	3	4	–2	–4	4	0	0

Think

10. Solve the following system of equations graphically,
$x + y = 2$, $x - y = 4$.

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SUMMARY

- Cartesian plane can be defined as
$$R \times R = \{(a, b) \mid a, b \in R\}$$
- The Cartesian plane is also called xy-plane.
- x-axis and y-axis divides the Cartesian plane, into four quadrants named I, II, III and IV.
- The set of all solutions of an equation is its solution set.
- In quadrant I, both x-coordinate and y-coordinate are positive.
- In quadrant II, x-coordinate is negative and y-coordinate is positive.
- In quadrant III, x-coordinate is negative and y-coordinate is also negative.
- In quadrant IV, x-coordinate is positive and y-coordinate is negative.
- To locate a point in the Cartesian plane, we must know its x-coordinate, which is the horizontal distance from the y-axis and the y-coordinate which is the vertical distance from the x-axis.

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UNIT 09

INTRODUCTION TO COORDINATE GEOMETRY

After reading this unit, the students will be able to:

- Define coordinate geometry.
- Derive distance formula to calculate distance between two points given in Cartesian plane.
- Use distance formula to find distance between two given points.
- Define collinear points. Distinguish between collinear and non-collinear points.
- Use distance formula to show that given three (or more) points are collinear.
- Use distance formula to show that the given three non-collinear points form:
  - an equilateral triangle,
  - an isosceles triangle,
  - a right angled triangle,
  - a scalene triangle.
- Use distance formula to show that given four non-collinear points form:
  - a square,
  - a rectangle,
  - a parallelogram.
- Recognize the formula to find the midpoint of the line joining two given points.
- Apply distance and midpoint formulae to solve/verify different standard results related to geometry.

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Introduction

In the year 1637 the French philosopher and mathematician Rene Descartes, (1596-1650) established a landmark in the field of mathematics when his book *La Geometrie* was published. In this book he showed the connection between algebra and geometry. By establishing this relationship, he was able to study geometric figures by examining various equations which represent the figures.

9.1 COORDINATE GEOMETRY

The study of geometric properties of figures by the study of their equations is called *coordinate* or *analytic* geometry.

We know that a plane is divided into four quadrants by two perpendicular lines called the axes intersecting at origin. We have also seen that there is a one to one correspondence between the points of the plane and the ordered pairs in

9.1.1 Distance formula between two points in a Cartesian plane

To find the distance between two points in a plane, we first see how to determine distance between two points on the real line.

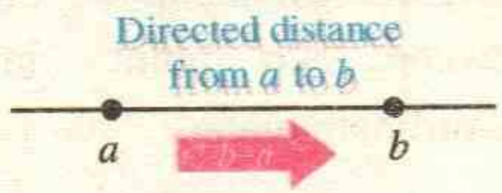
Suppose we are given two distinct points a & b on the real line then

Remember

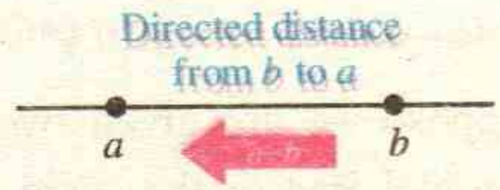
The students must remember that each ordered pair corresponds to exactly one point in the plane and each point in a plane corresponds to exactly one ordered pair.

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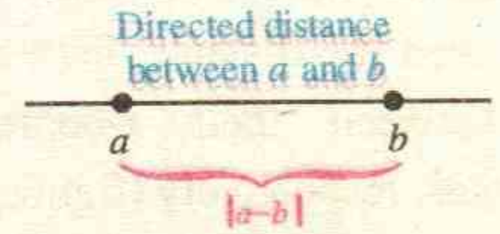
i. the directed distance from a to b is $b-a$



ii. the directed distance from b to a is $a-b$.



iii. the distance between a and b is $|a-b|$ *or* $|b-a|$.



In the given figure, we note that b is to the right of a, the directed distance from a to b (moving) to the right is positive. Moreover, a is to the left of b, the directed distance from b to a (moving to the left) is negative. The distance between two points on the real line can never be negative.

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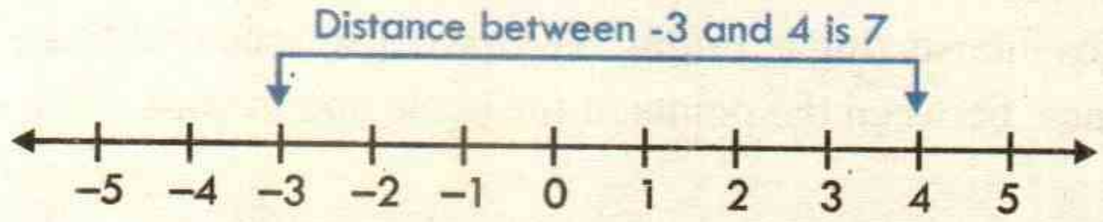
a) Distance between two points on the real line

The distance d between points x_1 & x_2 on the real line is given by $d = |x_2 - x_1| = \sqrt{(x_2 - x_1)^2}$. It is important to note that the order of subtraction with x_1 & x_2 does not matter in finding the distance between them since $|x_2 - x_1| = |x_1 - x_2|$ and $(x_2 - x_1)^2 = (x_1 - x_2)^2$.

Example 1 Finding distance on the real line.

Determine the distance between -3 and 4 on the real line. What is the directed distance from -3 to 4 and from 4 to -3?

Solution:



The distance between -3 and 4 is given by $|4-(-3)| = |4+3| = |7| = 7$.

or $|-3-4| = |-7| = 7$

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The directed distance from -3 to 4 is $4-(-3) = 4+3 = 7$.

The directed distance from 4 to -3 is $-3-4 = -7$.

Now, we establish the distance formula between two points in a Cartesian Co-ordinate plane.

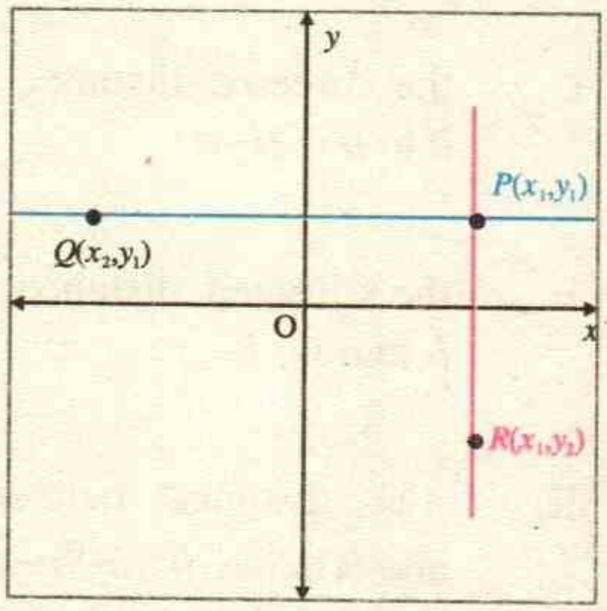


Figure . 1

b) Distance between two points in a plane

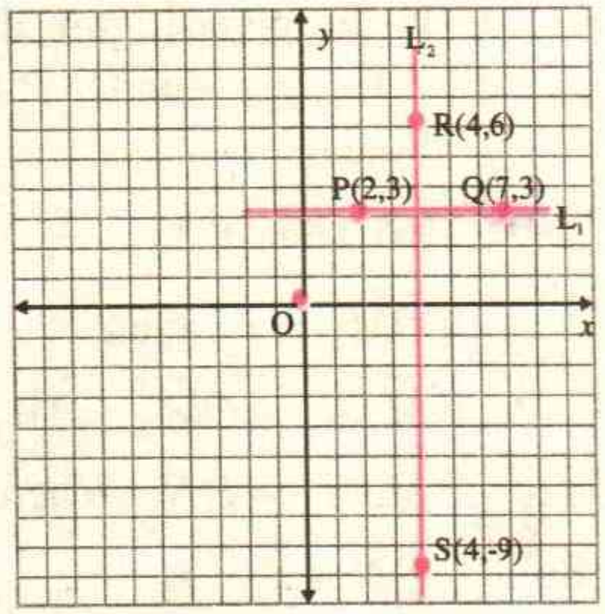
Suppose we are given two points on the same horizontal line or the same vertical line in the plane, then the distance between them is the distance between their x-coordinates or y-coordinates, respectively (figure 1).

Thus the distance between $P(x_1 y_1)$ and $Q(x_2, y_1)$ is $|x_2 - x_1|$ and the distance between $P(x_1 y_1)$ and $R(x_1 y_2)$ is $|y_2 - y_1|$.

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Example 2

Find the distance between $(2,3)$ & $(7,3)$ and the distance between $(4,6)$ & $(4,-9)$.



Solution:

In the figure, we see that $P(2,3)$ & $Q(7,3)$ lie on the same horizontal line so that distance between $P(2,3)$ & $Q(7,3)$ is $|7-2|=|5|=5$.

Also $R(4,6)$ and $S(4,-9)$ lie on the same vertical line L_2 so the distance between these points is $|6-(-9)|=|6+9|=|15|=15$.

9.1.2 Derivation of Distance Formula

In a coordiante plane, the distance, d between two points (x_1, y_1) and (x_2, y_2) is given by the following formula:

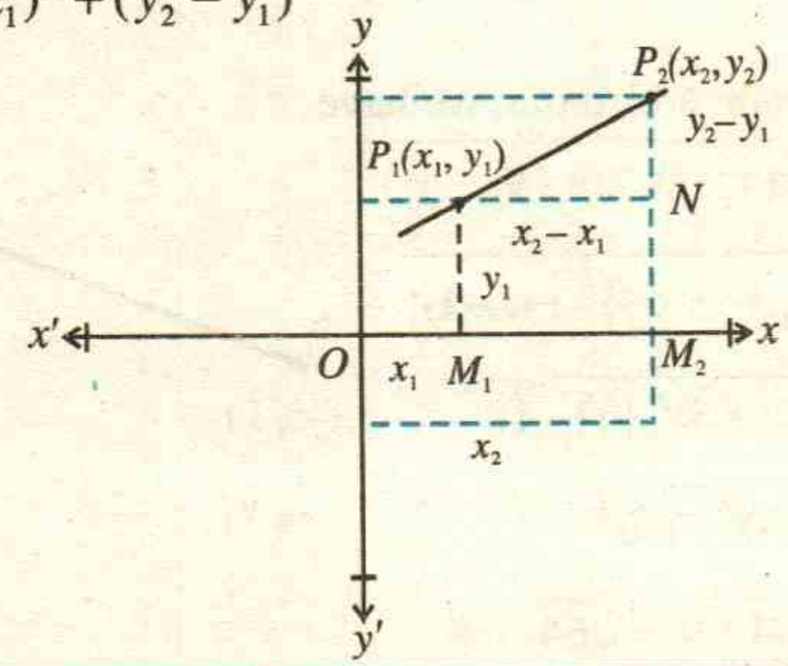
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$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

9.1.3 Derivation of Distance Formula

The distance $|P_1P_2|$ between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is given by

$$|P_1P_2|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$



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Proof:

In the given figure the two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are shown.

Let $\overline{OM_1} = x_1$

$\overline{OM_2} = x_2$, then

$\overline{M_1M_2} = \overline{OM_2} - \overline{OM_1} = x_2 - x_1$

but $\overline{M_1M_2} = \overline{P_1N}$ (Where $\overline{P_1N}$ is perpendicular to Y-axis)

So $\overline{P_1N} = x_2 - x_1$

Let $\overline{M_1P_1} = y_1$ and $\overline{M_2P_2} = y_2$ where both $\overline{M_1P_1}$ and $\overline{M_2P_2}$ are perpendiculars on X-axis.

then $\overline{NP_2} = \overline{M_2P_2} - \overline{M_2N}$

$= \overline{M_2P_2} - \overline{M_1P_1}$ as $(\overline{M_2N} = \overline{M_1P_1})$

$= y_2 - y_1$

Since P_1NP_2 is a right angled triangle, so by Pythagoras theorem

$|P_1P_2|^2 = |P_1N|^2 + |NP_2|^2$

$= (x_2 - x_1)^2 + (y_2 - y_1)^2$

Or $|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

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Example 3

Find the distance between points $A(-5,1)$ and $B(3,1)$.

Solution:

Using distance formula, we have

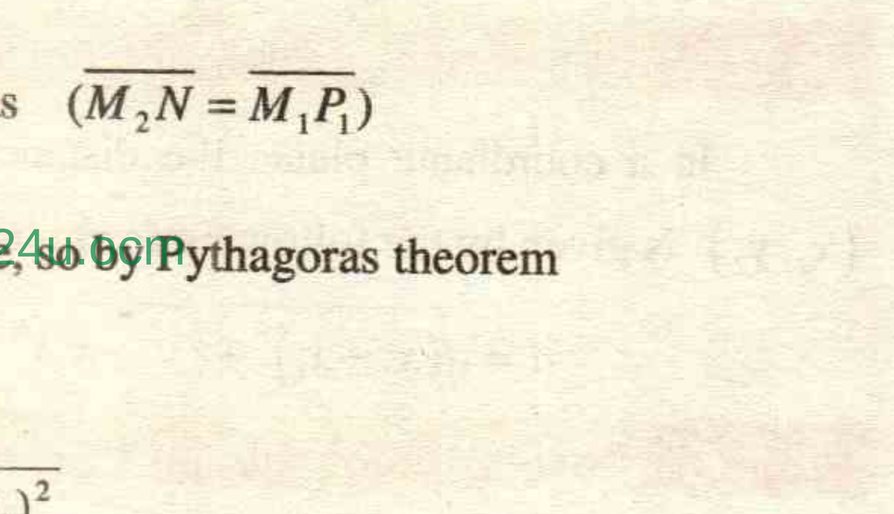
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$d = \sqrt{\{3-(-5)\}^2 + (1-1)^2}$$

$$= \sqrt{(3+5)^2 + (1-1)^2}$$

$$= \sqrt{(8)^2 + (0)^2}$$

$$= \sqrt{64+0} = \sqrt{64} = 8$$



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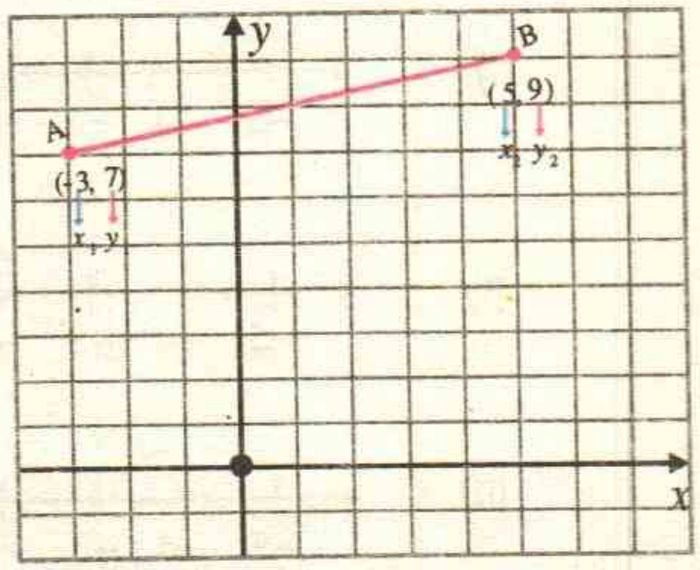
Example 4

Plot the points $(-3,7)$ & $(5,9)$ and find the distance between them.

Solution:

Let $A=(-3,7)$ & $B=(5,9)$ and suppose d is the distance between them using the formula, then

$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$, we have



$$d=\sqrt{(-5+3)^2+(9-7)^2}$$

$$=\sqrt{(-8)^2+(2)^2}$$

$$=\sqrt{64+4}=\sqrt{68}$$

$$=\sqrt{2\times2\times17}$$

$$=2\sqrt{17}$$

Many real-world situations can be modeled, or described mathematically, using equations in which two variables appear.

Example 5

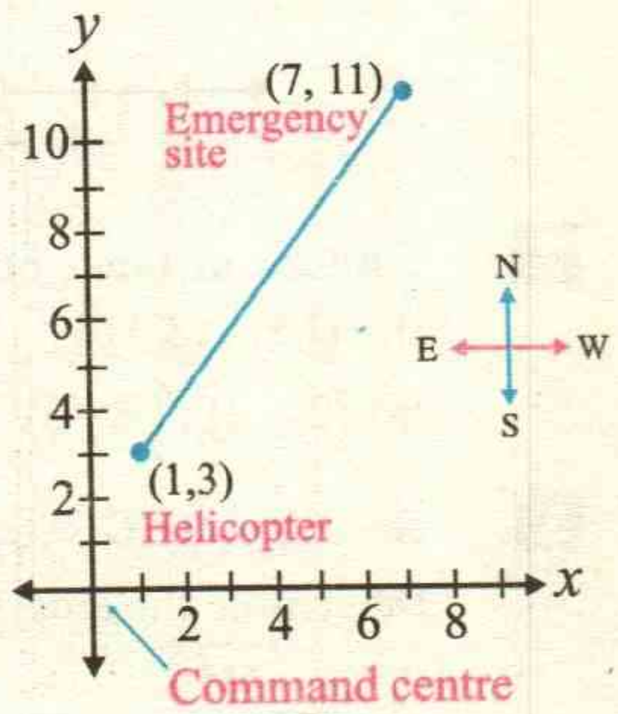
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A helicopter pilot located 1 mile west and 3 miles north of the command centre must respond to an emergency located 7 miles west and 11 miles north of the centre. How far must the helicopter travel to get to the emergency site?

Solution:

Let (0. 0) be the coordinates of the command centre and taking positive part of y-axis representing north. Then the helicopter's current coordinates are (1, 3).

The coordinates of the emergency site are (7, 11). Using the distance formula $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ we have



$$d=\sqrt{(7-1)^2+(11-3)^2}$$

$$d=\sqrt{6^2+8^2}=\sqrt{36+64}=\sqrt{100}=10$$

The helicopter must travel 10 miles to get to the emergency site.

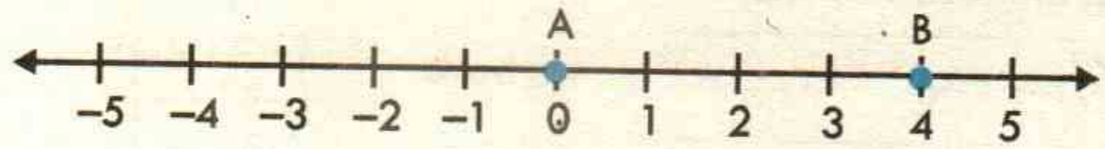
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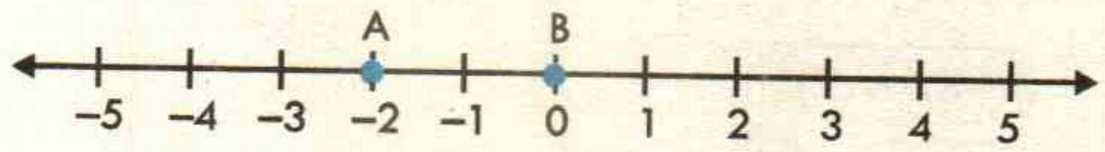
Exercise 9.1

1. **Find the length of AB in the following figures.**

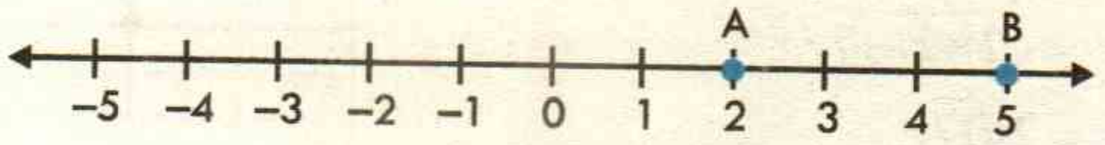
i)



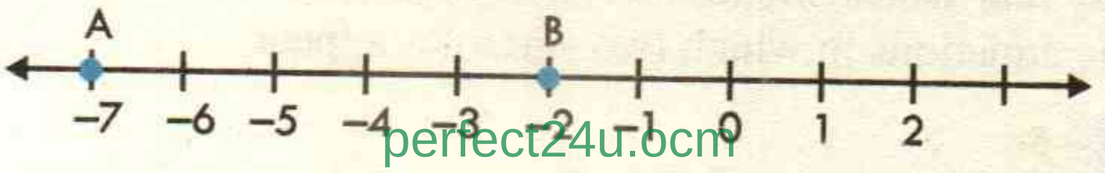
ii)



iii)

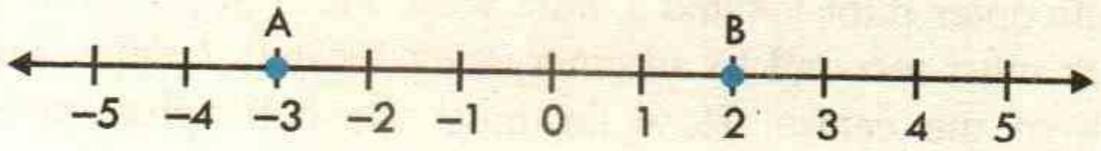


iv)

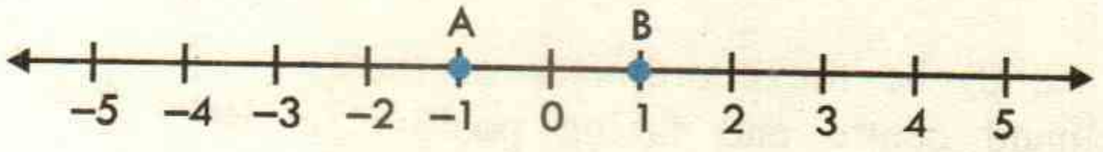


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v)



vi)



2. **Find the distance between each pair of points.**

i) $(1,1)$, $(3,3)$ ii) $(1,2)$, $(4,5)$

iii) $(2,-2)$, $(2,-3)$ iv) $(3,-5)$, $(5,-7)$

3. **Given points $O(0,0), A(3,4), B(-5,12), C(15,-8), D(11,-3), E(-9,-4)$.**
Determine the lengths of the following segments:

i) $\overline{OA}$ ii) $\overline{OB}$

iii) $\overline{OC}$ iv) $\overline{AD}$

v) $\overline{AB}$ vi) $\overline{AC}$

vii) $\overline{BE}$

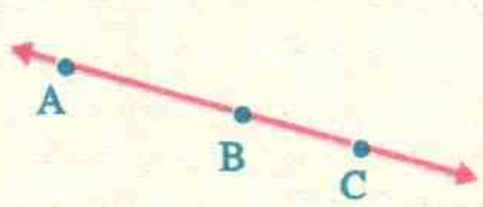
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9.2 COLLINEAR POINTS

9.2.1 Collinear and Non-collinear Points

Three or more points P_1, P_2, P_3 ..., are said to be collinear if they lie on the same line. If the points are not collinear then they are called non-collinear. That is non-collinear points do not lie on a line.

In the figure, points A,B,C are collinear while D,E,F are non-collinear.



Collinear Points



Non-collinear Points

Example 6

Prove that the points $A(5,-2), B(1,2)$ and $C(-2,5)$ are collinear.

Solution:

Given $A=(5,-2)$, $B=(1,2)$ & $C=(-2,5)$

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Using distance formula, we can write

$$|AB| = \sqrt{(1-5)^2+(2-(-2))^2}$$

$$= \sqrt{(-4)^2+(2+2)^2} = \sqrt{16+16} = \sqrt{32} = \sqrt{16\times 2} = 4\sqrt{2}$$

$$|BC| = \sqrt{(x_3-x_2)^2+(y_3-y_2)^2} = \sqrt{(-2-1)^2+(5-2)^2}$$

$$= \sqrt{(-3)^2+(3)^2} = \sqrt{9+9} = \sqrt{9\times 2} = 3\sqrt{2}$$

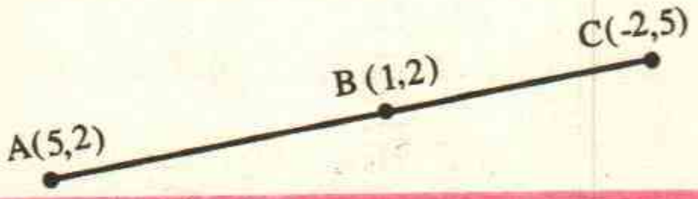
$$|AC| = \sqrt{(-2-5)^2+(5-(-2))^2} = \sqrt{(-7)^2+(7)^2} = \sqrt{49+49} = \sqrt{49\times 2} = 7\sqrt{2}$$

Since $4\sqrt{2}+3\sqrt{2}=7\sqrt{2}$ is obviously true

$\therefore \quad |AB|+|BC|=|AC|$

So, the three given points $A=(5,-2)$, $B=(1,2)$ & $C=(-2,5)$ are collinear.

Geometrically, we can arrange these points on a line as shown.



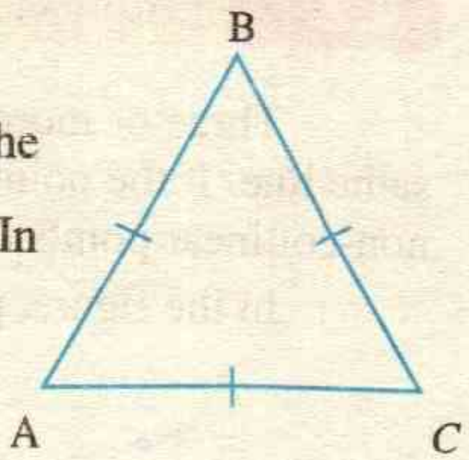
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9.2.2 Use of distance formula to show that the given three non-collinear points form

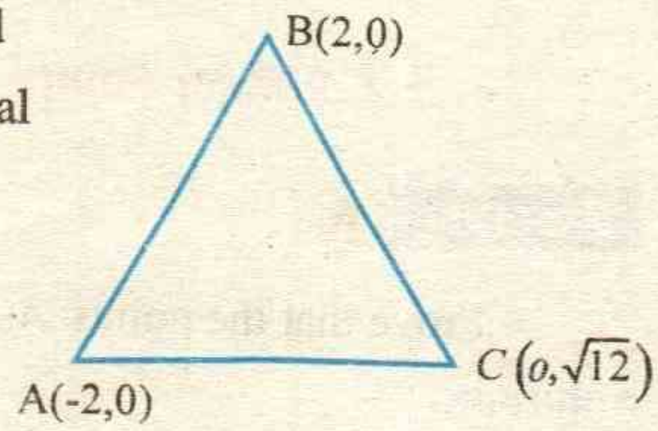
a) Equilateral Triangle

A triangle in which all the three sides and all the three angles are equal is called an equilateral triangle. In equilateral triangle measure of each angle is 60°.



Example 7

Prove that the points $A(-2,0), B(2,0)$ and $C(0, \sqrt{12})$ are the vertices of an equilateral triangle.



Solution:

By distance formula, first we find the lengths of all the sides.

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$$|AB| = \sqrt{(2+2)^2 + (0-0)^2} = \sqrt{(4)^2} = \sqrt{16} = 4$$

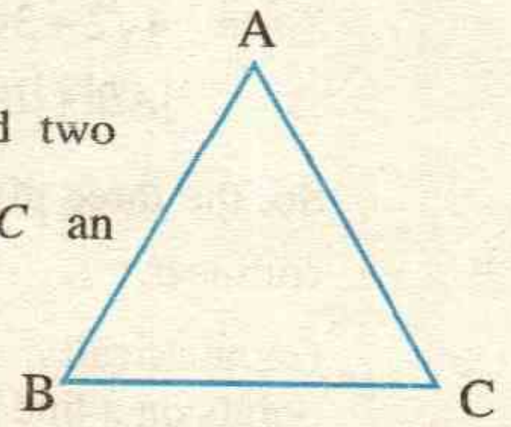
$$|BC| = \sqrt{(0-2)^2 + (\sqrt{12}-0)^2} = \sqrt{4+12} = \sqrt{16} = 4$$

$$|AC| = \sqrt{(0+2)^2 + (\sqrt{12}-0)^2} = \sqrt{4+12} = \sqrt{16} = 4$$

Since $|AB| = |BC| = |AC| = 4$, thus the points A, B and C are the vertices of an equilateral triangle.

b) Isosceles Triangle

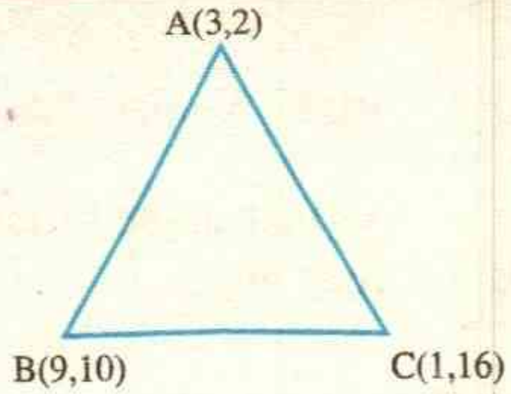
An isosceles triangle has two equal sides and two equal angles, opposite to the equal sides. Thus ABC an isosceles triangle if $|AB| = |AC|$ & $m\angle B = m\angle C$.



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Example 8

Show that the points $A(3,2), B(9,10)$ and $C(1,16)$ are the vertices of an isosceles triangle.



Solution:

Let $|AB|, |BC|$ and $|AC|$ denote the lengths of the side of the triangle. To determine $|AB|, |BC|$ and $|AC|$, distance formula is used.

$$|AB| = \sqrt{(9-3)^2 + (10-2)^2} = \sqrt{36+64} = \sqrt{100} = 10$$

$$|BC| = \sqrt{(1-9)^2 + (16-10)^2} = \sqrt{64+36} = \sqrt{100} = 10$$

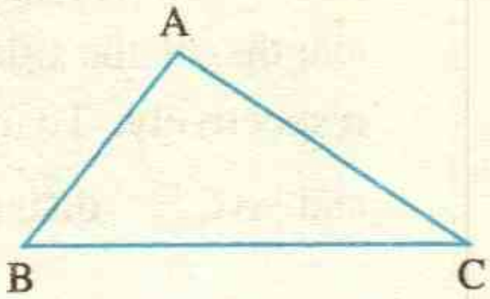
$$|AC| = \sqrt{(1-3)^2 + (16-2)^2} = \sqrt{4+196} = \sqrt{200} = 10\sqrt{2}$$

Since $|AB| = |BC| = 10$, thus the two sides AB and AC have equal length so A, B and C are the vertices of an isosceles triangle.

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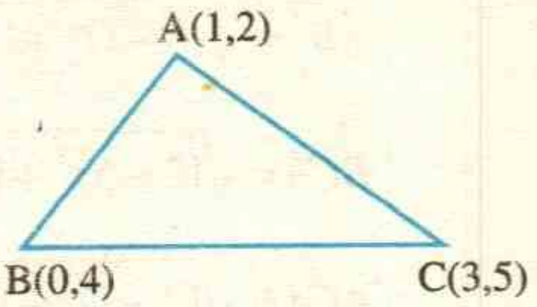
c) Scalene Triangle

If all the three sides of a triangle are different in lengths, it is called scalene triangle.



Example 9

Show that the points $A(1,2)$, $B(0,4)$ and $C(3,5)$ are the vertices of an scalene triangle.



Solution:

By distance formula, we first find the length of each side.

$$|AB| = \sqrt{(0-1)^2 + (4-2)^2} = \sqrt{(-1)^2 + (2)^2} = \sqrt{1+4} = \sqrt{5}$$

$$|BC| = \sqrt{(3-0)^2 + (5-4)^2} = \sqrt{(3)^2 + (1)^2} = \sqrt{9+1} = \sqrt{10}$$

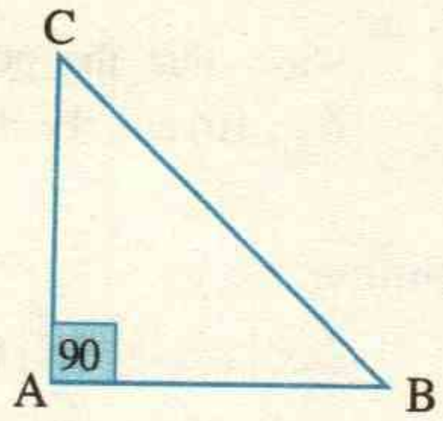
$$|AC| = \sqrt{(3-1)^2 + (5-2)^2} = \sqrt{(2)^2 + (3)^2} = \sqrt{4+9} = \sqrt{13}$$

Since $|AB| \neq |BC| \neq |AC|$ i.e. the lengths of all the three sides are different, Therefore A, B and C are the vertices of a scalene triangle.

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d) Right Angled Triangle

A right angled triangle has one angle equal to $90°$ i.e. right angle.



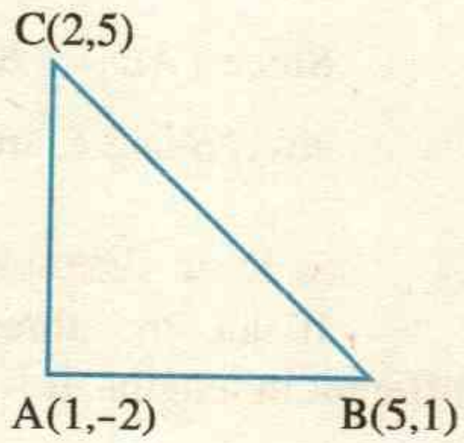
By Pythagoras theorem.

A triangle ABC is a right-angled triangle if $(Base)^2 + (Perp)^2 = (Hyp)^2$

Example 10

Construct the triangle ABC with the help of the points $A(1,-2)$, $B(5,1)$ and $C(2,5)$, and prove that the triangle is a right-angled triangle.

Solution:



Let $|AB|$, $|BC|$ and $|AC|$ denote the lengths of the sides AB, BC and AC respectively. To determine $|AB|$, $|BC|$ and $|AC|$, distance formula is used.

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$$|AB| = \sqrt{(5-1)^2 + (1+2)^2} = \sqrt{(4)^2 + (3)^2} = \sqrt{16+9} = \sqrt{25} = 5$$

$$|BC| = \sqrt{(2-5)^2 + (5-1)^2} = \sqrt{(-3)^2 + (4)^2} = \sqrt{9+16} = \sqrt{25} = 5$$

$$|AC| = \sqrt{(2-1)^2 + (5+2)^2} = \sqrt{(1)^2 + (7)^2} = \sqrt{1+49} = \sqrt{50} = 5\sqrt{2}$$

since $(5\sqrt{2})^2 = (5)^2 + (5)^2$

become $50 = 25 + 25 = 50$.

thus $|AB|^2 + |BC|^2 = |AC|^2$,

That is, we have $(Base)^2 + (Perp)^2 = (Hyp)^2$ which is the Pythagorean theorem.

Hence the triangle ABC is a right-angled triangle.

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9.2.3 Use Of Distance Formula To Show That The Given Four Non-Collinear Points Form

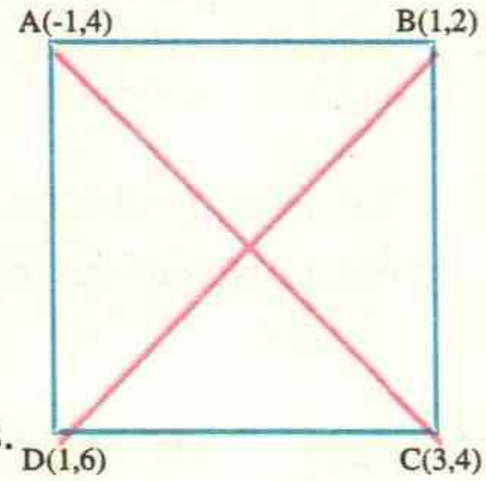
a) Square

Example 11

By means of distance formula, show that the points;

i) $A(-1, 4), B(1, 2), C(3, 4)$ and $D(1, 6)$ form a square.

ii) Verify that the diagonals have equal lengths.



Solution:

i) Let $|AB|, |BC|, |CD|$ and $|AD|$ denote the lengths of the sides of the square, then

$$|AB| = \sqrt{(1+1)^2 + (2-4)^2} = \sqrt{(2)^2 + (-2)^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$$

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$$|BC| = \sqrt{(3-1)^2 + (4-2)^2} = \sqrt{(2)^2 + (2)^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$$

$$|CD| = \sqrt{(1-3)^2 + (6-4)^2} = \sqrt{(-2)^2 + (2)^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$$

$$|AD| = \sqrt{(1+1)^2 + (6-4)^2} = \sqrt{(2)^2 + (2)^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$$

Thus $|AB| = |BC| = |CD| = |AD|$.

ii) Now AC and BD are two diagonals of the square $ABCD$.

$$|AC| = \sqrt{(3+1)^2 + (4-4)^2} = \sqrt{(4)^2 + 0} = \sqrt{16} = 4$$

$$|BD| = \sqrt{(1-1)^2 + (6-2)^2} = \sqrt{0 + (4)^2} = \sqrt{16} = 4$$

$\therefore$ diagonal $|AC|$ = diagonal $|BD| = 4$.

Thus the diagonals have equal lengths.

$\therefore$ The given points A, B, C & D form the vertices of a square.

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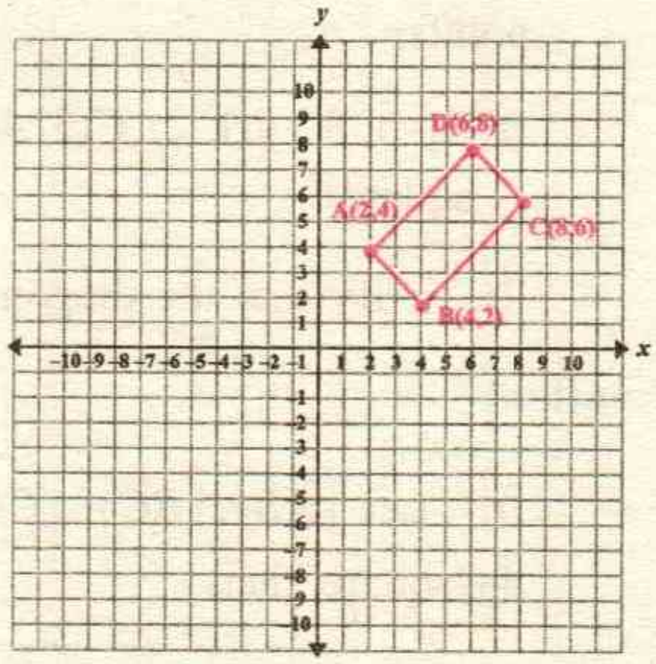
b) Rectangle

Example 12

Show that the points A(2, 4), B(4, 2) ,C(8, 6) and D(6, 8) are the vertices of a rectangle. Also plot the points.

Solution:

By using the distance formula, first we find the lengths of the sides of the *AB*, *BC*, *CD* and *AD*.



Thus , $|AB| = \sqrt{(4-2)^2 + (2-4)^2}$

$= \sqrt{(2)^2 + (-2)^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$

$|BC| = \sqrt{(8-4)^2 + (6-2)^2}$

$= \sqrt{(4)^2 + (4)^2} = \sqrt{16+16} = \sqrt{32} = 4\sqrt{2}$

$|CD| = \sqrt{(6-8)^2 + (8-6)^2} = \sqrt{(-2)^2 + (2)^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$

$|AD| = \sqrt{(6-2)^2 + (8-4)^2} = \sqrt{(4)^2 + (4)^2} = \sqrt{16+16} = \sqrt{32} = 4\sqrt{2}$

Since $|AB| = |CD|$ and $|BC| = |AD|$.

Note that, at this point we cannot say that the quadrilateral ABCD must be a rectangle. It can be just a parallelogram and not a rectangle. To complete the proof we will have to show that the diagonals $|AC|$ and $|BD|$ are equal in length.

$|AC| = \sqrt{(8-2)^2 + (6-4)^2} = \sqrt{(6)^2 + (2)^2} = \sqrt{36+4} = \sqrt{40} = 2\sqrt{10}$

$|BD| = \sqrt{(6-4)^2 + (8-2)^2} = = \sqrt{(2)^2 + (6)^2} = \sqrt{4+36} = \sqrt{40} = 2\sqrt{10}$

Since $|AC| = |BD|$. Thus the points *A*, *B*, *C* and *D* are the vertices of a rectangle.

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c) Parallelogram

Example 13

Show that the points F(−1, 5), G(3, 3),H(6,4) and J(2,2) are the vertices of a parallelogram. Also plot the points.

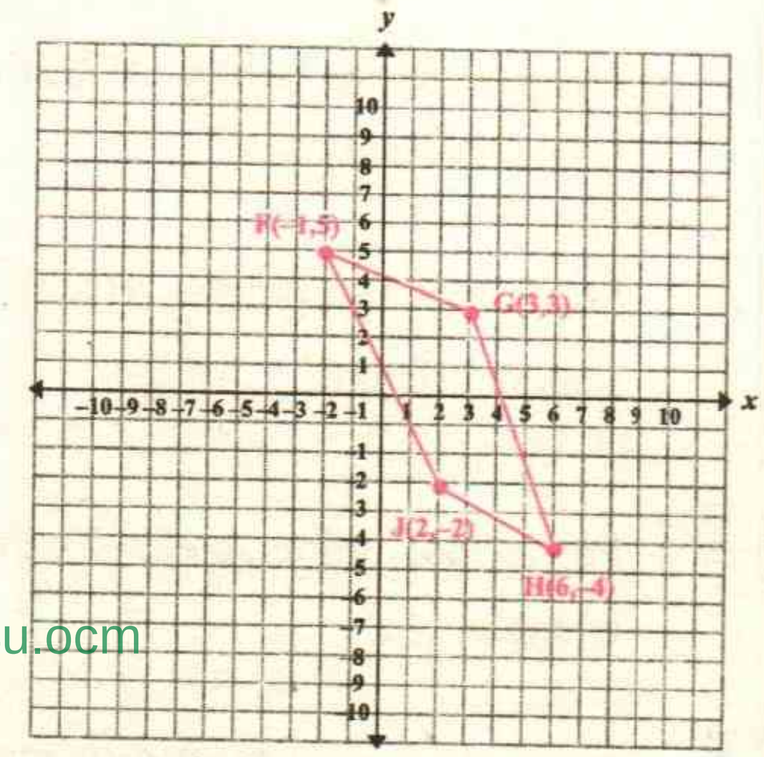
Solution:

By using distance formula, first we find the lengths of the sides of *FG*, *GH*, *JH* and *FJ*.

$$|FG| = \sqrt{(3+1)^2 + (3-5)^2}$$

$$= \sqrt{(4)^2 + (-2)^2} = \sqrt{16+4}$$

$$= \sqrt{20} = 2\sqrt{5}$$



$$|GH| = \sqrt{(6-3)^2 + (-4-3)^2}$$

$$= \sqrt{(3)^2 + (-7)^2} = \sqrt{9+49} = \sqrt{58}$$

$$|JH| = \sqrt{(6-2)^2 + (-4+2)^2}$$

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$$= \sqrt{4^2 + (-2)^2} = \sqrt{16+4} = 2\sqrt{5}$$

$$|FJ| = \sqrt{(2+1)^2 + (-2-5)^2}$$

$$= \sqrt{9+49} = \sqrt{3^2 + (-7)^2} = \sqrt{9+49} = \sqrt{58}$$

Thus $|FG| = |JH|$ and $|GH| = |FJ|$.

Now we find the diagonals $|FH|$ and $|GH|$.

$$|FH| = \sqrt{(6+1)^2 + (-4-5)^2}$$

$$= \sqrt{7^2 + (-9)^2} = \sqrt{49+81} = \sqrt{130}$$

$$|GJ| = \sqrt{(3-2)^2 + (3+2)^2}$$

$$= \sqrt{1+25} = \sqrt{26}$$

>	Greater Than
≥	Greater Than or Equal To
<	Less Than
≤	Less Than or Equal To

Since diagonals $|FH|$ and $|GJ|$ are not equal so the points F, G, H and J are the vertices of a parallelogram.

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Exercise 9.2

1. Prove that A(–4,–3), B(1,4), and C(6,11) are collinear.

2. Prove that $A(-1,3)$, $B(-4,7)$, C(0,4) is an isosceles triangle.

3. Show that the points $A(2,3), B(8,11)$ and $C(0,17)$ are the vertices of an isosceles triangle.

4. Show that the points $A(1,2)$, $B(3,4)$ *and* $C(0,-1)$ are the vertices of a scalene triangle.

5. Prove that points $A(-2,-2), B(4,-2)$, *and* $C(4,6)$ are vertices of a right triangle.

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6. Prove that $A(-2,0), B(6,0), C(6,6), D(-2,6)$ are vertices of a rectangle.

7. The vertices of the rectangle $ABCD$ are $A(2,0), B(5,0), C(5,4)$, and $D(2,4)$. How long is the diagonal AC?

8. Prove that $A(-4,-1)$, $B(1,0)$, $C(7,-3)$ *and* $D(2,-4)$ are vertices of a parallelogram.

9. Find b such that the points $A(2,b), B(5,5)$ and $C(-6,0)$ are the vertices of a right angled triangle with $\angle BAC = 90°$.

10. Given $A(-4,-2), B(1,-3)$ *and* $C(3,1)$, find the coordinate of D, in the 2nd quadrant such that quadrilateral $ABCD$ is a parallelogram.

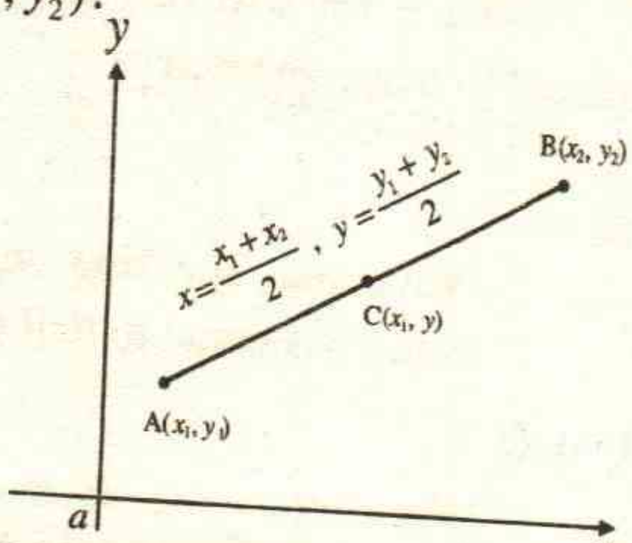
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9.3 MID-POINT FORMULA

We now look at a formula for finding the mid-point of the line segment obtained by joining two points $A(x_1, y_1)$ *and* $B(x_2, y_2)$.

If $C(x, y)$ denotes the mid-point of the line segment $\overline{AB}$ then

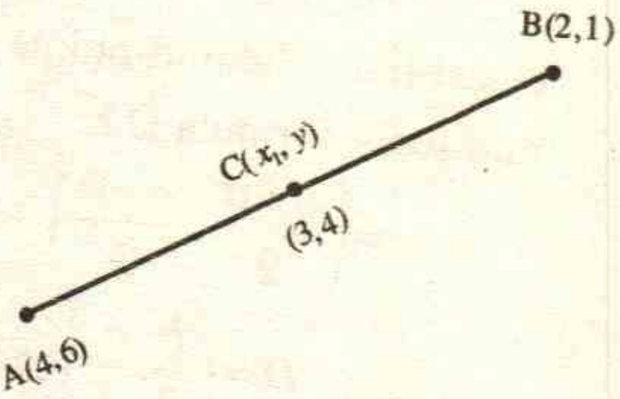
$$\left.\begin{aligned} x &= \frac{x_1 + x_2}{2} \\ y &= \frac{y_1 + y_2}{2} \end{aligned}\right\} \quad \ldots\ldots\ldots (i)$$



The above formula (i) is called the mid-point formula. From this formula it is seen that the x and y coordinates of the mid-point of a line segment are respectively, the average of the x coordinates and the average y coordinates of the end points of the line segment.

Example 14

Find the coordinates of the mid-point of the segment joining the points $A(4,6)$ and $B(2,1)$.



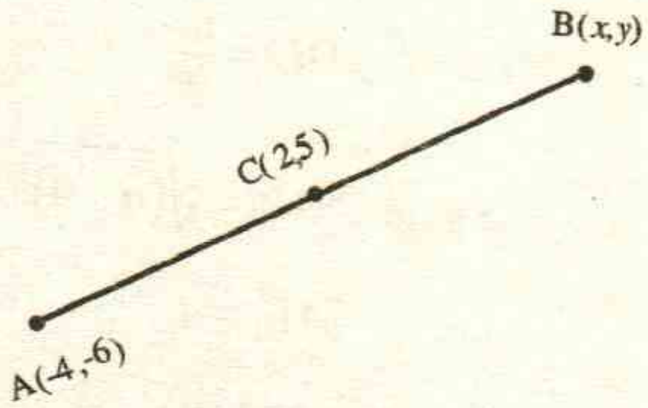
Solution:

Let $C(x, y)$ is the mid-point of $\overline{AB}$ then by the mid-point formula

$$C(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{4+2}{2}, \frac{2+6}{2}\right) = \left(\frac{6}{2}, \frac{8}{2}\right) = (3,4)$$

Example 15

The coordinates of midpoint of a line segment AB are (2,5) and that of A are (–4,–6). Find the coordinates of point B.



Solution:

Let the coordinates of B are (x, y) then by the mid-point formula, we can write

$$2 = \frac{-4+x}{2} \text{ and } 5 = \frac{-6+y}{2}$$

$$4 = -4+x \text{ and } 10 = -6+y$$

$x = 8$ and $y = 16$. So the coordinates of B are (8, 16).

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9.3.1 Application of distance and mid-point formula to solve/verify different standard results related to geometry

Here we will use co-ordinate geometry to prove some standard results related to plane geometry.

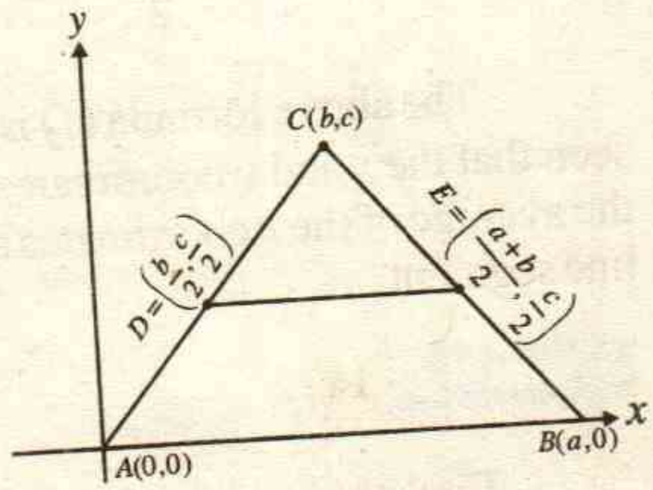
Result 1:

Prove that the line segment joining the mid-pints of two sides of a triangle is equal to half of the length of the third side.

Proof:

We choose the axes of coordinates as show and construct a triangle ABC with one vertex at the origin and one side along x-axis, so that

$A=(0,0)$, $B=(a,0)$ *and* $C=(b,c)$. Let D and E are the mid-points of $\overline{AC}$ and $\overline{BC}$. By mid-point formula D has co-ordinate's



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$$=\left(\frac{b+0}{2},\frac{c+0}{2}\right)$$

i.e. $D=\left(\frac{b}{2},\frac{c}{2}\right)$. Similarly $E=\left(\frac{a+b}{2},\frac{c+0}{2}\right)$ or $E=\left(\frac{a+b}{2},\frac{c}{2}\right)$

By distance formula

$$|DE|=\sqrt{\left(\frac{a+b}{2}-\frac{b}{2}\right)^2+\left(\frac{c}{2}-\frac{c}{2}\right)^2}=\sqrt{\left(\frac{a+b-b}{2}\right)^2+0}=\sqrt{\left(\frac{a}{2}\right)^2}$$

i.e. $|DE|=\frac{a}{2}$ ..... (i)

Again $|AB|=\sqrt{(a-0)^2+(0+0)^2}=\sqrt{a^2}=a$

i.e. $|AB|=a$ ..... (ii)

Form (i) and (ii)

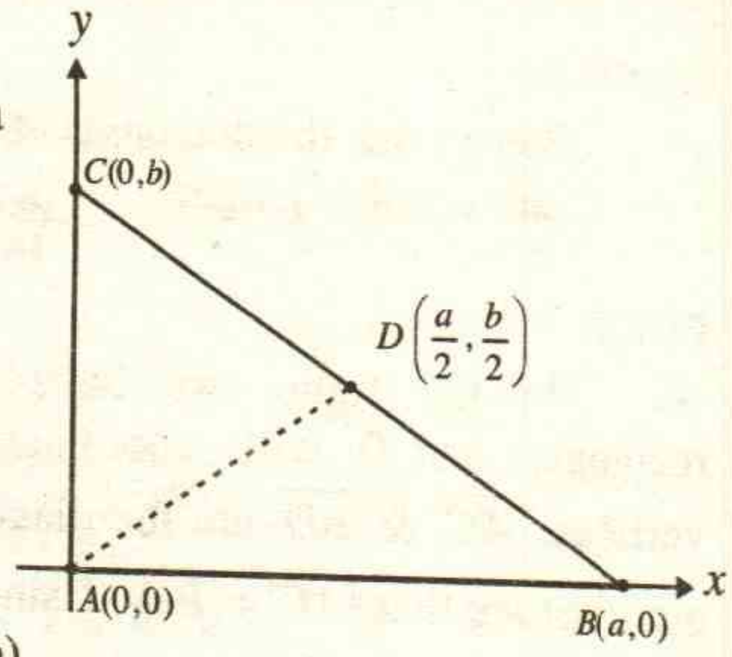
$$|DE|=\frac{a}{2}=\frac{1}{2}a=\frac{1}{2}|AB|$$

i.e. $|DE|=\frac{1}{2}|AB|$. This proves the required result.

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Result 2:

The mid-point of the hypotenuse of a right triangle is equidistant from the vertices.



Proof:

The triangle ABC is a right triangle. $\overline{BC}$ is the hypotenuse and D is its mid-point. The co-ordinates of the vertices are

$$A=(0,0), \qquad B=(a,0), \qquad C=(0,b)$$

Coordinates of of $D=\left(\frac{a+0}{2},\frac{0+b}{2}\right)$ by mid-point formula.

Coordinates of of $D=\left(\frac{a}{2},\frac{b}{2}\right)$

We are to prove that $|AD|=|BD|=|CD|$.

Using distance formula, we have

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$$|AD|=\sqrt{\left(\frac{a}{2}-0\right)^2+\left(\frac{b}{2}-0\right)^2}=\sqrt{\frac{a^2}{4}+\frac{b^2}{4}}=\sqrt{\frac{a^2+b^2}{4}} \quad \ldots\ldots\ldots \text{(i)}$$

$$|BD|=\sqrt{\left(a-\frac{a}{2}\right)^2+\left(0-\frac{b}{2}\right)^2}=\sqrt{\left(\frac{2a-a}{2}\right)^2+\left(-\frac{b}{2}\right)^2}$$

$$=\sqrt{\left(\frac{a}{2}\right)^2+\left(-\frac{b}{2}\right)^2}=\sqrt{\frac{a^2}{4}+\frac{b^2}{4}}=\sqrt{\frac{a^2+b^2}{4}} \quad \ldots\ldots\ldots \text{(ii)}$$

$$|CD|=\sqrt{\left(\frac{a}{2}-0\right)^2+\left(\frac{b}{2}-b\right)^2}=\sqrt{\frac{a^2}{4}+\left(\frac{b-2b}{2}\right)^2}$$

$$|CD|=\sqrt{\frac{a^2}{4}+\left(-\frac{b}{2}\right)^2}=\sqrt{\frac{a^2}{4}+\frac{b^2}{4}}=\sqrt{\frac{a^2+b^2}{4}} \quad \ldots\ldots\ldots \text{(iii)}$$

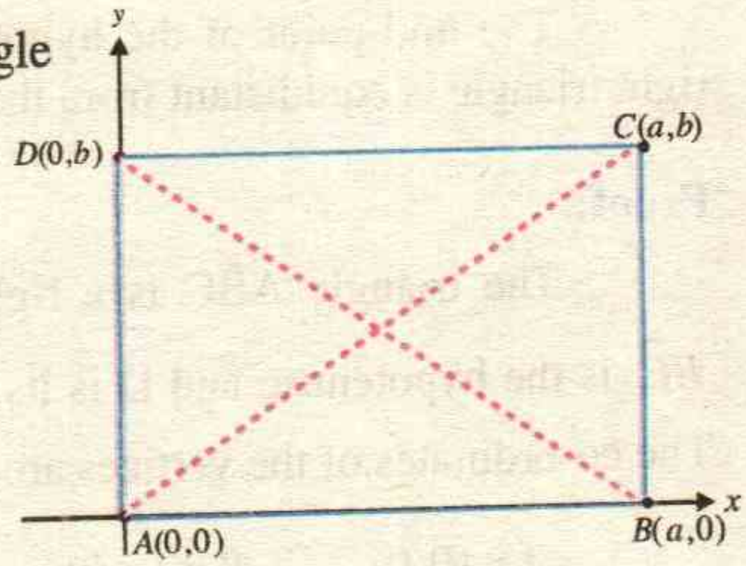
Comparing (i), (ii) and (iii) we have $|AD|=|BD|=|CD|$ proved.

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Result 3:

Verify that the diagonals of any rectangle are equal in length.



Proof:

In the figure we have shown a rectangle $ABCD$ with coordinates of its vertices. $\overline{AC}$ & $\overline{BD}$ are the diagonals. We are to prove that $|AC| = |BD|$. Using distance formula.

$$|AC| = \sqrt{(a-0)^2 + (b-0)^2} = \sqrt{a^2 + b^2} \quad \ldots\ldots \text{(i)}$$

$$|BD| = \sqrt{(a-0)^2 + (0-b)^2} = \sqrt{a^2 + b^2} \quad \ldots\ldots \text{(ii)}$$

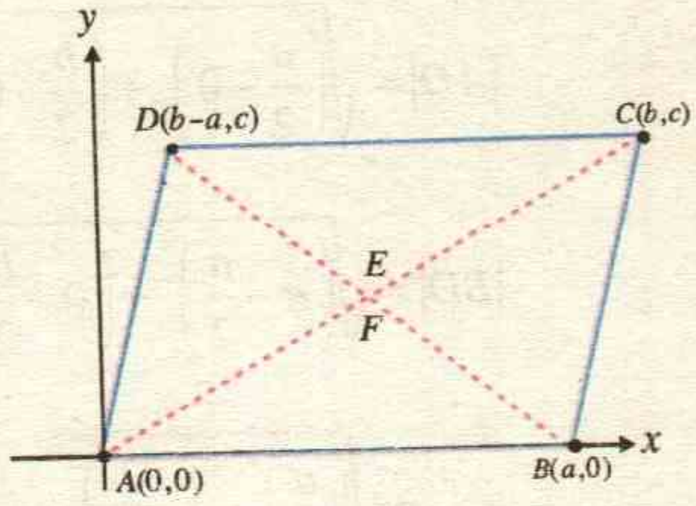
Form (i) and (ii)

$|AC| = |BD|$. This proves the required result.

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Result 4:

Show that diagonals of a parallelogram bisect each other.



Proof:

Let $ABCD$ is a parallelogram whose vertices are shown i.e.

$A = (0,0)$, $B = (a,0)$,

$C = (b,c)$, $D = (b-a,c)$

Let the mid-point of the diagonal $\overline{AC}$ is E then

Coordinates of $E = \left(\frac{b+0}{2}, \frac{c+0}{2}\right) = \left(\frac{b}{2}, \frac{c}{2}\right)$ By mid-point formula

Let the mid-point of the diagonal $\overline{BD}$ is F then

Coordinates of $F = \left(\frac{a+b-a}{2}, \frac{0+c}{2}\right) = \left(\frac{b}{2}, \frac{c}{2}\right)$

It means that E & F are same.

$\therefore$ $|AE| = |EC|$ and $|BF| = |FC|$

$\therefore$ Diagonals of a parallelogram bisect each other.

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Result 5:

Prove that in a right angled triangle the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of two legs.

Proof:

In the diagram, we have shown a right triangle with right angle at A. we have taken the coordinates of the vertices as

$A=(0,0)$, $B=(a,0)$ *and* $C=(0,b)$.

Now by distance formula

$$|AB|=\sqrt{(a-0)^2+(0-0)^2}=\sqrt{a^2} \quad \ldots\ldots \text{(i)}$$

$$|AC|=\sqrt{(0-0)^2+(b-0)^2}=\sqrt{b^2} \quad \ldots\ldots \text{(ii)}$$

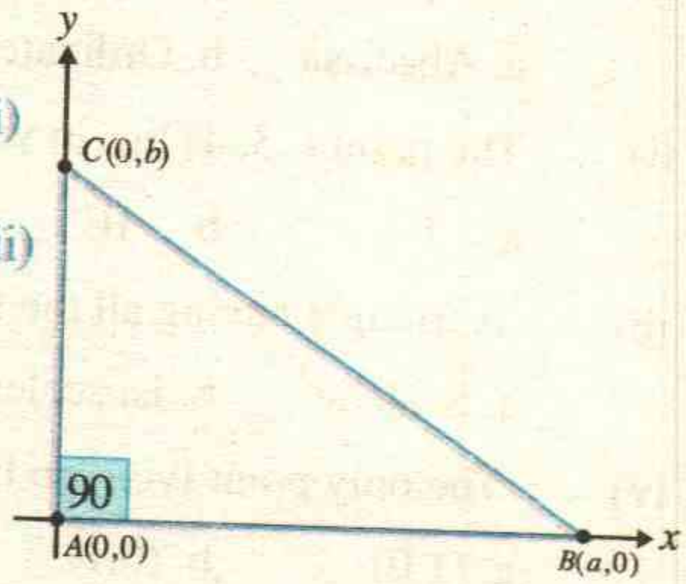
$$|BC|=\sqrt{(a-0)^2+(0-b)^2}=\sqrt{a^2+b^2}$$

$$\therefore \quad |BC|^2=a^2+b^2$$

$$\therefore \quad |BC|^2=|AB|^2+|AC|^2$$

i.e. (Hypotenuse)^2 = (one leg)^2 + (other leg)^2

This proves the required result.



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Exercise 9.3

1. Find the coordinates of the midpoint of the segment with the given endpoints.

i) (8,−5) and (−2,9)

ii) (7,6) and (3, 2).

iii) (−2,3) and (−9,−6)

iv) $(a+b,a-b)$ *and* $(-a,b)$

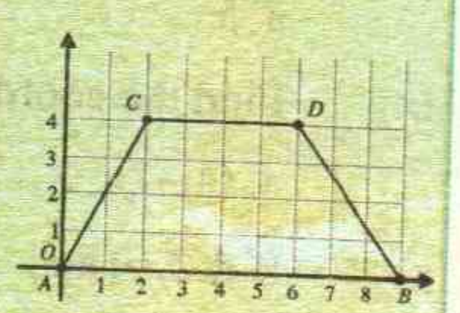
2. The mid-point and one end of a line segment are (3,7) and (4,2) respectively. Find the other end point.

3. The midpoints of the sides of a triangle are (2,5), (4,2), (1,1). Find the coordinates of the three vertices.

4. The distance between two points with coordinates $(1,1)$ *and* $(4,y)$ is 5. Find all possible values for y.

Activity

Assign coordinates to the vertices *A, B, C & D* of the trapezoid shown in the figure and then find mid-points of the segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ & $\overline{DA}$. Show these mid-points in the diagram.



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REVIEW EXERCISE 9

1. Select the correct answer.

i) The point having its coordinates (0,0) is called

a. Abscissa b. Ordinate c. Origin d. Critical point

ii) The point (–3, 4) lies in which quadrant?

a. I b. II c. III d. IV

iii) A triangle having all the three sides equal is

a. Scalene b. Isosceles c. Equilateral d. None of these

iv) The only point lying on both the axes is

a. (1,0) b. (0,1) c. (–1,0) d. (0,0)

v) Find the midpoint of a segment with endpoints at (5, 1) and (1, –3).

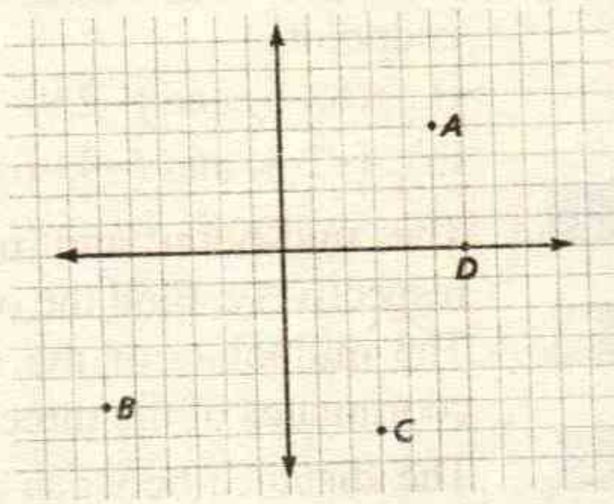
a) (–1, 3) b) (–3, 3) c) (–2, 2) d) (3, –1)

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vi) What are the coordinates of point (2, –3) after it is reflected over the x-axis?

a) (2, 3) b) (–2, 3) c) (–2, –3) d) (2, –3)

For vii-ix, use the given figure,



vii) Which of the following are the coordinates of point A?

a. (4, 5) b. (5,4)

c. (5,–4) d. (4, –5)

viii) What is the distance between points B and D?

a. 5 b. 11 c. 12 d. 13

ix) Find the coordinates of the midpoint of the line joining A and C.

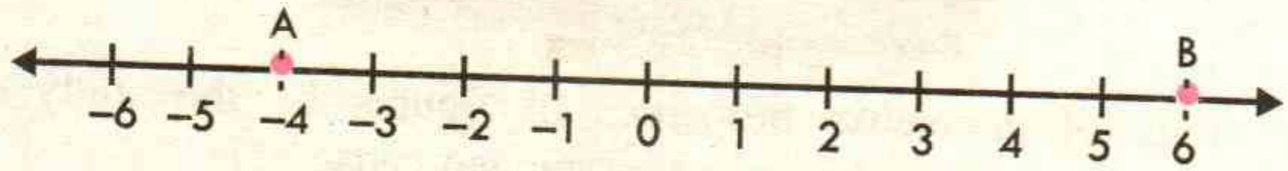
a. (4,–1) b. (–1,4) c. (5, –2) d. (–2, 5)

x) If $A(3,0)$ and $B(0,3)$ are any two points in a plane, then $|AB| =$

a) 18 b) $\sqrt{18}$ c) $9\sqrt{2}$ d) zero

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2. Find the distance between A and B on the number line below.



3. What is the distance between two points with coordinates of $(1,-5)$ *and* $(-5,7)$?

4. Using distance formula, show that the points $(4,-3),(2,\ 0),(-2,\ 6)$ are collinear.

5. Find the point on the x-axis which is equidistant from (0, 1) and (3, 3).

6. A segment has one endpoint at (15, 22) and a midpoint at (5,18), what are the coordinates of the other endpoint?

7. Prove that (2, 1), (0, 0), (−1, 2), and (1, 3) are the vertices of a rectangle.

8. Prove that $A(-1,0)$, $B(3,3)$, $C(6,-1)$, *and* $D(2,-4)$ is a square.

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Think

9. Whether the triangle with the vertices $(6,\ 5)$, $(2,\ -4)$, *and* $(5,\ -1)$ is an isosceles triangle ?

Activity

You have a quadrilateral with vertices $A(0,0)$, $B(9,0)$, $C(2,4)$, $D(6,4)$. Find the mid-points of their diagonals. Does diagonals cut at the mid-point. Show it on graph paper

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SUMMARY

- The study of geometric properties of figures by the study of their equations is called coordinate or analytic geometry.
- This one-to-one correspondence between the set of real numbers $\mathbb{R}$ and points on the number line denoted by xx' with the point 0 as its origin.
- The first coordinate x is called abscissa while the second coordinate y is called ordinate.
- In a coordiante plane, the distance, d between two points (x_1, y_1) and (x_2, y_2) is given by the following formula:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

- Three or more points P_1, P_2, P_3 ..., are said to be collinear if they lie on the same line.
- A triangle in which all the three sides and all the three angles are equal is called an equilateral triangle.
- In equilateral triangle measure of each angle is 60°. An isosceles triangle has two equal sides and two equal angles, opposite to the equal sides.
- A right angled triangle has one angle equal to 90°.
- By Pythagorean theorem , $(Base)^2 + (Perp)^2 = (Hyp)^2$.
- If all the three sides of a triangle are different in lengths, it is called scalene triangle.
- The midpoint of a segment with endpoints (x_1, y_1) and (x_2, y_2) has the following coordinates:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

- The line segment joining the mid-points of two sides of a triangle is equal to half of the length of the third side.
- The diagonals of any rectangle are equal in length.
- The mid-point of the hypotenuse of a right triangle is equidistant from the vertices.
- The diagonals of a parallelogram bisect each other.
- A trapezoid is a quadrilateral in which two & only two opposite sides are parallel.

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UNIT 10

CONGRUENT TRIANGLES

After reading this unit, the students will be able to:

Prove the following theorems along with corollaries and apply them to solve appropriate problems.

- In any correspondence of two triangles, if one side and any two angles of one triangle are congruent to the corresponding side and angles of the other, the two triangles are congruent.
- If two angles of a triangle are congruent then the sides opposite to them are also congruent.
- In a correspondence of two triangles, if three sides of one triangle are congruent to the corresponding three sides of the other, the two triangles are congruent.
- If in the correspondence of two right-angled triangles, the hypotenuse and one side of one are congruent to the hypotenuse and the corresponding side of the other, then the triangles are congruent.

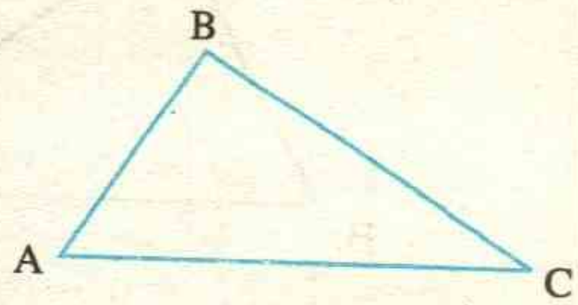
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Introduction

Two plane figures are congruent if they have the same size & shape. Such figures can always be made to coincide. This means that if two such figures were to be cut from paper, it will always be possible to make one exactly match the other. So congruent figures are called exact duplicate of each other. The process of matching two figure is called superposition. The angles and sides which would match are called corresponding or homologous parts. In our practical life, spare parts of particular purpose and of particular machinery are made congruent to the original parts. When two figures A & B are congruent, it is denoted by $A \cong B$.

Here we will talk about the congruence of triangles. A triangle has six elements, three sides and three angles. For example:

The six elements of triangle ABC are $\overline{AB}$, $\overline{BC}$, $\overline{AC}$, $\angle BAC$ or $\angle A$, $\angle CBA$ or $\angle B$, and $\angle ACB$ or $\angle C$.



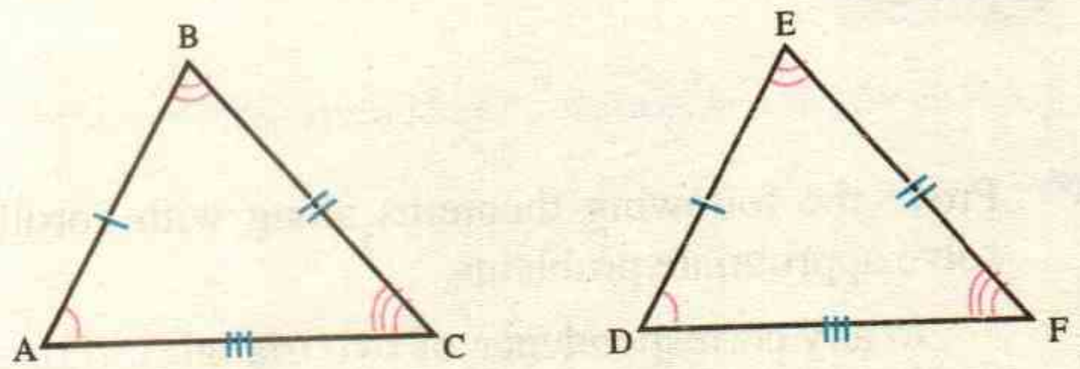
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Definition:

Two triangles ΔABC and ΔDEF are said to be congruent if the following six conditions all hold:

i) $\angle A \cong \angle D$
ii) $\angle B \cong \angle E$
iii) $\angle C \cong \angle F$
iv) $\overline{AB} \cong \overline{DE}$
v) $\overline{BC} \cong \overline{EF}$
vi) $\overline{AC} \cong \overline{DF}$



In this case, we write $\Delta ABC \cong \Delta DEF$

Note

Properties of congruent triangles;

- $\Delta ABC \cong \Delta ABC$ (Identity congruence) i.e. every triangle is congruent to itself.
- If $\Delta ABC \cong \Delta DEF$ then $\Delta DEF \cong \Delta ABC$ (Symmetric property of congruence).
- If $\Delta ABC \cong \Delta DEF$ and $\Delta DEF \cong \Delta XYZ$. then $\Delta ABC \cong \Delta XYZ$ (Transitive property of congruence).

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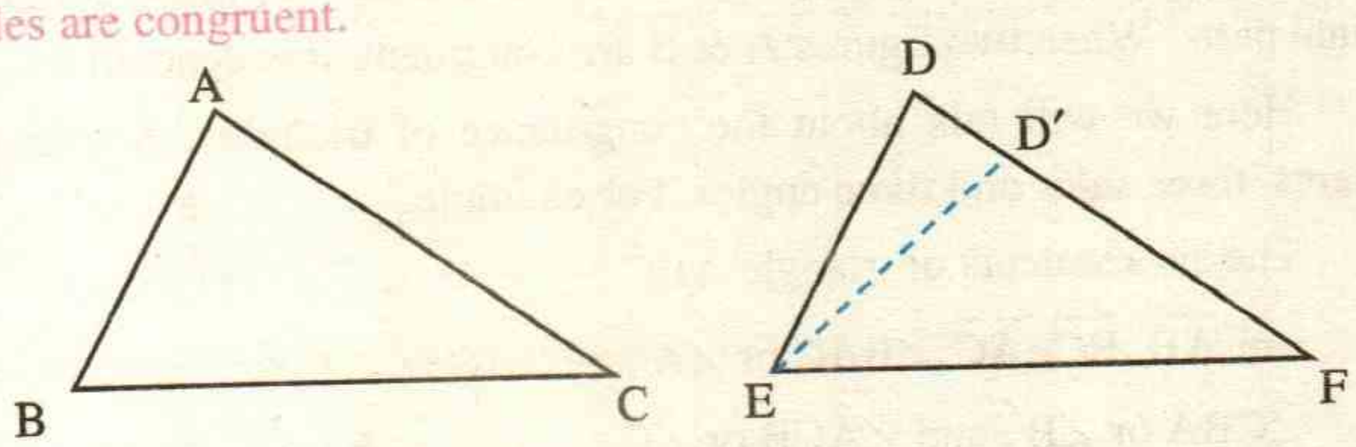
SAS (Side-Angle-Side) (Postulate)

If two sides and their included angle in one triangle are conguent to two sides and their included angle in another triangle, then the two triangles are conguent.

THEOREM 10.1

AAS (Angle-Angle-Side) Congruence Theorem

If two angles and a nonincluded side of one traingle are congruent to the corresponding angles and nonincluded side of another triangle, then the triangles are congruent.



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Given:

In $\Delta ABC \leftrightarrow \Delta DEF$

$\overline{BC} \cong \overline{EF}$, $\angle B \cong \angle E$ and $\angle A \cong \angle D$

To Prove:

$\Delta ABC \cong \Delta DEF$

Construction:

Suppose $\overline{AC}$ is not congruent to $\overline{DF}$. Take a point D′ on $\overline{DF}$, such that $\overline{AC} \cong \overline{D'F}$ join D′ to E.

Proof:

Statements	Reasons
In $\Delta ABC \leftrightarrow \Delta D'EF$	
$\overline{BC} \cong \overline{EF}$	(Given)
$\angle C \cong \angle F$	$\angle B \cong \angle E$ and $\angle A \cong \angle D$
$\overline{AC} \cong \overline{D'F}$	(Supposition)
$\therefore \quad \Delta ABC \cong \Delta D'EF$	(S.A.S ≅ S.A.S)
Hence $\angle B \cong \angle D'EF$	(Corresponding angles of congruent Δs)
But $\angle B \cong \angle DEF$	(Given)
$\therefore \quad \angle D'EF \cong \angle DEF$	(Transitive Property)
This implies $\overline{ED} \cong \overline{ED'}$	
And thus $\overline{AC} \cong \overline{DF}$,	
In $\Delta ABC \leftrightarrow \Delta DEF$	
$\overline{BC} \cong \overline{EF}$	(Given)
$\overline{AC} \cong \overline{DF}$	(Proved)
$\angle C \cong \angle F$	(Given)
$\therefore \quad \Delta ABC \cong \Delta DEF$	(S.A.S ≅ S.A.S)

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Corollary

ASA (Angle-Side-Angle)

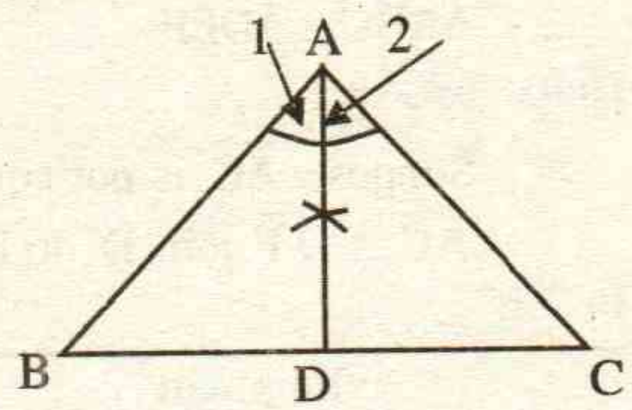
If two angles and their included side in one triangle are congruent to two angles and their included side in another triangle, then the two triangles are congruent.

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THEOREM 10.2

Converse of the Isosceles triangle theorem

If two angles of a triangle are congruent, then the sides opposite to those angles are congruent.



Given:

In ΔABC, $\angle B \cong \angle C$.

To Prove:

$\overline{AB} \cong \overline{AC}$

Construction:

Draw bisector of $\angle A$ cutting $\overline{BC}$ at the point D.

Proof:

Statements	Reasons
In $\Delta ABD \leftrightarrow \Delta ACD$	
$\overline{AD} \cong \overline{AD}$	(Common)
$\angle 1 \cong \angle 2$	(Construction)
$\angle B \cong \angle C$	(Given)
$\therefore \quad \Delta ABD \cong \Delta ACD$	($A.A.S \cong A.A.S$)
Hence $\overline{AB} \cong \overline{AC}$	(Corresponding sides of congruent Δs)

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THEOREM 10.3

SSS (Side-Side-Side)

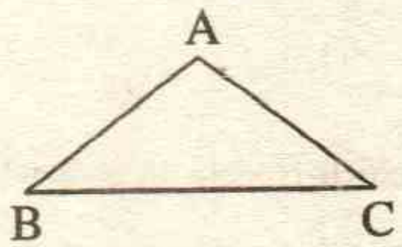
In a correspondernce of two triangles, if three sides of one triangle are congruent to the corresponding three sides of the other, the two triangles are congruent.

Given:

In $\Delta ABC \leftrightarrow \Delta DEF$

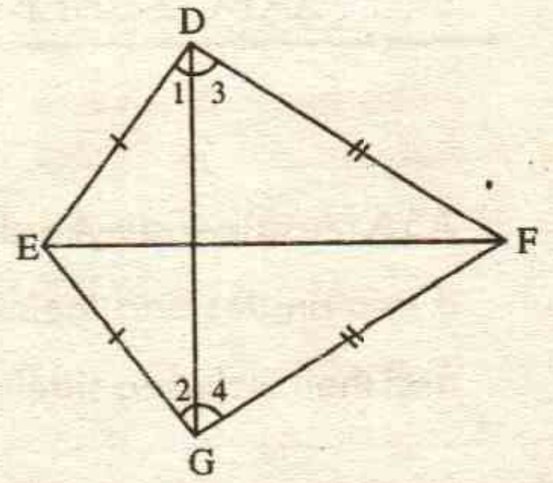
$\overline{AB} \cong \overline{DE}$

$\overline{BC} \cong \overline{EF}$ and $\overline{CA} \cong \overline{FD}$



To Prove:

$\Delta ABC \cong \Delta DEF$



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Construction:

Suppose the side $\overline{BC}$ is not shorter than any of the other two sides of the triangle ABC. Construct ΔGEF with G and D on opposite sides of $\overline{EF}$ such that $\overline{GE} \cong \overline{AB}$.

Join G and D. let the angles be named by the numbers as shown in figure.

Proof:

Statements	Reasons
In the correspondence $\Delta ABC \leftrightarrow \Delta GEF$	
$\overline{BC} \cong \overline{EF}$	(Given)
$\angle B \cong \angle GEF$	(Construction)
$\overline{AB} \cong \overline{GE}$	(Construction)
$\therefore\ \Delta ABC \cong \Delta GEF$	(S.A.S $\cong$ S.A.S)
So $\overline{GF} \cong \overline{AC}$	(Corresponding sides of congruent triangles)
But $\overline{DF} \cong \overline{AC}$	(Given)
$\therefore\ \overline{GF} \cong \overline{DF}$ ---- (i)	(Transitive Property)
In ΔFDG,	
$\angle 3 \cong \angle 4$ ---- (ii)	(ΔDFG is isosceles)
In ΔEGD,	
$\overline{DE} \cong \overline{EG}$	each is congruent to $\overline{AB}$
$\therefore\ \angle 1 \cong \angle 2$ ---- (iii)	
$\therefore\ \angle 3 + \angle 1 \cong \angle 4 + \angle 2$	By (ii) and (iii)
$\therefore\ m\angle D \cong m\angle G$	Addition of angles postulate
In the correspondence $\Delta DEF \leftrightarrow \Delta GEF$	
$\overline{DE} \cong \overline{GE}$	Proved
$\angle D \cong \angle G$	Proved
$\overline{DE} \cong \overline{FG}$	By (i)
$\therefore\ \Delta DEF \cong \Delta GEF$	S.A.S
But $\Delta ABC \cong \Delta GEF$	Proved
$\therefore\ \Delta ABC \cong \Delta GEF$	

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Note This theorem will be referred to as S.S.S $\cong$ S.S.S

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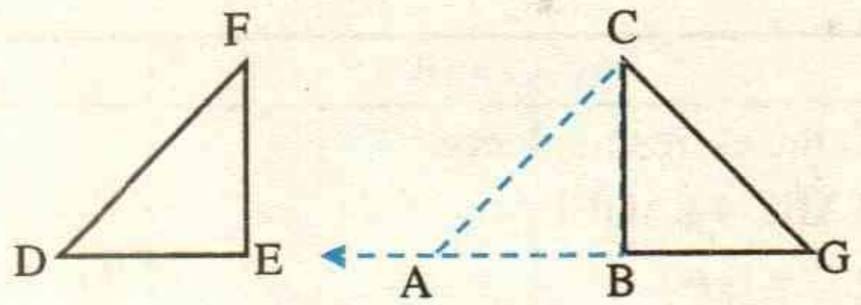
THEOREM 10.4

HL (Hypotenuse-Leg) Congruence Theorem

If in the correspondence of two right angled triangles, the hypotenose and one side of one are respectively congruent to the hypotenuse and the corresponding side of the other, the triangles are congruent.

Given:

In $\Delta ABC \leftrightarrow \Delta DEF$
$\angle ABC \cong \angle E$
(each is right angle)
$\overline{AC} \cong \overline{DF}$ *and* $BC \cong \overline{EF}$



To Prove:

$\Delta ABC \cong \Delta DEF$

Construction:

Produce AB to the point G such that $\overline{BG} \cong \overline{DE}$ join C and G.

Proof:

Statements	Reasons
In ΔGBC, $m\angle CBG = 90°$	(Supplementary angle of the right angle $\angle ABC$) (Given)
In the correspondence $\Delta GBC \leftrightarrow \Delta DEF$	
$\overline{GB} \cong \overline{DE}$	Construction
$\angle GBC \cong \angle DEF$	Each is right angle
$\overline{BC} \cong \overline{EF}$	Given
$\therefore$ $\Delta GBC \cong \Delta DEF$	S.A.S $\cong$ S.A.S
$\overline{GC} \cong \overline{DF}$	corresponding sides of congruent Δs.
But $\overline{DF} \cong \overline{AC}$	given
$\therefore$ $\overline{GC} \cong \overline{AC}$	
It follows that $\angle A \cong \angle G$	Angles opposite to the congruent sides of the ΔCAG.
In the correspondence $\Delta ABC \leftrightarrow \Delta GBC$	
$\overline{AC} \cong \overline{GC}$	Proved
$\angle A \cong \angle G$	Proved
$\angle ABC \cong \angle GBC$	Each is right angle
$\therefore$ $\Delta ABC \leftrightarrow \Delta GBC$	S.A.S $\cong$ S.A.S
But $\Delta GBC \cong \Delta DEF$	Proved
$\therefore$ $\Delta ABC \cong \Delta DEF$	

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Note This theorem will be referred to as H.L $\cong$ H.L

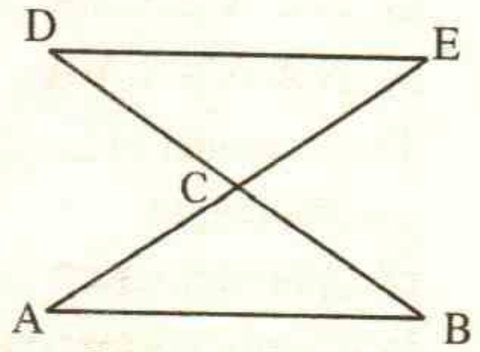
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Exercise 10.1

1. Prove that the perpendiculars drawn from the endpoints of the base of an isosceles triangle to their opposite sides are congruent.

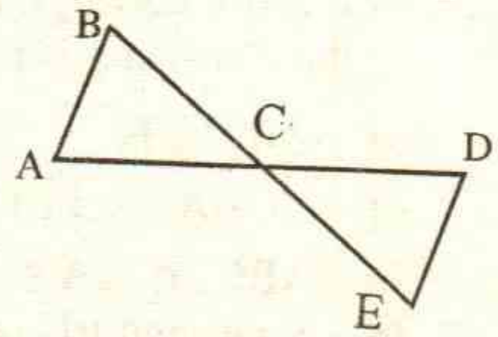
2. In the given figure $\overline{AC} \cong \overline{CE}$ and $\angle B \cong \angle D$. Prove that $\overline{BC} \cong \overline{CD}$



3. Given: C is the midpoint of $\overline{BE}$.

$\angle B \cong \angle E$

Prove: $\Delta ABC \cong \Delta DEC$



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4. Prove that the median from the vertex angle of an isosceles triangle is perpendicular to the base and bisects the vertex angle.

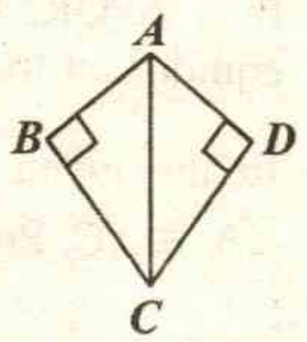
5. Prove that the bisector of the vertex angle of an isosceles triangle is the perpendicular bisector of the base.

6. PQRS is a square. X, Y and Z are the mid-points of $\overline{PQ}$, $\overline{QR}$ and $\overline{RS}$ respectively. Prove that $\Delta PXY \cong \Delta SZY$.

7. Given: $\overline{AB} \perp \overline{BC}$, $\overline{AD} \perp \overline{DC}$

$\overline{AB} \cong \overline{AD}$

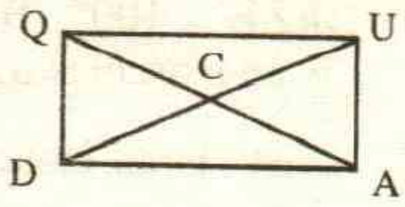
Prove that: $\Delta ABC \cong \Delta ADC$.



8. QUAD is a rectangle. Find x.

$QC = x$

$DC = 3x - 8$



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REVIEW EXERCISE 10

1 Select the correct answer.

i) Which of the following is not a sufficient condition for the congruency of two triangles?

a) A.S.A ≅ A.S.A
b) H.S ≅ H.S
c) S.A.A ≅ S.A.A
d) A.A.A ≅ A.A.A

ii) The diagonal of _____ does not divide it into two congruent triangles.

a) rectangle
b) square
c) parallelogram
d) trapezium

iii) In a given correspondence of two triangles, if $\Delta ABC \cong \Delta DEF$, then which of the following is not correct?

a) $\angle B \cong \angle E$
b) $\overline{CA} \cong \overline{FD}$
c) $\angle CBA \cong \angle FED$
d) $\angle ABC \cong \angle EFD$

iv) In ΔABC, if $\angle A \cong \angle B$ then bisector of _____ divides the ΔABC into two congruent triangles.

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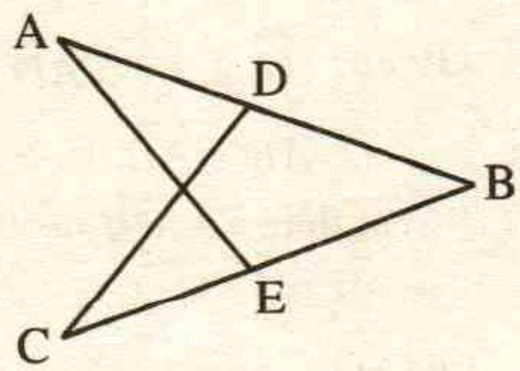
a) $\angle A$
b) $\angle B$
c) $\angle C$
d) any one of its angles

v) Which of the following is a legitimate reason for declaring two triangles to be congruent?

a) SSA ≅ SSA
b) SAS ≅ SAS
c) AAA ≅ AAA
d) all of the these

2. In a ΔPQR, the bisectors of $\angle Q$ and $\angle R$ meet at O. Prove that O is equidistant from the sides of the triangle.

3. In the given figure $\overline{AB} \cong \overline{CB}$ and $\angle A \cong \angle C$. Prove that $\overline{AE} \cong \overline{CD}$.



4. ABC is a triangle in which $m\angle A = 35°$ and $m\angle B = 100°$, $\overline{BD} \perp \overline{AC}$. Prove that ΔBDC is an isosceles triangle.

5. ABC is an isosceles triangle. D is the mid-point of base $\overline{BC}$. Prove that $\overline{AD}$ bisects $\angle A$ and $\overline{AD} \perp \overline{BC}$.

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SUMMARY

- When two geometric figures are *identical* (they can be placed exactly on top of each other), the figures are called ***congruent.*** Congruent figures are exact duplicates of each other. They have exactly the same size and shape.
- A triangle has six elements, three sides and three angles.
- CPCTC means Corresponding parts of congruent triangles are congruent.
- If two sides and their included angle in one triangle are conguent to two sides and their included angle in another triangle, then the two triangles are conguent.
- If two angles and a non-included side of one traingle are congruent to the corresponding angles and non-included side of another triangle, then the triangles are congruent.
- A triangle in which all the three sides and all the three angles are equal is called an equilateral triangle.
- If two angles and their included side in one triangle are congruent to two angles and their included side in another triangle, then the two triangles are congruent.
- If two angles of a trianlge are congruent, then the sides opposite to those angles are congruent.
- If the sides of one trianlge are congruent to the sides of another triangle, then the two triangles are congruent.
- If the hypotenuse and leg of a right triangle are congruent to the hypotenuse and corresponding leg of another right triangle, then the two triangles are congruent.

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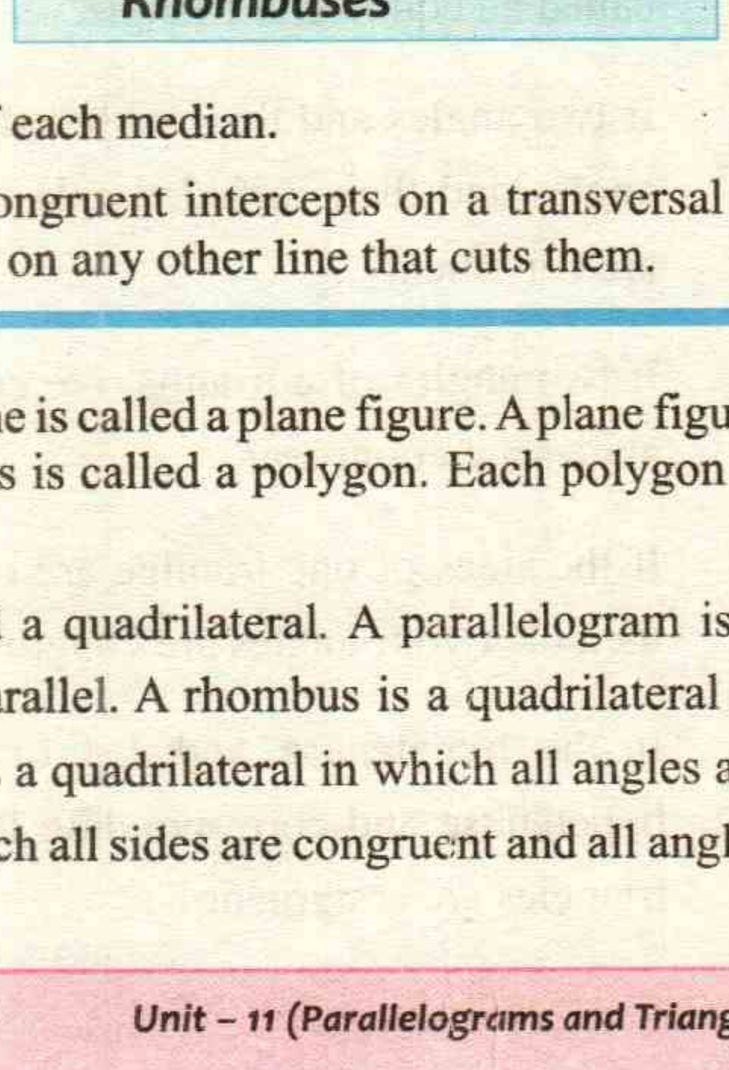
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UNIT 11

PARALLELOGRAMS AND TRIANGLES

After reading this unit, the students will be able to:

- Prove the following theorems along with corollaries and apply them to solve appropriate problems.
- In a parallelogram:
  - the opposite sides are congruent,
  - the opposite angles are congruent,
  - the diagonals bisect each other.
- If two opposite sides of a quadrilateral are congruent and parallel, it is a parallelogram.
- The line segment, joining the midpoints of two sides of a triangle, is parallel to the third side and is equal to one half of its length.
- The medians of a triangle are concurrent and their point of concurrency is the point of trisection of each median.
- If three or more parallel lines make congruent intercepts on a transversal they also intercept congruent segments on any other line that cuts them.



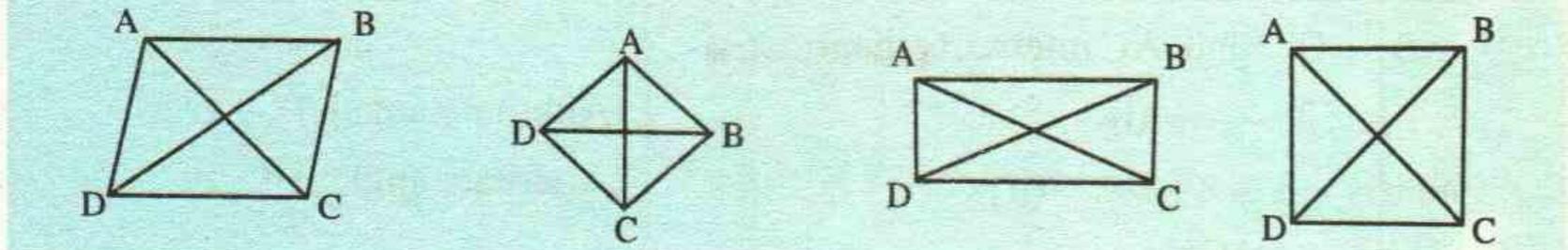
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In geometry, a figure that lies in a plane is called a plane figure. A plane figure with three or more straight edges as its sides is called a polygon. Each polygon is named after the number of sides it contains.

A polygon with four sides is called a quadrilateral. A parallelogram is a quadrilateral in which opposite sides are parallel. A rhombus is a quadrilateral in which all sides are congruent. A rectangle is a quadrilateral in which all angles are congruent. A Square is a quadrilateral in which all sides are congruent and all angles are congruent.

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From the definition it follows that in fact rectangle, rhombus and Square are parallelograms.



A parallelogram is a quadrilateral with both pairs of opposite sides parallel. The term "parallelogram PQRS" can be written as PQRS.

In PQRS, PQ || RS

QR || PS by definition.

Tit-Bit

The sum of four angles of a quadrilateral is 360°.

The theorem below describes the properties of parallelogram.

THEOREM 11.1

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In a parallelogram,

- The opposite sides are congruent.
- The opposite angles are congruent.
- The diagonals bisect each other.

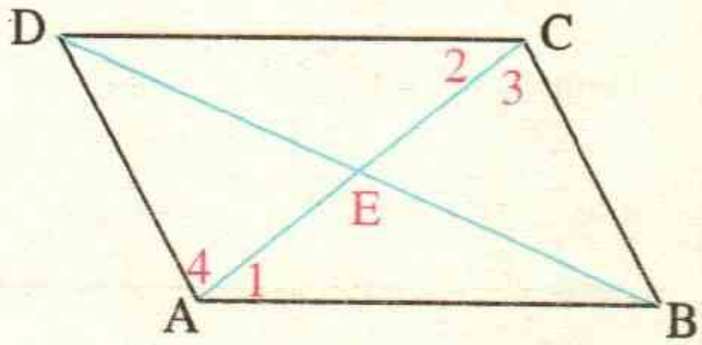
Given:

ABCD is a parallelogram

i.e. $\overline{AB} \parallel \overline{DC}$

and $\overline{AD} \parallel \overline{BC}$

$\overline{AC}$ and $\overline{BD}$ are its diagonals.



To Prove:

$\overline{AB} \cong \overline{DC}$ and $\overline{AD} \cong \overline{BC}$

$\angle A \cong \angle C$ and $\angle B \cong \angle D$

$\overline{AE} \cong \overline{CE}$ and $\overline{DE} \cong \overline{BE}$

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Proof:

Statements	Reasons
∵ $\overline{AB} \parallel \overline{DC}$ and $\overline{AC}$ intersects them	
∴ $\angle 1 \cong \angle 2$ -------- (i)	(Alternate angels)
Similarly $\angle 3 \cong \angle 4$ -------(ii)	(Alternate angles)
In $\Delta ABC \leftrightarrow \Delta CDA$	
$\overline{AC} \cong \overline{AC}$	(Common)
$\angle 3 \cong \angle 4$	(Proved)
$\angle 1 \cong \angle 2$	(Proved)
∴ $\Delta ABC \cong \Delta CDA$	$(A \cdot S \cdot A \cong A \cdot S \cdot A)$
Hence $\overline{AB} \cong \overline{CD}$	(corresponding sides of congruent triangles)
and $\overline{BC} \cong \overline{DA}$	(corresponding sides of congruent triangles)
also $\angle B \cong \angle D$	(corresponding angles of congruent triangles)
and $m\angle 1 + m\angle 4 = m\angle 2 + m\angle 3$	(from (i) and (ii))
or $\angle A \cong \angle C$	
In $\Delta ABE \leftrightarrow \Delta CDE$	
$\overline{AB} \cong \overline{CD}$	(Proved)
$\angle 1 \cong \angle 2$	(Proved)
$\angle AEB \cong \angle CED$	(vertical angles)
∴ $\Delta ABE \cong \Delta CDE$	$(S \cdot A \cdot A \cong S \cdot A \cdot A)$
Hence $\overline{AE} \cong \overline{CE}$	(corresponding sides of congruent triangles)
and $\overline{BE} \cong \overline{DE}$	(corresponding sides of congruent triangles)

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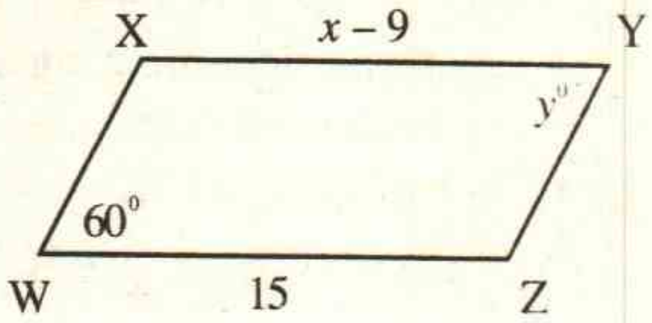
Corollary

A diagonal of a parallelogram divides the parallelogram into two congruent triangles.

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Example 1

Quadrilateral WXYZ is a parallelogram. Find the value of x and y.



Solution:

$XY = WZ$ ($\because$ opposite sides of a parallelogram are equal)

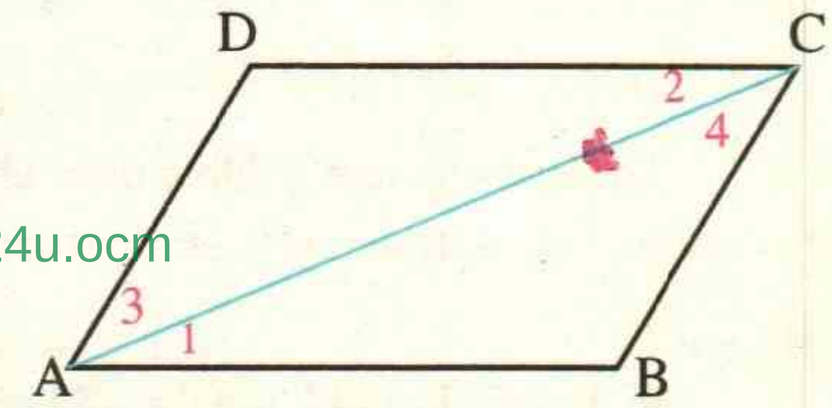
$x - 9 = 15$

$x = 24$

$y = 60$ ($\because$ opposite angles of a parallelogram are equal)

THEOREM 11.2

If two opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram.



Given:

ABCD is a quadrilateral in which $\overline{AB} \cong \overline{DC}$ and $\overline{AB} \parallel \overline{DC}$.

To Prove:

ABCD is a parallelogram.

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Construction:

Join A to C.

Proof:

Statements	Reasons
In $\Delta ABC \leftrightarrow \Delta CDA$	
$\overline{AB} \cong \overline{DC}$	(Given)
$\overline{AC} \cong \overline{AC}$	(Common)
$\angle 1 \cong \angle 2$	(Alternate angles)
$\therefore \Delta ABC \cong \Delta CDA$	(S.A.S $\cong$ S.A.S)
Hence $\angle 3 \cong \angle 4$	(Corresponding angles of congruent triangles)
But these are alternate angles	
$\therefore$ $\overline{AD} \parallel \overline{BC}$	
$\because$ ABCD has both pairs of opposite sides parallel.	
$\therefore$ ABCD is a parallelogram.	

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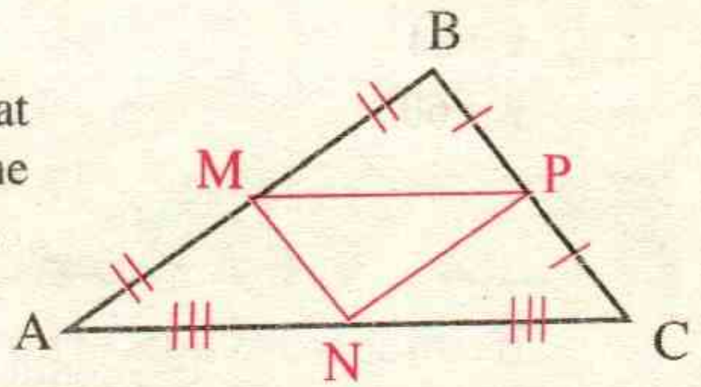
Remember

If a quadrilateral is a parallelogram, then its

- opposite sides are congruent.
- opposite angles are congruent.
- consecutive angles are supplementary.
- diagonal bisects each other.

The triangle Mid-Segment Theorem

A mid segment of a triangle is a segment that connects the midpoints of two sides of the triangle. Every triangle has three mid segments. The mid segments of ΔABC at the right are MP, MN, and NP.



THEOREM 11.3

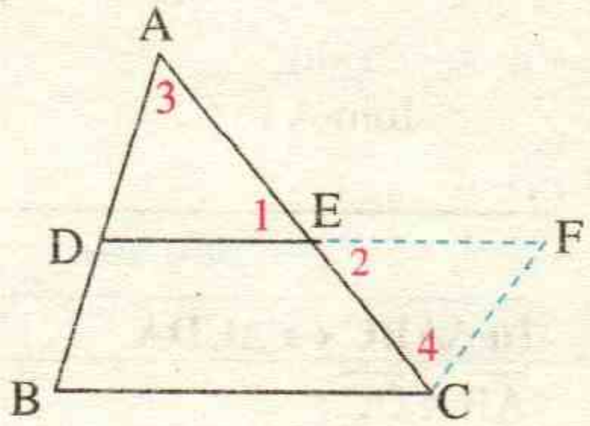
The triangle Midsegment Theorem

The line segment, joining the midpoints of two sides of a triangle, is parallel to the third side and is equal to one half of its length.

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Given:

In ΔABC, D and E are the mid points of $\overline{AB}$ and $\overline{AC}$ respectively. $\overline{DE}$ joins them.



To Prove:

$\overline{DE} \parallel \overline{BC}$ and $m\overline{DE} = \frac{1}{2} m\overline{BC}$.

Construction:

On $\overrightarrow{DE}$, takes F, such that $\overline{DE} \cong \overline{EF}$. Join F to C.

Proof:

Statements	Reasons
In $\Delta ADE \leftrightarrow \Delta CFE$	
$\overline{DE} \cong \overline{EF}$	(Construction)
$\overline{AE} \cong \overline{CE}$	(Given)
$\angle 1 \cong \angle 2$	(Vertical angles)
$\therefore \quad \Delta ADE \cong \Delta CFE$	(S.A.S $\cong$ S.A.S)

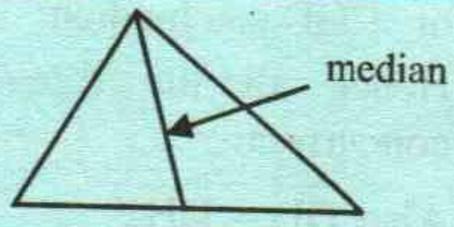
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Hence	(Corresponding sides of congruent triangles)
But $\overline{AD} \cong \overline{BD}$	(Given)
$\therefore$ $\overline{BD} \cong \overline{CF}$	(Transitive Property)
Also $\overline{AB} \parallel \overline{CF}$	(Alternate angles are congruent)
or $\overline{BD} \parallel \overline{CF}$	
i.e. BCFD is a parallelogram	(Opposite sides parallel)
Which follows that	
$\overline{DE} \parallel \overline{BC}$	
and $m\overline{DF} = m\overline{BC}$	(Opposite sides of a parallelogram)
But $m\overline{DE} = \frac{1}{2} m\overline{DF}$	(Construction)
$\therefore$ $m\overline{DE} = \frac{1}{2} m\overline{BC}$	($\because$ $m\overline{DF} = m\overline{BC}$)

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A **median** of a triangle is a line segment from a vertex to the midpoint of the opposite side.

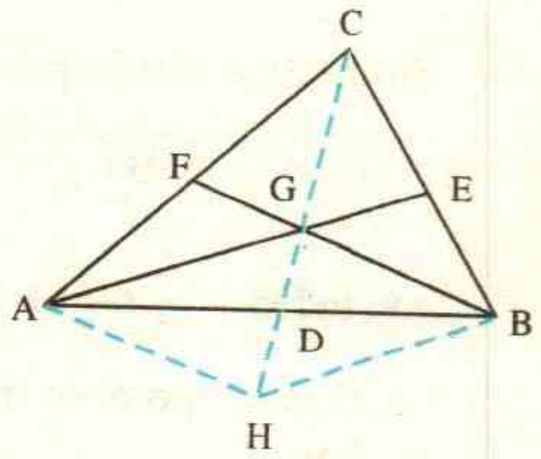


THEOREM 11.4

The medians of a triangle are concurrent and their point of concurrency is the point of trisection of each median.

Given:

In ΔABC, E and F are the mid points of $\overline{BC}$ and $\overline{CA}$ respectively. $\overline{AE}$ and $\overline{BF}$ intersect each other at G. $\overrightarrow{CG}$ is drawn to meet $\overline{AB}$ at D.



To Prove:

i) Medians are concurrent i.e. $\overline{CGD}$ is also a median.

ii) G is the point of trisection of each median.

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Construction:

Take a point H on $\overrightarrow{CG}$, such that $\overline{CG} = \overline{GH}$. Join H to A and B.

Proof:

Statements	Reasons
In ΔACH	
$\overline{FG} \parallel \overline{AH}$	($\overline{FG}$ is the segment joining mid points of $\overline{CA}$ and $\overline{CH}$)
i.e. $\overline{FB} \parallel \overline{AH}$	
Similarly in ΔBCH $\overline{GE} \parallel \overline{HB}$	(G and E are the mid points of $\overline{CH}$ and $\overline{CB}$)
or $\overline{AE} \parallel \overline{HB}$	
$\therefore$ AHBG is a parallelogram.	(Opposite sides are parallel)
i.e. $\overline{AD} \cong \overline{BD}$	(Diagonals of a parallelogram bisect each other)
or D is the mid point of $\overline{AB}$.	
or $\overline{CGD}$ is a median.	
Hence all the three medians are concurrent.	
Also $\overline{GD} \cong \overline{DH}$	(Diagonals of a parallelogram bisect each other)
or $m\overline{GD} = \frac{1}{2} m\overline{GH}$	
$= \frac{1}{2} m\overline{GC}$	($\because \overline{CG} \cong \overline{GH}$ / construction)
	$\therefore$ G is the point of trisection of median $\overline{CD}$.
Similarly it can be proved that	
$m\overline{GE} = \frac{1}{2} m\overline{AG}$	
and $m\overline{FG} = \frac{1}{2} m\overline{GB}$	
i.e. G is the point of trisection of each median	

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Parallel and perpendicular lines

Two lines that do not intersect and are coplanar are parallel lines.

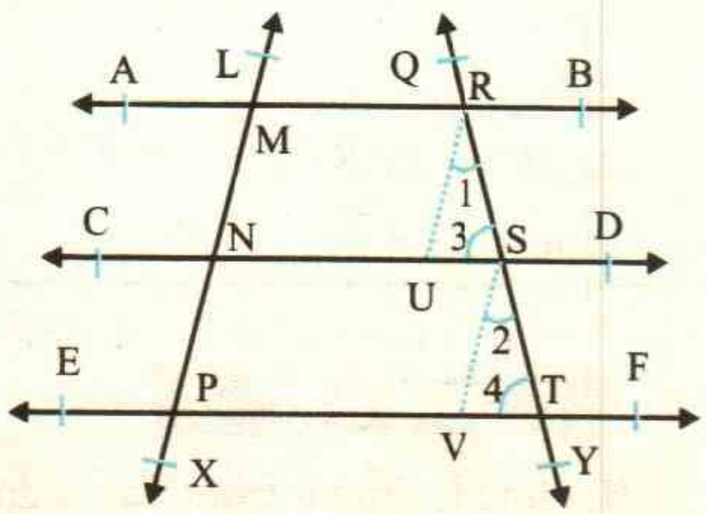
If two lines intersect to form a right angle then they are perpendicular lines.

THEOREM 11.5

If three or more parallel lines make congruent intercepts on a transversal, they also intercept congruent segments on any other line that cuts them.

Given:

Lines $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD} \parallel \overleftrightarrow{EF}$. The transversal $\overleftrightarrow{LX}$ intercepts $\overleftrightarrow{AB}$, $\overleftrightarrow{CD}$ and $\overleftrightarrow{EF}$ at the points M,N and P respectively such that $\overline{MN} \cong \overline{NP}$ while another transversal $\overleftrightarrow{QY}$ intercepts them at the points R,S and T respectively.



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To Prove:

$\overline{RS} \cong \overline{ST}$

Construction:

From R, draw $\overline{RU} \parallel \overline{LX}$ which meets $\overleftrightarrow{CD}$ at U and from S, draw $\overline{RV} \parallel \overline{LX}$ which meets $\overleftrightarrow{EF}$ at V. As shown in the figure, label the angles as $\angle 1$, $\angle 2$, $\angle 3$ and $\angle 4$.

Proof:

Statements	Reasons
$\overline{RU} \parallel \overline{LX}$	Construction
$\overline{AB} \parallel \overline{CD}$	Given
$\therefore$ MNUR is a parallelogram	By definition

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Thus $\overline{MN} \cong \overline{RU}$ .....(i)	Opposite sides of a parallelogram
Similarly $\overline{NP} \cong \overline{SV}$ .....(ii)	
But $\overline{MN} \cong \overline{NP}$ .....(iii)	Given
$\therefore \quad \overline{RU} \cong \overline{SV}$	From (i),(ii) and (iii)
$\therefore \quad \overline{RU} \parallel \overline{SV}$	Both are congruent to $\overline{LX}$
Now in $\Delta RUS \leftrightarrow \Delta SVT$	
$\overline{RU} \cong \overline{SV}$	(Proved)
$\angle 1 \cong \angle 2$	Corresponding angles
$\angle 3 \cong \angle 4$	Corresponding angles
$\therefore \Delta RUS \cong \Delta SVT$	S.A.A $\cong$ S.A.A
Hence $\overline{RS} \cong \overline{ST}$	CPCTC

Corollary - 1

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If a line is drawn from the midpoint of one side of a triangle and parallel to other side, it will bisect the third side

Corollary - 2

If a line is drawn parallel from the midpoint of a non-parallel side of a Trapezoid to the parallel sides, it will bisect the other non-parallel side.

Note

A trapezoid is a quadrilateral with exactly one pair of parallel sides. The parallel sides are the bases.

Corollary - 3

If one side of a triangle is divided into congruent segments, the line drawn from the points of division parallel to the other side will make congruent segments on third side.

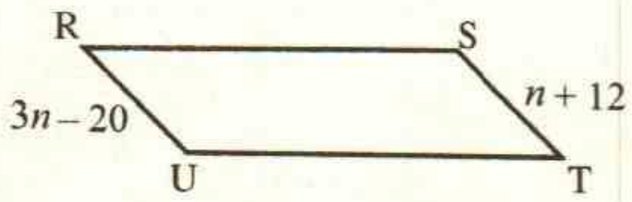
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Exercise 11.1

1. Measure of one of the angles of parallelogram is 70°. Find the measure of the remaining angles.

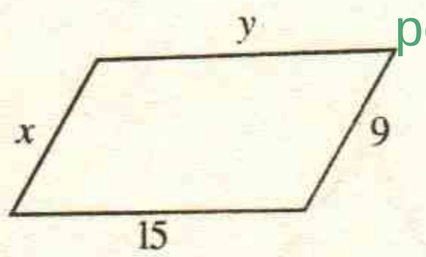
2. Measure of one of the exterior angles of a parallelogram is 125°. Find the measure of all of its interior angles.

3. RSTU is a parallelogram. Find ST.



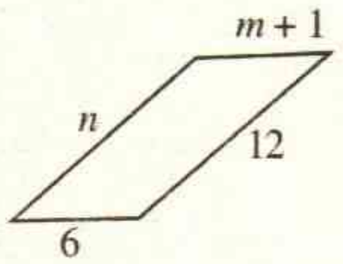
4. Find the value of each variable in the parallelogram using the properties.

i)

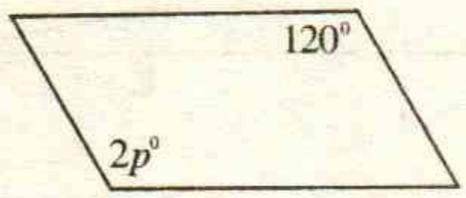


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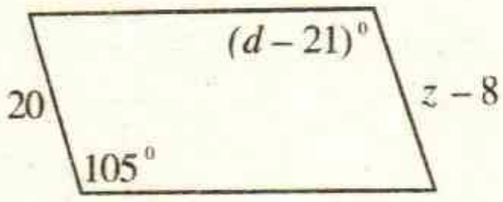
ii)



iii)

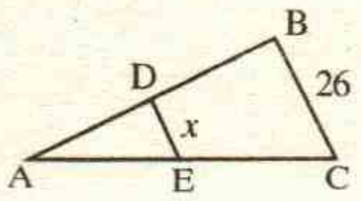


iv)

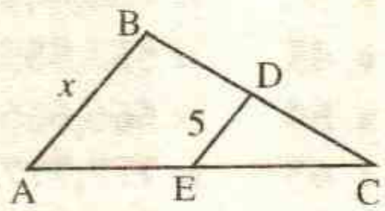


5. $\overline{DE}$ is a midsegment of ΔABC. Find the value of x.

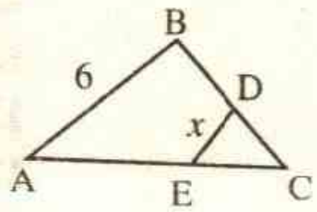
i)



ii)



iii)



6. Prove that diagonals of a rhombus bisect its angles.

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7. In rhombus MNOP, $m\angle N = 60°$. What is $m\angle O$?

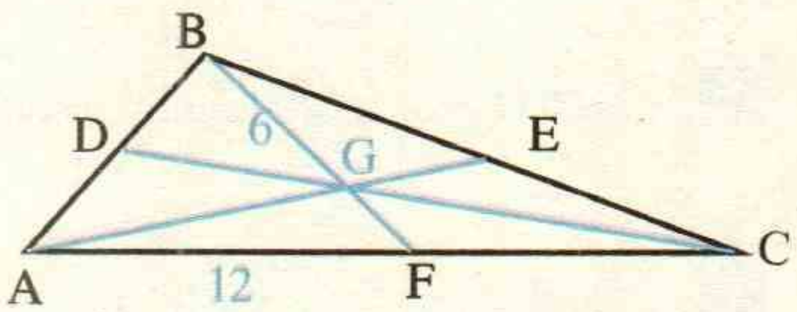
8. Prove that diagonals of a rectangle are congruent.

9. In a ΔABC, D and E are two points on $\overline{AB}$ and $\overline{AC}$, such that $m\overline{AD} = \frac{1}{4} m\overline{AB}$ and $m\overline{AE} = \frac{1}{4} m\overline{AC}$. Prove that $m\overline{DE} = \frac{1}{4} m\overline{BC}$.

10. G is the centroid of ΔABC, BG = 6, AF = 12 and AE = 15. Find the length of the segment.

i) $\overline{FC}$ ii) $\overline{BF}$

iii) $\overline{AG}$ iv) $\overline{GE}$

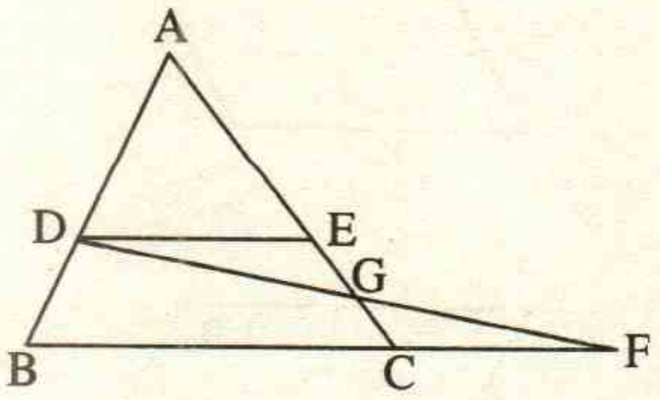


11. Prove that the four triangles formed by joining the mid points of the three sides of a triangle are congruent to one another.

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12. In the given ΔABC, D and E are the mid-points of $\overline{AB}$ and $\overline{AC}$, and $m\overline{CF} = \frac{1}{2} m\overline{BC}$.

Prove that $m\overline{CG} = \frac{1}{3} m\overline{AG}$.



DID YOU KNOW?

12,345,679 • 9	= 111,111,111
12,345,679 • 18	= 222,222,222
12,345,679 • 27	= 333,333,333
12,345,679 • 36	= 444,444,444
12,345,679 • 45	= 555,555,555
12,345,679 • 54	= 666,666,666
12,345,679 • 63	= 777,777,777
12,345,679 • 72	= 888,888,888
12,345,679 • 81	= 999,999,999

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REVIEW EXERCISE 11

1. Select the correct answer.

i) Which quadrilateral must have diagonals that are congruent and perpendicular?

a) Rhombus b) Square
c) Trapezoid d) Parallelogram

ii) How many equilateral triangles can be made by joining the midpoints of the sides of an equilateral triangle.

a) 2 b) 3
c) 4 d) cannot be determined

iii) Which statement must be true?

a) If a parallelogram is not a rectangle, then it is not a square.
b) If a parallelogram is not a square, then it is not a rectangle.
c) All rectangles are squares.
d) If a parallelogram is a rectangle, then it is a square.

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iv) Which quadrilateral's diagonals are perpendicular to each other and bisect the figure's opposite angles?

a) Trapezoid b) Rectangle
c) Rhombus d) Parallelogram

v) If opposite angles of a quadrilateral are equal in measure and none of them is a right angle, then the quadrilateral is a

a) Square b) Parallelogram
c) Trapezoid d) Rectangle

vi) Medians of a triangle are divided by the point of concurrency in the ratio

a) 2 : 1 b) 2 : 3
c) 1 : 3 d) None of these

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vii) Centroid is the point of concurrency of

a) medians of a triangle b) Angle bisectors of a triangle
c) Altitudes of a triangle d) Perpendicular bisectors of a triangle.

viii) If sum of the measures of $\angle A$ and $\angle C$ of a parallelogram ABCD is 120°, then $m\angle B =$

a) 25° b) 50°
c) 65° d) None of these

ix) Diagonals of a square are _____ to each other.

a) Perpendicular b) Not congruent
c) Congruent d) Parallel

x) Sum of the measures of interior angles of a quadrilateral is

a) 2 right angles b) 4 right angles
c) 3 right angles d) None of these

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2. Measure of one of the angles of parallelogram is 60°. Find the measure of the remaining angles.

3. Measure of one of the exterior angles of a parallelogram is 130°. Find the measure of all of its interior angles.

4. Prove that the line segments joining the mid-points of the sides of a quadrilateral taken in order, form a parallelogram.

5. In rhombus MNOP, $m\angle N = 70°$. What is $m\angle O$?

6. Measure of one of the angles of parallelogram is 50°. Find the measure of the remaining angles.

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SUMMARY

- A polygon with four sides is called a **quadrilateral**.
- A **parallelogram** is a quadrilateral in which opposite sides are parallel.
- A **rhombus** is a quadrilateral in which all sides are congruent.
- A **rectangle** is a quadrilateral in which all angles are congruent.
- A **Square** is a quadrilateral in which all sides are congruent and all angles are congruent.
- In fact rectangle, rhombus and Square are parallelograms.
- A polygon with three sides is called a triangle.
- The opposite sides of a parallelogram are congruent.
- The opposite angles of a parallelogram are congruent.
- The diagonals of a parallelogram bisect each other.
- A diagonal of a parallelogram divides the parallelogram into two congruent triangles.
- If two pairs of opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram.
- The line segment, joining the midpoints of two sides of a triangle, is parallel to the third side and is equal to one half of its length.
- If three or more parallel lines make congruent intercepts on a transversal, they also intercept congruent segments on any other line that cuts them.
- If a line is drawn from the midpoint of one side of a triangle and parallel to other side, it will bisect the third side.
- If a line is drawn parallel from the midpoint of a non-parallel side of a Trapezoid to the parallel sides, it will bisect the other non-parallel side.
- If one side of a triangle is divided into congruent segments, the line drawn from the points of division parallel to the other side will make congruent segments on third side.
- The point through which all the medians of the sides of a triangle pass is called centroid of the triangle.

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UNIT 12

LINE BISECTORS AND ANGLE BISECTORS

After reading this unit, the students will be able to:

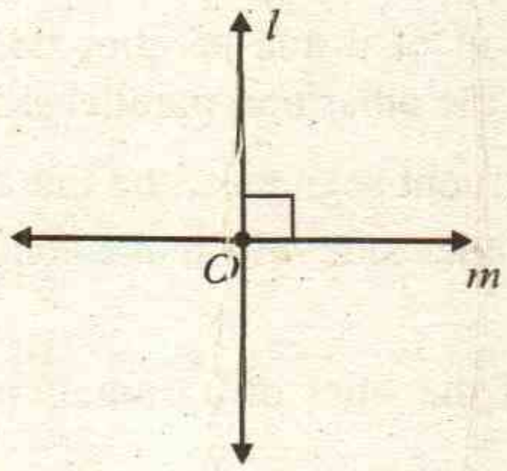
Prove the following theorems along with corollaries and apply them to solve appropriate problems.

- Any point on the right bisector of a line segment is equidistant from its end points.
- Any point equidistant from the points of a line segment is on the right bisector of it.
- The right bisectors of the sides of a triangle are concurrent.
- Any point on the bisector of an angle is equidistant from its arms.
- Any point inside an angle, equidistant from its arms, is on the bisector of it.
- The bisectors of the angles of a triangle are concurrent.

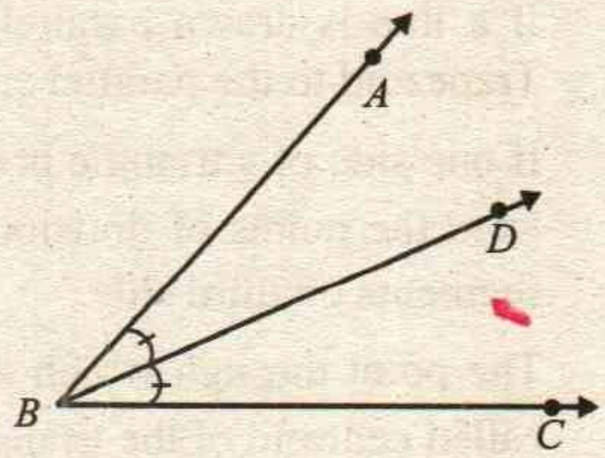
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Definitions

A segment bisector is a line that divides a segment into two congruent parts. The point where a bisector intersects a segment is the midpoint of the segment. A bisector that is perpendicular to a segment is called a perpendicular bisector. An angle bisector is a line or ray that divides an angle into two congruent angles.



l is perpendicular to m.



DB is the angle bisector of $\angle ABC$.

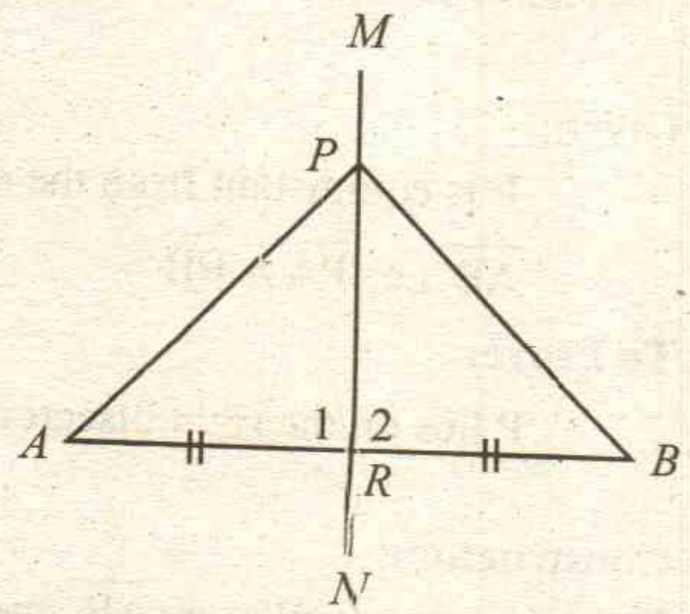
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THEOREM 12.1

Any point on the right bisector of a line segment is equidistant from end points of the segment.

Given:

A line MN intersects the line segment AB at the point R such that MN is the right bisector of $\overline{AB}$ which cuts $\overline{AB}$ at R. Let P be a point on MN.



Construction:

Join P to the points A and B.

To Prove:

P is equidistant from A and B i.e. $m\overline{PA} = m\overline{PB}$

Proof:

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Statements	Reasons
In $\Delta PRA \leftrightarrow \Delta PRB$	
$PR \cong PR$	(Common)
$AR \cong BR$	(Given)
$\angle 1 \cong \angle 2$	(Right angles)
$\therefore \Delta PRA \cong \Delta PRB$	(S.A.S $\cong$ S.A.S)
Hence $\overline{PA} \cong \overline{PB}$	(Corresponding sides of congruent triangles)

i.e. P is equidistant from A and B. Since P is arbitrary so any point taken on MN will be equidistant from A and B.

Example 1

$\overleftrightarrow{BD}$ is the perpendicular bisector of $\overline{AC}$. Find AD.

Solution:

$AD = CD$ perpendicular bisector theorem

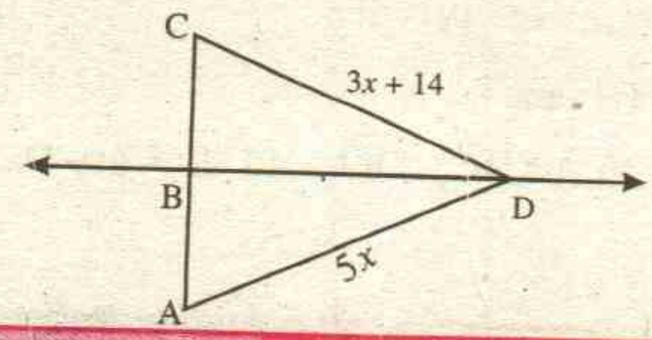
$5x = 3x + 14$ substitute.

$x = 7$ solve for x.

$AD = 5x$

$= 5(7)$

$= 35$



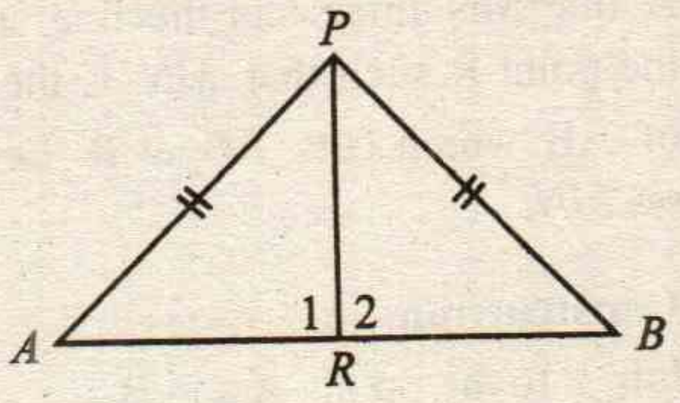
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THEOREM 12.2

Any point equidistant from the end points of a line segment is on the right bisector of it.

Given:

P is equidistant from the end points of $\overline{AB}$ i.e. $\overline{PA} \cong \overline{PB}$



To Prove:

P lies on the right bisector of $\overline{AB}$.

Construction:

Draw a perpendicular on $\overline{AB}$, meeting $\overline{AB}$ at R.

Proof:

Statements	Reasons
In $\Delta PRA \leftrightarrow \Delta PRB$	
$\overline{PA} \cong \overline{PB}$	(Given)
$\overline{PR} \cong \overline{PR}$	(Common)
$m\angle 1 = m\angle 2 = 90°$	(Construction)
$\therefore \Delta PRA \cong \Delta PRB$	(S.S.S $\cong$ S.S.S)
$\therefore \overline{PR} \cong \overline{PR}$	(Corresponding angles of congruent triangles)
$\therefore$ $\overline{PR}$ is the right bisector of $\overline{AB}$.	(By def., and Construction)
or P lies on the right bisector of $\overline{PR}$.	

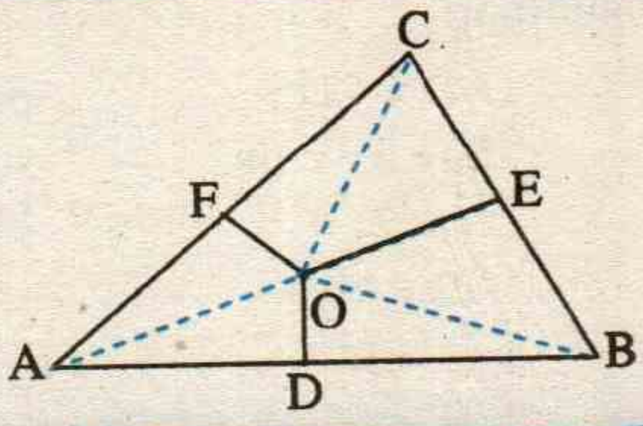
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THEOREM 12.3

The right bisectors of the sides of a triangle are concurrent.

Given:

In ΔABC, $\overline{DO}$ and $\overline{EO}$ are the right bisectors



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of $\overline{AB}$ and $\overline{BC}$ respectively, and intersect each other at O. O is joined to F, the mid-point of $\overline{AC}$.

To Prove:

The right bisectors of the sides AB, BC, AC of a triangle are concurrent at O.

Construction:

Draw the right bisectors of $\overline{AB}$ and $\overline{BC}$ which intersect each other at O. Join O to A, B and C.

Proof:

Statements	Reasons
In ΔAOB	
O lies on the right bisector of $\overline{AB}$.	(Given)
$\therefore \quad \overline{AO} \cong \overline{BO}$ -------- (i)	(Point on the right bisector of a line segment is equidistant from its end points.)
Similarly in ΔBOC	
$\overline{BO} \cong \overline{CO}$ -------- (ii)	(O is on the right bisector of $\overline{BC}$)
Hence $\overline{AO} \cong \overline{CO}$	(From (i) and (ii))
or, O is on the right bisector of $\overline{AC}$	(O is equidistant from A and C)
i.e. O lies on all the right bisectors $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$.	

Hence the right bisectors of the three sides of a triangle are concurrent at O.

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THEOREM 12.4

Any point on the bisector of an angle is equidistant from its arms.

Given:

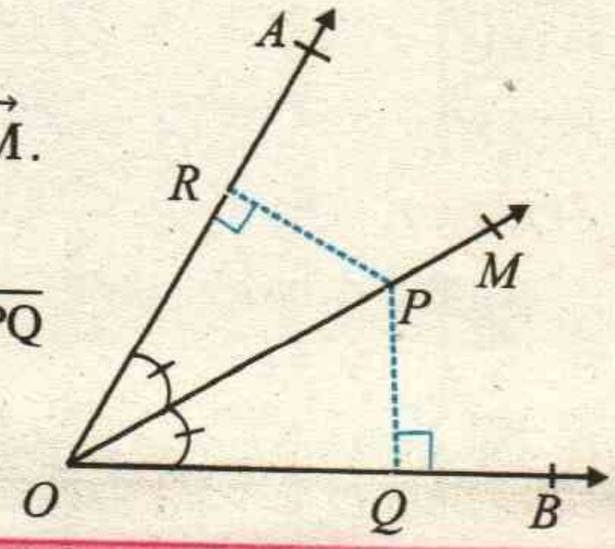
$\overrightarrow{OM}$ is the bisector of $\angle AOB$. P is any point on $\overrightarrow{OM}$.

To Prove:

P is equidistant from the arms of $\angle AOB$ i.e. $\overline{PR} \cong \overline{PQ}$

Construction:

Draw $\overline{PR} \perp \overrightarrow{OA}$ and $\overline{PQ} \perp \overline{OB}$



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Proof:

Statements	Reasons
In $\Delta PRO \leftrightarrow \Delta POQ$	
$\angle POR \cong \angle POQ$	(Given)
$\angle PQO \cong \angle PRO$	(Right angles)
$\overline{PO} \cong \overline{PO}$	(Common)
$\therefore \Delta PRO \cong \Delta POQ$	$(A \cdot A \cdot S \cong A \cdot A \cdot S)$
Hence $PR \cong PQ$	CPCTC

i.e. P is equidistant from $\overrightarrow{BA}$ and $\overrightarrow{BC}$, the arms of $\angle AOB$.

THEOREM 12.5

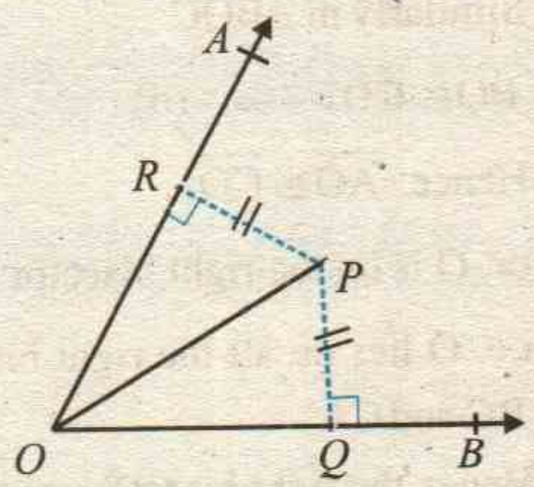
Any point inside an angle, equidistant from its arms, is on its bisector.

Given:

P is a point lies inside $\angle AOB$, such that $\overline{PR} \cong \overline{PQ}$,

where $\overline{PR} \perp \overrightarrow{OA}$ and $\overline{PQ} \perp \overrightarrow{OB}$

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To Prove:

OP is on the bisector of $\angle AOB$.

Construction:

Join P to O. Draw perpendicular PR and PQ on OA and OB

Proof:

Statement	Reason
In $\Delta OPR \leftrightarrow \Delta OPQ$	
$\overline{PO} \cong \overline{PO}$	(Common)
$\overline{PR} \cong \overline{PQ}$	(Given)
$\angle PRO \cong \angle PQO$	(Right angles)
$\therefore \Delta OPR \cong \Delta OPQ$	(S.S.A $\cong$ S.S.A)
Hence $\angle POR \cong \angle POQ$	CPCTC
or, $\overline{OP}$ is the bisector of $\angle AOB$.	

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Example 2

For what values of x does P lies on the bisector of $\angle A$?

Solution:

From the converse of the angle bisector theorem P lies on the bisector of $\angle A$ if P is equidistant from the sides of $\angle A$, so when BP = CP.

BP = CP	set segment lengths equal
$x + 3 = 2x - 1$	substitute expressions for segments lengths
$4 = x$	solve for x

Thus the point P lies on the bisector of $\angle A$ when $x = 4$.

An angle bisector of a triangle is the bisector of the interior angle of the triangle.

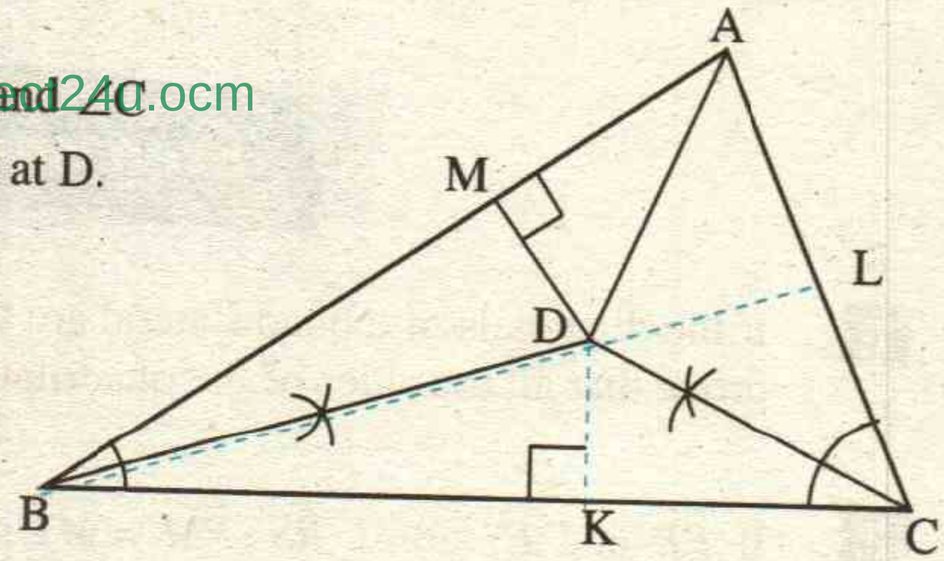
THEOREM 12.6

The bisectors of the angles of a triangle are concurrent.

Given:

$\overline{BD}$ and $\overline{CD}$ are the bisectors of $\angle B$ and $\angle C$ of ΔABC, which intersect each other at D.
D is joined to A.

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To Prove:

$\overline{DA}$ is the bisector of $\angle A$.

Construction:

From D, draw $\overline{DK} \perp \overline{BC}$,
$\overline{DL} \perp \overline{CA}$ and $\overline{DM} \perp \overline{AB}$.

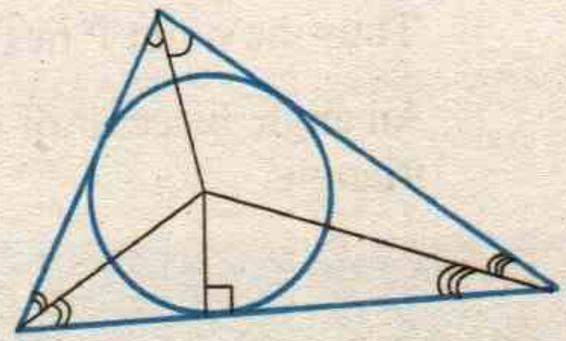
Proof:

Statements	Reasons
∵ D lies on the bisector of $\angle B$	(Given)
$\therefore \overline{DK} \cong \overline{DM}$ --------- (i)	(Distance of D from the arms of $\angle B$)
Similarly D lies on the bisector of $\angle C$	
$\therefore \overline{DK} \cong \overline{DL}$ ---------- (ii)	(Distance of D from the arms of $\angle C$)

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Hence $\overline{DL} \cong \overline{DM}$	(From (i) and (ii))
i.e. D lies on the bisector of $\angle A$	($\because$ D is equidistant from L and M)
or, $\overline{AD}$ is the bisector of $\angle A$.	
or, the bisectors of the angles of the triangle ABC are concurrent.	

The point of concurrency of the three angle bisectors of a triangle is called the **incenter** of the triangle. The incenter always lies inside the triangle.



The incenter is equidistant from the three sides of the triangle, a circle drawn using incenter and the distance to one side as radius is said to be inscribed within the triangle.

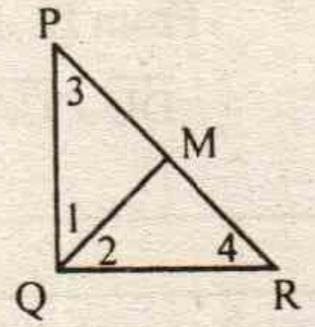
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Exercise 12.1

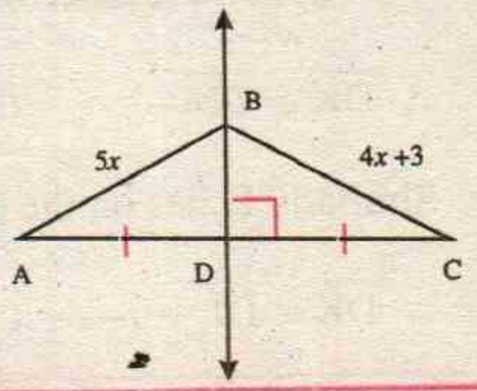
1. If the diagonals of a quadrilateral are the right bisectors of each other, then prove that all the sides of the quadrilateral are congruent.

2. If $\overline{PR}$ and $\overline{TS}$ are $\perp \overline{RS}$, $\overline{PM} = \overline{MT}$,
Prove that $\Delta PRM \cong \Delta MTS$

3. If $\angle 3 \cong \angle 4$ *and* $\overline{QM}$ bisects $\angle PQR$, prove that M is the midpoint of $\overline{PR}$.



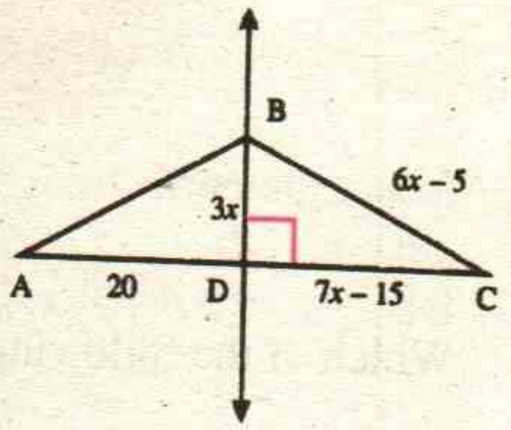
4. Find the length of AB.



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5. In the diagram, $\overline{BD}$ is the perpendicular bisector of $\overline{AC}$.



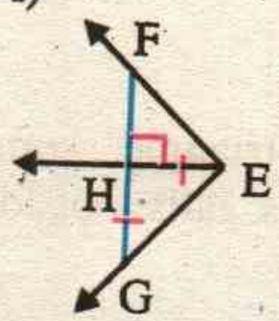
i) What segment lengths are equal?

ii) What is the value of x?

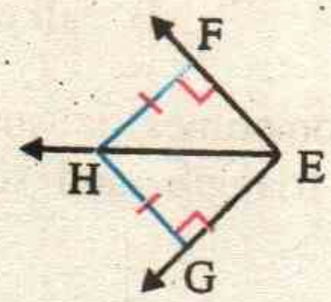
iii) Find AB.

6. Can we conclude that $\overline{EH}$ bisects $\angle FEG$?

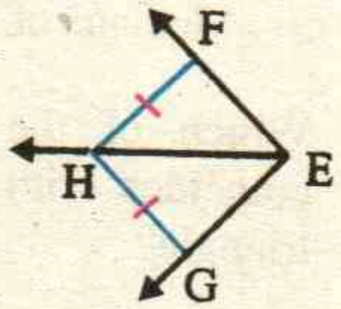
i)



ii)



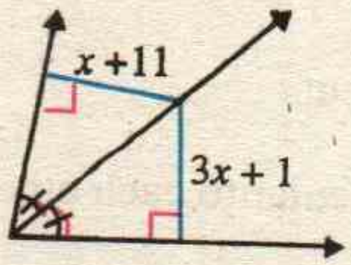
iii)



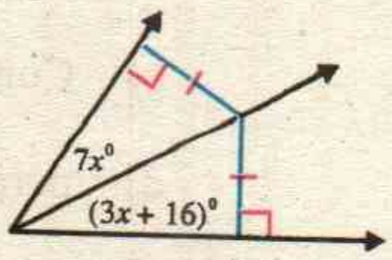
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7. Find the value of x.

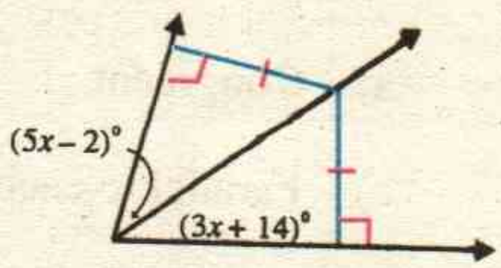
i)



ii)

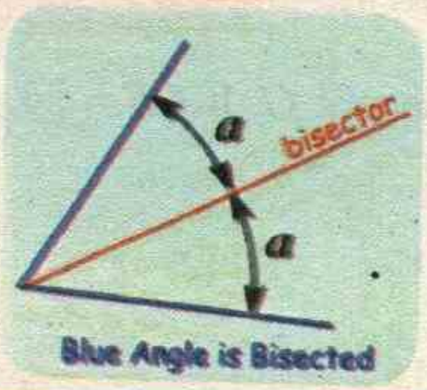


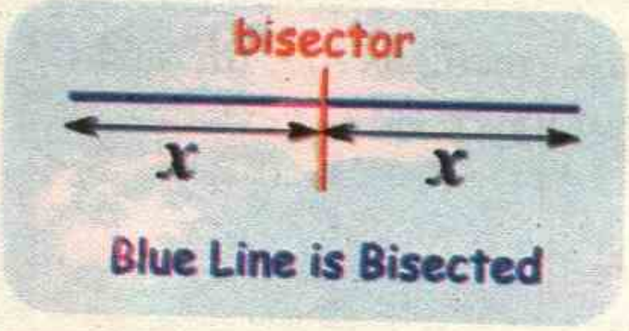
iii)



8. Prove that the diagonals of a square are the right bisectors of each other.

Tid-Bit





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REVIEW EXERCISE 12

1. Select the correct answer and write the corresponding letter a, b, c or d in the box.

i) Which of the following are concurrent?

a) Angles bisectors of a triangle

b) Perpendicular bisectors of the sides of a triangle.

c) Medians of a triangle

d) all of these

ii) Which of the following sometimes is inside a triangle, sometimes coincides with a side of a triangle, and sometimes falls outside of a triangle?

a) the base

b) the altitude

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c) the median

d) the angle bisector

iii) Perpendicular bisectors of a triangle are

a) Congruent

b) Concurrent

c) Parallel to each other

d) perpendicular to each other

iv) In which triangle the perpendicular bisector of the base passes through its vertex angle.

a) Right angled

b) Scalene

c) Isosceles

d) Acute-angled

v) In ΔABC, medians AD, BE, and CF intersect at G. If $CF = 24$, what is the length of FG?

a) 8

b) 12

c) 10

d) 16

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vi) The angle bisectors of a triangle meet at a point which is equidistant from _____ of the triangle.

a) the vertices b) the sides

c) midpoints of the sides d) all of these

vii) In an equilateral triangle, all the perpendicular bisectors are

a) congurent b) concurrent

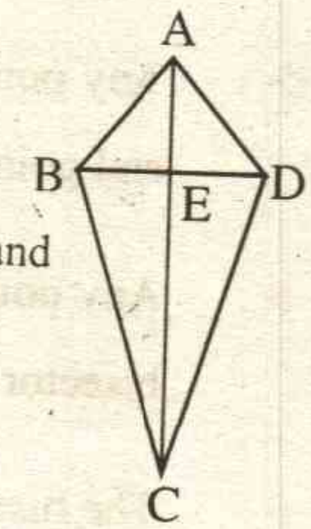
c) the angle bisectors as well d) parallel

viii) Point of intersection of the angle bisectors of a triangle is equidistant from _____ of the triangle.

a) the vertices b) the sides

c) mid points of the sides d) all of these

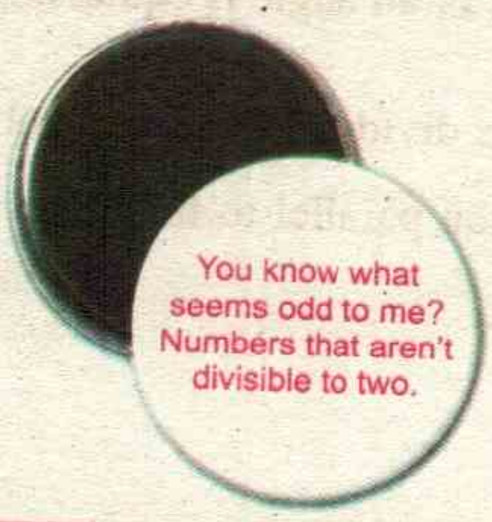
2. In the given figure $\overline{AB} \cong \overline{AD}$ and $\overline{BC} \cong \overline{DC}$. Prove that $\overline{AC} \perp$ BD and $\overline{BE} \cong \overline{DE}$.



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3. Prove that the diagonals of a rhombus are the right bisectors of each other.

4. Prove that bisectors of the base angles of an isosceles triangle intersect each other at the right bisector of the base.



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SUMMARY

- A segment bisector is a line that divides a segment into two congruent parts.
- The point where a bisector intersects a segment is the midpoint of the segment.
- A bisector that is perpendicular to a segment is called a perpendicular bisector.
- An angle bisector is a line or ray that divides an angle into two congruent angles.
- Any point on the right bisector of a line segment is equidistant from end points of the segment.
- Any point equidistant from the end points of a line segment is on the right bisector of it.
- The bisectors of the angles of a triangle are concurrent.
- The right bisectors of the sides of a triangle are concurrent.
- Any point on the bisector of an angle is equidistant from its arms.
- If one side of a triangle is divided into congruent segments, the line drawn from the points of division parallel to the other side will make congruent segments on third side.

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UNIT 13

SIDES AND ANGLES OF A TRIANGLE

After reading this unit, the students will be able to:

Prove the following theorems along with corollaries and apply them to solve appropriate problems.

- If two sides of a triangle are unequal in length, the longer side has an angle of greater measure opposite to it.
- If two angles of a triangle are unequal in measure, the side opposite to the greater angle is longer than the side opposite to the smaller angle.
- The sum of the lengths of any two sides of a triangle is greater than the length of the third side.
- From a point, out-side a line, the perpendicular is the shortest distance from the point to the line.

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THEOREM 13.1

If two sides of a triangle are unequal in length, the longer side has an angle of greater measure opposite to it.

Given:

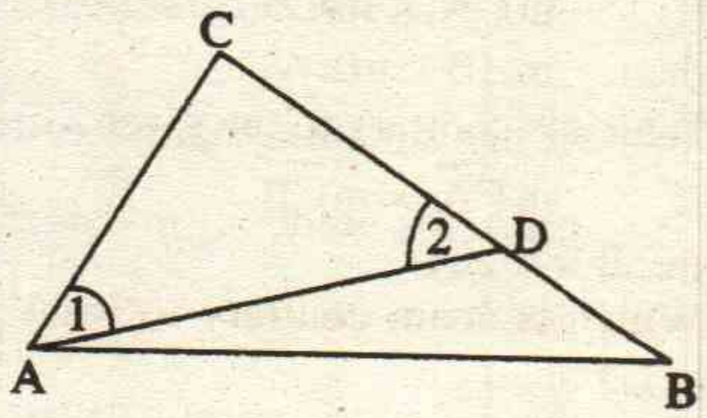
ΔABC with $m\overline{BC} > m\overline{AC}$

To Prove:

$m\angle CAB > m\angle B$

Construction:

Take a point D on a point $\overline{CB}$ such that $\overline{CD} \cong \overline{CA}$. Join D to A.



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Proof:

Statements	Reasons
In ΔADC	
$\overline{CA} \cong \overline{CD}$	(Construction)
$\therefore \quad \angle 1 \cong \angle 2$	(Angles opposite to congruent sides of ΔADC)
But $m\angle 2 > m\angle B$	(Exterior angle of ΔABD)
$\therefore\ m\angle 1 > m\angle B$---- (i)	($\because \angle 1 \cong \angle 2$)
Also $m\angle CAB > m\angle 1$ ------ (ii)	($\because m\angle CAB = m\angle 1 + m\angle DAB$)
$\therefore\ m\angle CAB > m\angle B$	(From (i) and (ii))

THEOREM 13.2

If two angles of a triangle are unequal in measure, the side opposite the greater angle is longer than the side opposite to the smaller angle.

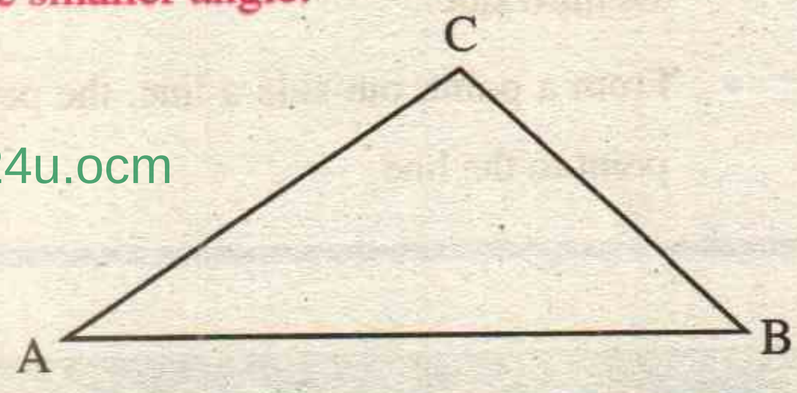
Given:

ΔABC in which $m\angle B > m\angle A$

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To Prove:

$m\overline{CA} > m\overline{CB}$



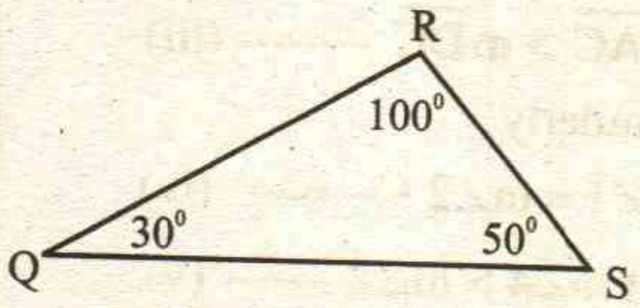
Proof:

Statements	Reasons
There are three possibilities	
i) $m\overline{CA} < m\overline{CB}$	
or ii) $m\overline{CA} = m\overline{CB}$	(Trichotomy Property)
or iii) $m\overline{CA} > m\overline{CB}$	
If $m\overline{CA} < m\overline{CB}$	
then $m\angle B < m\angle A$	(Angle opposite to smaller side)
Which is contrary to the given.	
If $m\overline{CA} = m\overline{CB}$	
$m\angle B = m\angle A$	(Base angles of an isosceles triangle)
Which is again contrary to what is given.	
Hence the only possibility that holds is $m\overline{CA} > m\overline{CB}$	

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Example 1

List the sides of the given triangle in increasing order.



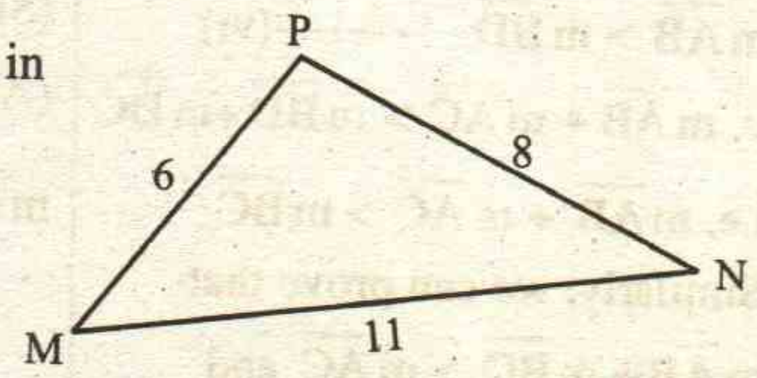
Solution:

As $30 < 50 < 100$,

then $RS < QR < QS$.

Example 2

List the angles of the given triangle in increasing order.



Solution:

As $6 < 8 < 11$,

then $m\angle N < m\angle M < m\angle P$

THEOREM 13.3

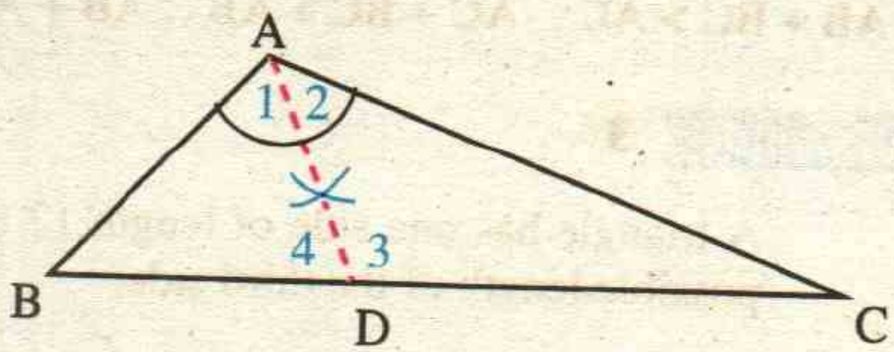
Triangle Inequality Theorem

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The sum of the lengths of any two sides of a triangle is greater than the length of the third side.

Given:

A triangle ABC.



To Prove:

$m\overline{AB} + m\overline{AC} > m\overline{BC}$

$m\overline{AB} + m\overline{BC} > m\overline{AC}$

$m\overline{AC} + m\overline{BC} > m\overline{AB}$

Construction:

Bisect $\angle A$. Let the bisector meets $\overline{BC}$ at D.

Proof:

Statements	Reasons
$m\angle 1 = m\angle 2$ ---------- (i)	($\overline{AD}$ is the bisector of $\angle A$)
but $m\angle 3 > m\angle 1$ ------ (ii)	(Exterior angle of ΔABD)
$\therefore\ m\angle 3 > m\angle 2$	(From (i) and (ii))

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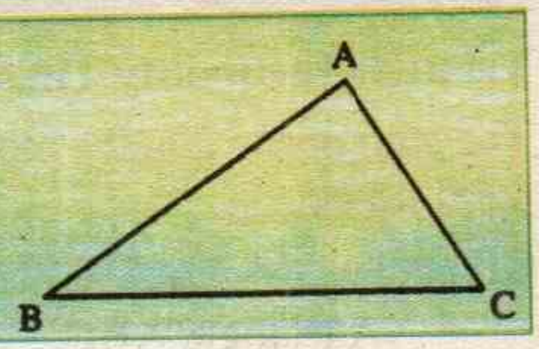
This implies	
$m\overline{AC} > m\overline{DC}$ -------- (iii)	(Side opposite to bigger angle)
Similarly	
$m\angle 1 = m\angle 2$ ---------- (iv)	($\overline{AD}$ is the bisector of $\angle A$)
but $m\angle 4 > m\angle 2$ ------ (v)	(Exterior angle of ΔACD)
$\therefore\ m\angle 4 > m\angle 1$	(From (iv) and (v))
This implies	
$m\overline{AB} > m\overline{BD}$ ---------(vi)	(Side opposite to bigger angle)
$\therefore\ m\overline{AB} + m\overline{AC} > m\overline{BD} + m\overline{DC}$	(Adding (iii) and (vi))
i.e. $m\overline{AB} + m\overline{AC} > m\overline{BC}$	$m\overline{BC} = m\overline{BD} + m\overline{DC}$
Similarly, we can prove that	
$m\overline{AB} + m\overline{BC} > m\overline{AC}$ and	
$m\overline{AC} + m\overline{BC} > m\overline{AB}$	

Triangle inequality Theorem

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The sum of the lengths of any two sides of a triangle is greater than the length of the third side.

$AB + BC > AC$, $AC + BC > AB$, $AB + AC > BC$.



Example 3

A triangle has one side of length 12 and another of length 8. Describe the possible length of the third side.

Solution:

Let x represent length of the third side.

Draw diagram to help visualize the small and large values of x.

Then use the triangle inequality theorem to write and solve inequalities.

Small values of x

$x + 8 > 12$

$x > 4$

Large values of x

$8 + 12 > x$

$20 > x$

$\Rightarrow\ x < 20$

The length of the third side must be greater than 4 and less than 20.

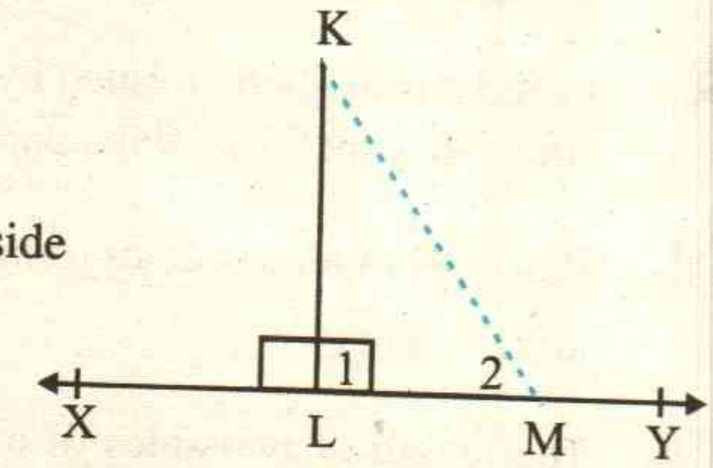
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THEOREM 13.4

From a point outside a line, the perpendicular is the shortest distance from the point to the line.

Given:

$\overline{KL} \perp \overleftrightarrow{XY}$, where K is a point outside $\overleftrightarrow{XY}$.



To Prove:

$m\overline{KL}$ is the shortest of all the distances from K to $\overleftrightarrow{XY}$

Construction:

Take a point M on $\overleftrightarrow{XY}$ other than L. Join M to K. ΔKLM is formed.

Proof:

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Statements	Reasons
In ΔKLM	
$m\angle 1 = 90°$	(Given)
$\therefore \quad m\angle 1 > m\angle 2$	(Only one angle can be a right angle in a triangle)
or $\quad m\overline{KM} > m\overline{KL}$	(Side opposite to greater angle)
or $\quad m\overline{KL} < m\overline{KM}$	(Reflexive Property)

Similarly it can be proved that the distance of K from any point taken on $\overleftrightarrow{XY}$ except L will be greater than $\overline{KL}$.

Hence perpendicular $\overline{KL}$ is the shortest distance of point K from $\overleftrightarrow{XY}$.

DID YOU KNOW?

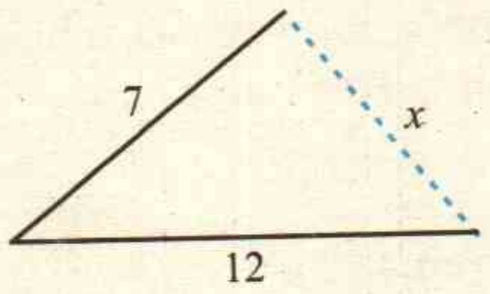
2,520 is the smallest number that can be exactly divided by all the numbers 1 to 10.

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Exercise 13.1

1. In the given figure, what values of x will make a triangle possible?

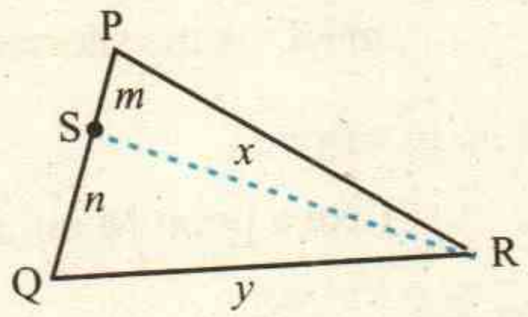


2. Could a triangle be formed by fastening three sticks that are 6 in.,10 in., and 3 in. long?

3. Base $\overline{QR}$ of an isosceles triangle PQR is extended to S. Prove that $m\overline{PS} > m\overline{PQ}$.

4. The length of two sides of a triangle are 11 and 23, If the third side is x, find the range of possible values for x.

5. If S is any point on the side PQ of ΔPQR, and S is joined to R, prove that $PQ + QR > PS + SR$.



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6. If two angles of a triangle are 45^0 and 70^0 respectively, i) which is the greater of the two opposite sides? ii)which is the shortest side of the triangle? iii) which is the longest?

7. If in ΔRST, $RS > ST$,and $m\angle S = 60^0$, i) which is the smallest side of the triangle? ii) which is the longest?

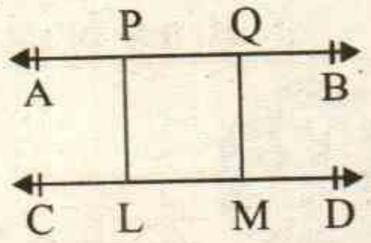
8. Which of the following sets of lengths could be the lengths of the sides of a triangle:

(a) 2 cm, 2 cm, 2 cm (b) 3 m, 4 m, 5 m

(c) 5 cm, 8 cm, 2 cm (d) 3 m, 3m, 2 m

(e) $1\frac{1}{2}m, 5m, 3\frac{1}{2}m$ (f) $2\frac{1}{2}cm,\ 3\frac{1}{2}cm,\ 4\frac{1}{2}cm$

9. $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$. If $\overline{PL}$ and $\overline{QM}$ are the shortest distances between $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ then prove that $\overline{PL} \parallel \overline{QM}$.



10. Prove that the perimeter of a triangle is greater than the sum of the measures of its altitudes.

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REVIEW EXERCISE 13

1. Select the correct answer.

i) An exterior angle of a triangle measures 120°. If measure of one of its remote interior angles is 40°, the measure of the second angle is

a) 40° b) 80°

c) 70° d) 120°

ii) In ΔABC, $m\angle A = 90°$, $m\angle B = 53°$, and $m\angle C = 37°$. Which expression correctly relates the lengths of the sides of this triangle?

a) AB <BC <CA b) AC <BC <AB

c) AB <AC <BC d) BC<AC <AB.

iii) In the figure P lies outside $\overleftrightarrow{AB}$. $m\overline{PR}$ will be the shortest distance if $m\angle PRA$ is

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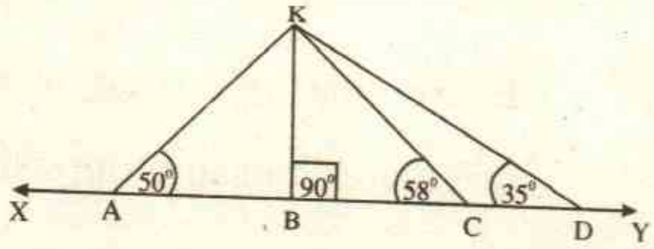
a) 180° b) 45°

c) 100° d) 90°

iv) In the given figure the point K lies outside $\overleftrightarrow{XY}$. Which of the following represents the shortest distance?

a) $m\overline{KD}$ b) $m\overline{KC}$

c) $m\overline{KA}$ d) $m\overline{KB}$



v) Measures of two sides of a triangle are 10 and 14. Which of the following can be its third side?

a) 2 b) 4

c) 22 d) 24

vi) In ΔABC, $m\angle A = 50°$ and $m\angle B = 30°$, Which of the following is correct?

a) $m\overline{BC} > m\overline{AB}$ b) $m\overline{AB} > m\overline{CA}$

c) $m\overline{BC} < m\overline{CA}$ d) $m\overline{AB} < m\overline{CA}$

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vii) Which of the following represents the sides of a triangle?

a) 3,4,and 5 b) 3,4,and 7

c) 3,4,and 8 d) 3,4,and 1

viii) In ΔKLM, $m\angle K = 45°$, $m\angle L = 55°$, and $m\angle M = 80°$. Which one of the following is the longest side?

a) KL b) LM

c) KM d) None of these

2. If two sides of a triangle are 8 in.and 12 in., what are the limiting values of the third side?

3. Can a triangle have its sides equal to 7 in.,5 in., and 12 in. ?

4. Prove that a line drawn from the vertex of an isosceles triangle to any point in the base is shorter than either of the equal sides.

5. D is a point inside a ΔABC. Prove that

$$m\overline{DA} + m\overline{DB} + m\overline{DC} > \frac{1}{2}(m\overline{AB} + m\overline{BC} + m\overline{CA})$$

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SUMMARY

- If two sides of a triangle are unequal in length, the longer side has an angle of greater measure opposite to it.
- If two angles of a triangle are unequal in measure, the side opposite the greater angle is longer than the side opposite to the smaller angle.
- The sum of the lengths of any two sides of a triangle is greater than the length of the third side.
- From a point outside a line, the perpendicular is the shortest distance from the point to the line.

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UNIT 14

RATIO AND PROPORTION

After reading this unit, the students will be able to:

Prove the following theorems along with corollaries and apply them to solve appropriate problems.

- A line parallel to one side of a triangle, intersecting the other two sides, divides them proportionally.
- If a line segment intersects the two sides of a triangle in the same ratio then it is parallel to the third side.
- The internal bisector of an angle of a triangle divides the side opposite to it in the ratio of the lengths of the sides containing the angle.
- If two triangles are similar, the measures of their corresponding sides are proportional.

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1. Students are already familiar with the use of ratio in Algebra and Arithmetic. They know that if (for example);

Area of $\Delta ABC = 18 cm^2$, and

Area of $\Delta DEF = 24 cm^2$, then

The ratio between their areas is 18 : 24 or 3 : 4.

If we take four quantities a, b, c, d such that the ratio between a and b is the same as the ratio between c and d, then the quantities are said to be in proportion.

i.e. $a : b :: c : d$ is a proportion or $\frac{a}{b} = \frac{c}{d}$.



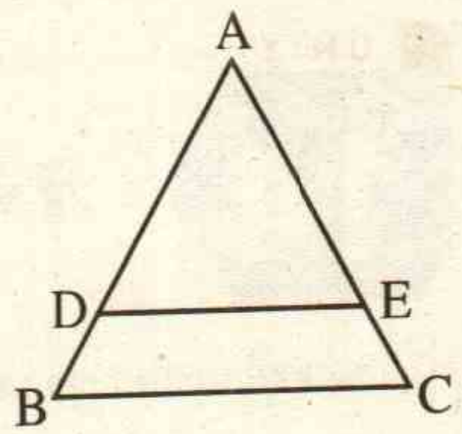
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2. If in ΔABC

$$\frac{m\overline{AD}}{m\overline{DB}} = \frac{m\overline{AE}}{m\overline{EC}}, \text{ then}$$

$$\frac{m\overline{AB}}{m\overline{DB}} = \frac{m\overline{AC}}{m\overline{EC}}, \quad \text{(Componendo property)}$$

and $$\frac{m\overline{AD}}{m\overline{AB}} = \frac{m\overline{AE}}{m\overline{AC}}$$ (Componendo and Invertendo property)



3. If in a correspondence $\Delta ABC \leftrightarrow \Delta DEF$, the corresponding angles are congruent then the triangles are similar i.e. $\Delta ABC \sim \Delta DEF$. Correspondence in this case is called 'Similarity'.

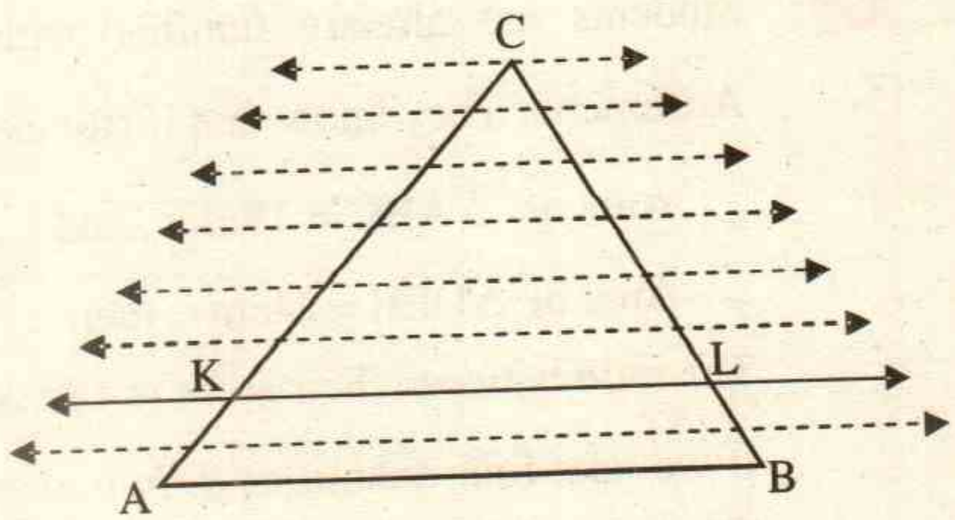
THEOREM 14.1

A line parallel to one side of a triangle, intersecting the other two sides, divides them proportionally.

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Given:

A triangle ABC in which $\overleftrightarrow{KL} \parallel \overline{AB}$. $\overleftrightarrow{KL}$ meets $\overline{CA}$ and $\overline{CB}$ at distinct points K and L respectively.



To Prove:

$$\frac{m\overline{CK}}{m\overline{KA}} = \frac{m\overline{CL}}{m\overline{LB}}$$

Construction:

Suppose the unit of length is taken such that $m\overline{CK} = p$ and $m\overline{KA} = q$ units (where p and q are whole numbers, $q \neq 0$)

i.e. $$\frac{m\overline{CK}}{m\overline{KA}} = \frac{p}{q}$$

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Divide $\overline{CK}$ into p congruent segments and $\overline{KA}$ into q congruent segments.

From each dividing point, draw lines parallel to $\overleftrightarrow{KL}$.

Proof:

Statements	Reasons
The parallel lines divide $\overline{CA}$ into $(p+q)$ congruent segments.	(Construction)
$\overline{CB}$ is another transversal which cuts these parallel lines.	
$\therefore$ these parallel lines divide $\overline{CB}$ also into $(p+q)$ congruent segments.	(Same number of congruent segments are intercepted on any other transversal)
i.e. $\overline{CL}$ is divided into p congruent segments and $\overline{LB}$ is divided into q congruent segments.	
Suppose the length of each congruent segments is 'a' unit.	
$\therefore \quad \frac{m\overline{CL}}{m\overline{LB}} = \frac{a \times p}{a \times q}$	
or $\frac{m\overline{CL}}{m\overline{LB}} = \frac{p}{q}$ -------- (i)	
But $\frac{m\overline{CK}}{m\overline{KA}} = \frac{p}{q}$ -------- (ii)	(Construction)
$\therefore \quad \frac{m\overline{CK}}{m\overline{KA}} = \frac{m\overline{CL}}{m\overline{CB}}$	(from (i) and (ii))

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Corollary

In ΔABC, if $\overleftrightarrow{KL} \,||\, \overline{AB}$, then

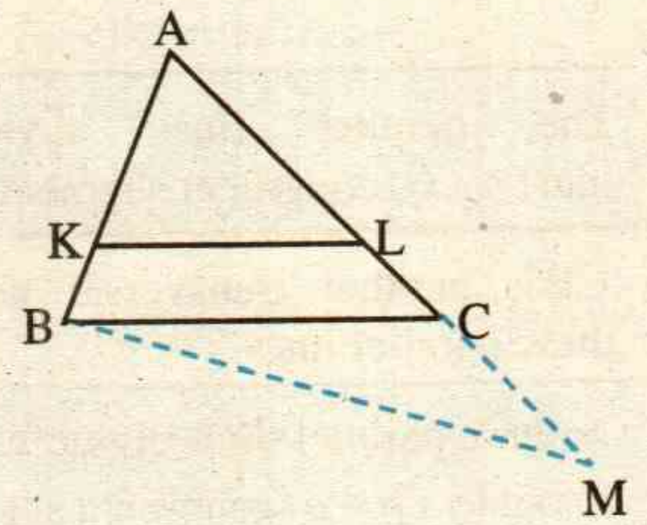
i) $\frac{m\overline{CA}}{m\overline{KA}} = \frac{m\overline{CB}}{m\overline{LB}}$ and ii) $\frac{m\overline{CK}}{m\overline{CA}} = \frac{m\overline{CL}}{m\overline{CB}}$

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THEOREM 14.2

If a line segment intersects the two sides of a triangle in the same ratio then it is parallel to the third side.



Given:

$\overleftrightarrow{KL}$ intersects two sides $\overline{AB}$ and $\overline{AC}$ of the ΔABC at K and L such that $\frac{m\overline{AK}}{m\overline{KB}} = \frac{m\overline{AL}}{m\overline{LC}}$

To Prove:

$\overline{KL} \parallel \overline{BC}$

Proof:

Statements	Reasons
Suppose $\overline{KL}$ is not parallel to $\overline{BC}$, then let $\overline{BM}$ be drawn through B, parallel to $\overline{KL}$, meeting $\overrightarrow{AC}$ at M.	
$\therefore \quad \frac{m\overline{AK}}{m\overline{KB}} = \frac{m\overline{AL}}{m\overline{LM}}$ --------- (i)	(Proportional segments are cut by line ∥ to one side of the triangle)
But $\frac{m\overline{AK}}{m\overline{KB}} = \frac{m\overline{AL}}{m\overline{LC}}$ --------- (ii)	(Given)
Thus $\frac{m\overline{AL}}{m\overline{LM}} = \frac{m\overline{AL}}{m\overline{LC}}$	(From (i) and (ii))
or $m\overline{LM} = m\overline{LC}$	(Denominators of equal fractions)

This is possible only when C and M coincide. Hence our supposition is wrong and $\overline{KL} \parallel \overline{BC}$

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Corollary - 1

In ΔABC, if $\frac{m\overline{AK}}{m\overline{AB}} = \frac{m\overline{AL}}{m\overline{AC}}$, then $\overline{KL} \parallel \overline{BC}$

Corollary - 2

In ΔABC, if $\frac{m\overline{AB}}{m\overline{KB}} = \frac{m\overline{AC}}{m\overline{LC}}$, then $\overline{KL} \parallel \overline{BC}$

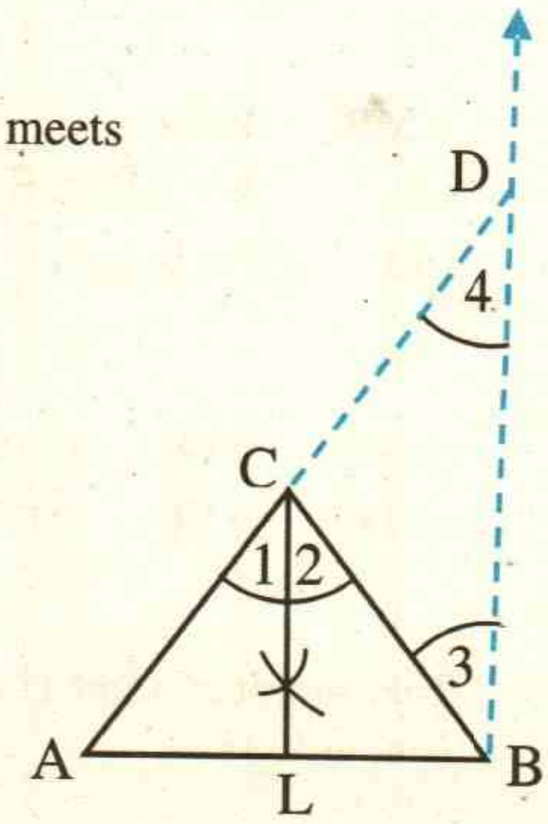
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THEOREM 14.3

The internal bisector of an angle of a triangle divides the side opposite to it in the ratio of the lengths of the sides containing the angles.

Given:

In ΔABC, $\overline{CL}$ is the bisector of $\angle C$, which meets the opposite side $\overline{AB}$ at L.



To Prove:

$$\frac{m\overline{AL}}{m\overline{BL}} = \frac{m\overline{AC}}{m\overline{BC}}$$

Construction:

Through B, draw $\overrightarrow{BD} \parallel \overline{LC}$, which intersects $\overline{AC}$ at D.

Proof:

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Statements	Reasons
$\overline{LC} \parallel \overrightarrow{BD}$	(Construction)
$\therefore \quad \angle 2 \cong \angle 3$ ------ (i)	(Alternate angles)
and $\quad \angle 1 \cong \angle 4$ ------ (ii)	(Corresponding angles)
But $\quad \angle 1 \cong \angle 2$	(Given)
$\therefore \quad \angle 3 \cong \angle 4$	(Transitive property)
Hence $\overline{BC} \cong \overline{DC}$	(Opposite sides of congruent angles)
Now in ΔABD	
$\overline{CL} \parallel \overline{DB}$	(Construction)
$\therefore \quad \frac{m\overline{AL}}{m\overline{BL}} = \frac{m\overline{AC}}{m\overline{DC}}$	(Proportional segments are cut by line ∥ to one side of the triangle)
or $\quad \frac{m\overline{AL}}{m\overline{BL}} = \frac{m\overline{AC}}{m\overline{BC}}$	($\because \quad \overline{BC} \cong \overline{DC}$)

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THEOREM 14.4

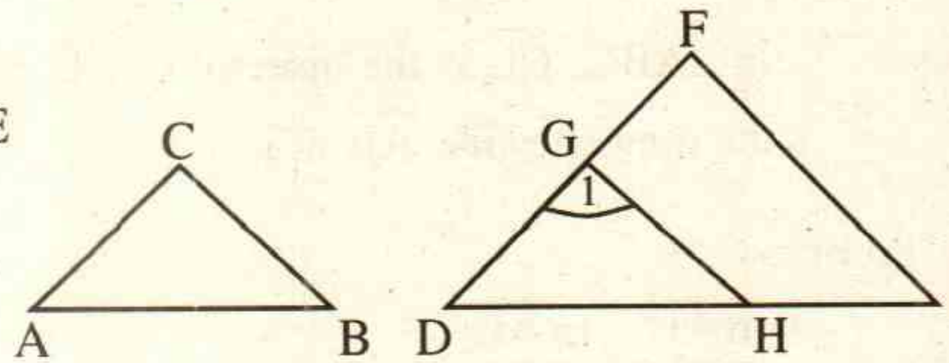
If two triangles are similar, then the measures of their corresponding sides are proportional.

Given:

$\Delta ABC \sim \Delta DEF$

i.e. $\angle A \cong \angle D, \angle B \cong \angle E$

and $\angle C \cong \angle F$.



To Prove:

$$\frac{m\overline{AB}}{m\overline{DE}} = \frac{m\overline{BC}}{m\overline{EF}} = \frac{m\overline{AC}}{m\overline{DF}}$$

Construction:

Take points G and H on $\overline{DF}$ and $\overline{DE}$ such that $\overline{AC} \cong \overline{DG}$ and $\overline{AB} \cong \overline{DH}$. Join G to H.

Proof:

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Statements	Reasons
In $\Delta ABC \leftrightarrow \Delta DHG$	
$\overline{AB} \cong \overline{DH}$	(Construction)
$\overline{AC} \cong \overline{DG}$	(Construction)
$\angle A \cong \angle D$	(Given)
$\therefore\ \Delta ABC \cong \Delta DHG$	(S.A.S $\cong$ S.A.S)
Hence $\angle C \cong \angle 1$	(Corresponding angles of congruent Δs)
But $\angle C \cong \angle F$	(Given)
$\therefore\ \angle 1 \cong \angle F$	(Transitive property)
But these are corresponding angles.	
$\therefore\ \overline{GH} \parallel \overline{FE}$	
Hence $\frac{m\overline{DG}}{m\overline{DF}} = \frac{m\overline{DH}}{m\overline{DE}}$	(Remaining sides proportional)

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or $\frac{m\overline{AC}}{m\overline{DF}} = \frac{m\overline{AB}}{m\overline{DE}}$ ---- (i)	($\because \overline{AC} \cong \overline{DG}$ and $\overline{AB} = \overline{DH}$)
Similarly it can be proved that	
$\frac{m\overline{AC}}{m\overline{DF}} = \frac{m\overline{BC}}{m\overline{EF}}$ ----- (ii)	
Hence $\frac{m\overline{AB}}{m\overline{DE}} = \frac{m\overline{BC}}{m\overline{EF}} = \frac{m\overline{AC}}{m\overline{DF}}$	(from (i) and (ii))

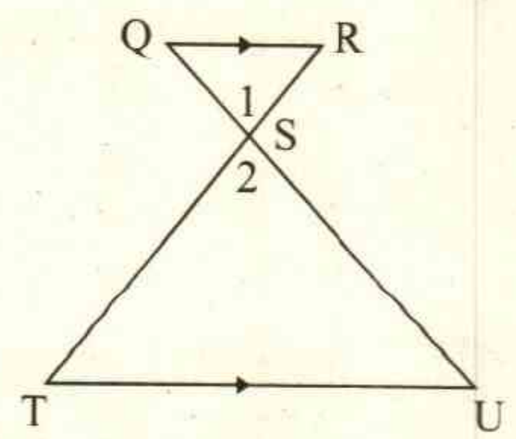
Exercise 14.1

1. The three angles of a triangle are in the ratio 1: 2: 6. What is the measure of the smallest angle?

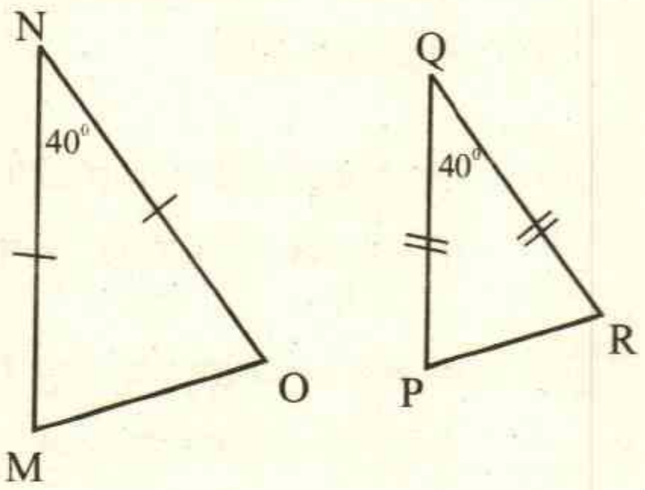
2. The measures of the three angles of a triangle are in the ratio 2:3:4. Find the measure of the largest angle of the triangle.

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3. In the given figure, show that ΔQRS and ΔTUS are similar. $(\overline{QR} \parallel \overline{TU})$



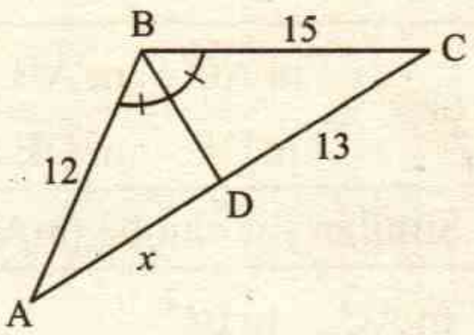
4. In the given figure show that $\Delta MNO \sim \Delta PQR$.



5. In ΔABC, DE is parallel to BC.
If mAD =1.5cm, mBD = 3cm, mAE = 1.3cm, then find mCE.

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6. In the given figure, find the value of x.



7. In ΔABC, $\overline{AD}$ is the bisector of $\angle A$. If $m\overline{AB} = 9$ cm, $m\overline{AC} = 10$ cm and $m\overline{BC} = 12$ cm, find the measures of $\overline{DC}$ and $\overline{BD}$.

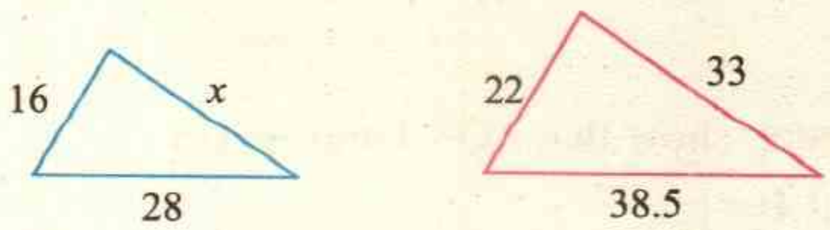
8. $\overline{PS}$ is the bisector of $\angle P$ in ΔPQR. If $m\overline{QS} = 3$cm, $m\overline{SR} = 7$ cm and $m\overline{PQ} + m\overline{PR} = 20$ cm, find the measure of $\overline{PQ}$ and $\overline{PR}$.

9. ABC is a triangle in which $m\overline{BC} = m\overline{CA} = 2m\overline{AB}$. Internal bisector of $\angle A$ meets $\overline{BC}$ at D and $\overline{DE}$ is drawn parallel to $\overline{CA}$ meeting $\overline{AB}$ at E. Find $m\overline{DE} : m\overline{AB}$.

10. Measures of the sides of a triangle are 6.5 cm, 7.8 cm and 9.1 cm. Find the lengths of the segments into which the smallest side is divided by the internal bisector of the opposite angle.

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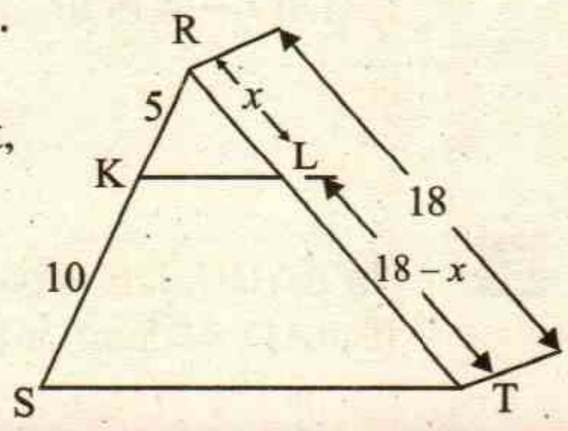
11. The given triangles are similar, find x.



12. If $\Delta ABC \sim \Delta DEF$ and $m\overline{AC} = 12$ cm, $m\overline{AB} = 4$ cm, $m\overline{BC} = 10$ cm and $m\overline{DF} = 9$ cm, then find the measures of $\overline{DE}$ and $\overline{EF}$.

13. Diagonals of a trapezoid ABCD intersect each other at O. If $\Delta AOB \sim \Delta COD$, then prove that $\overline{AB} \,||\, \overline{CD}$.

14. In the accompanying figure, the line segment, $\overline{KL}$, is drawn parallel to $\overline{ST}$, intersecting $\overline{RS}$ at K and $\overline{RT}$ at L in RST. If RK = 5, KS = 10, and RT = 18, then find RL.



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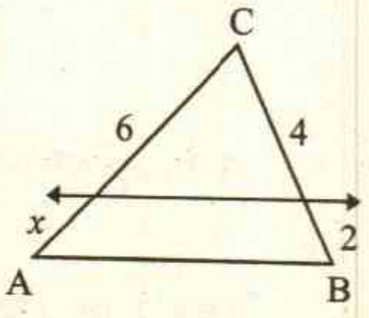
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REVIEW EXERCISE 14

1. Select the correct answer.

i) In the given figure compute x?

a) 7 b) 9 c) 2 d) 3



ii) Which of the following is not valid for proving triangles similarity?

a) SSS b) AA c) SSA d.) SAS

iii) In a mathematics class with 32 students, the ratio of girls to boys is 5 to 3. How many more girls are there than boys?

a) 2 b) 12 c) 8 d) 20

iv) The measure of a line segment joining the midpoints of $\overline{AB}$ and $\overline{AC}$ of ΔABC is 3.5 cm. Find m $\overline{BC}$ =

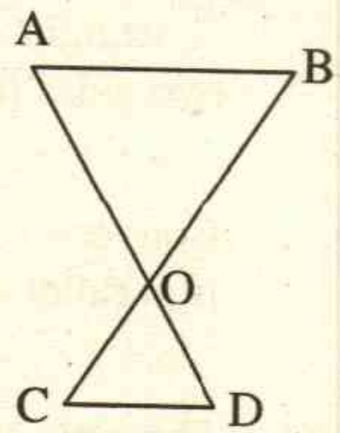
a) 4.5 cm b) 5.5 cm c) 6 cm d) 7 cm

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v) In the figure $\overline{AB} \parallel \overline{CD}$. If $\overline{AD}$ and $\overline{BC}$ intersects at O. Then ΔAOB and ΔDOC are

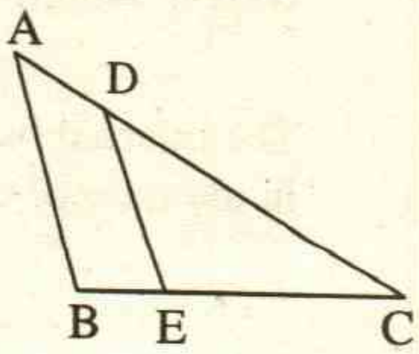
a) Congruent b) similar

c) Not similar d) none of these



2. Prove that a line passing through the midpoint of one side of a triangle and parallel to a second side bisects the third side of the triangle.

3. In the figure $\overline{AB} \parallel \overline{DE}$. If m $\overline{BE}$ = 4.1cm, m $\overline{EC}$ = 12.3cm and m $\overline{DC}$ = 16.5cm, find the measure of $\overline{AC}$.



4. Let $\overline{PQ}$ and $\overline{RS}$ intersect at O and $\frac{m\overline{PO}}{m\overline{OQ}} = \frac{m\overline{RO}}{m\overline{OS}}$.

Prove that $\Delta OPR \sim \Delta OQS$.

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SUMMARY

- Ratio is a comparison of two quantities. The ratio of a to b can be expressed $\frac{a}{b}$, where b is not zero. This ratio can also be written as a:b.

- A Proportion is an equation statisfing that two ratios are equal.

- Two triangles are similar if
  - Two angles of one triangle are equal to two angles of other triangle;
  - All the corresponding sides of two triangles are proportional;
  - An angles of one triangle is equal to an angle of other triangle,and the sides which include the equal angle of both triangles are proportional

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- A straight line drawn parallel to one side of a triangle, divides the other two sides proportionally.

- Conversely, if a line divides two sides of a triangle proportionally, then it is parallel to the third side.

- The ratio of the areas of any two similar figures is equal to the square of the ratio of any two corresponding lengths of the figures. If any two corresponding lengths of two similar figures are l_1 and l_2, while corresponding areas are $A_1$and A_2, then $A_1 / A_2 = (l_1 / l_2)^2$

- The internal bisector of an angle of a triangle divides the side opposite to it in the ratio of the lengths of the sides containing the angles.

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UNIT 15

PYTHAGORAS' THEOREM

After reading this unit, the students will be able to:

Prove the following theorems along with corollaries and apply them to solve appropriate problems.

- In a right-angled triangle, the square of the length of hypotenuse is equal to the sum of the squares of the lengths of the other two sides. (Pythagoras' theorem).
- If the square of one side of a triangle is equal to the sum of the squares of the other two sides then the triangle is a right angled triangle (converse to Pythagoras' theorem).

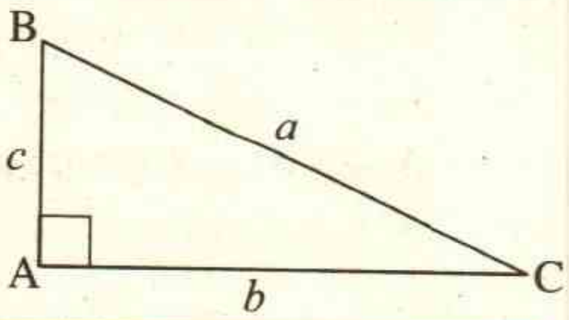
One of the most famous theorems in mathematics is the Pythagorean Theorem, named for the ancient Greek mathematician Pythagoras (around 500 B.C.). This theorem can be used to find information about the lengths of the sides of a right angle triangle.

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Note

We always denote the side opposite to angle A as a, the side opposite to angle B by b and the side opposite to angle C by c.

The following figure shows a right-angle triangle with BAC = 90°. The side opposite the right-angle A is called the hypotenuse. It is the longest side of a right-angled triangle.

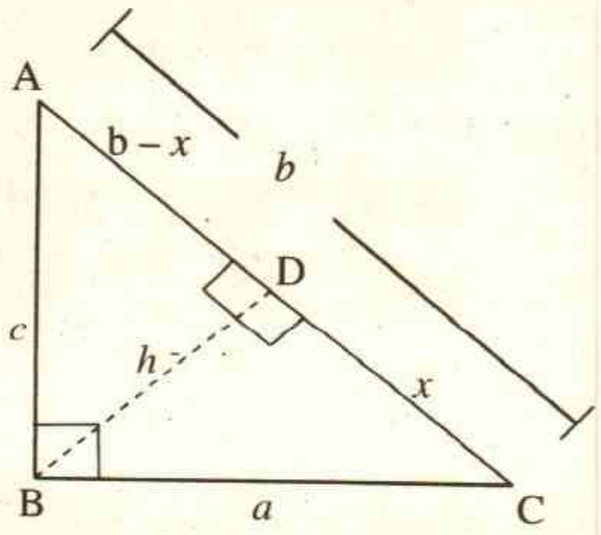


THEOREM 15.1 (Pythagoras' theorem)

In a right-angled triangle, the square of the length of hypotenuse is equal to the sum of the squares of the lengths of the other two sides.

Given:

ABC is a right-angled triangle, having right angle at B. a, b and c are the measures of sides $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ respectively.



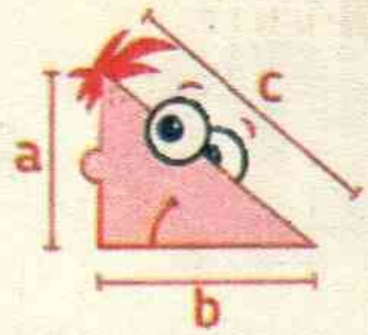
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To Prove:

$b^2 = c^2 + a^2$

Construction:

Draw $\overline{BD} \perp \overline{AC}$. Let $m\overline{BD} = h$ and $m\overline{CD} = x$. Therefore $m\overline{AD} = b - x$.



$a^2 + b^2 = c^2$

Proof:

In $\Delta ABC \leftrightarrow \Delta BDC$

$\angle ABC \cong \angle BDC$ (Right angles)

$\angle C \cong \angle C$ (Common / Self-congruent)

and $\angle CAB \cong \angle CBD$ (Complement of $\angle C$)

$\therefore$ $\Delta ABC \sim \Delta BDC$ (By definition)

Hence $\frac{b}{a} = \frac{a}{x}$ (Corresponding sides of similar triangles)

i.e. $\boxed{bx = a^2}$ ------ (i)

Again in $\Delta ABC \leftrightarrow \Delta ADB$

$\angle ABC \cong \angle ADB$ (Right angles)

$\angle A \cong \angle A$ (Common / Self-congruent)

and $\angle BCA \cong \angle DBA$ (Complement of $\angle A$)

$\therefore$ $\Delta ABC \sim \Delta ADB$ (By definition)

Hence $\frac{b}{c} = \frac{c}{b-x}$ (Corresponding sides of similar triangles)

i.e. $b(b-x) = c^2$ or $\boxed{b^2 - bx = c^2}$ ----- (ii)

Adding equation (i) and (ii), we get

$b^2 - bx + bx = c^2 + a^2$ or $\boxed{b^2 = c^2 + a^2}$

Example 1

Find the length of the hypotenuse of the right-angle triangle whose one length is 8cm and other length is 6cm.

Solution:

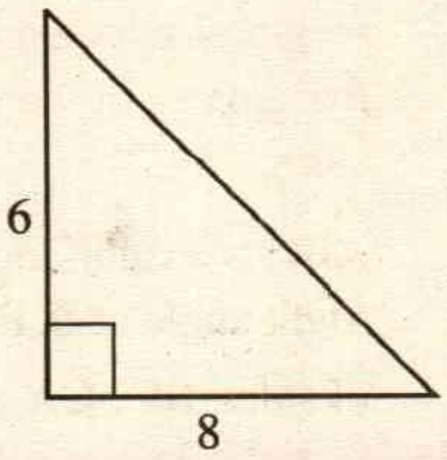
$(\text{Hypotenuse})^2 = (\text{perpendicular})^2 + (\text{base})^2$

$x^2 = (8\text{ cm})^2 + (6\text{cm})^2$

$x^2 = 64\text{cm}^2 + 36\text{ cm}^2$

$x^2 = 100\text{ cm}^2$

$x = 10\text{ cm}$



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Example 2

The top of the ladder rests against wall 23 feet above the ground. The base of the ladder is 6 feet away from the wall. What is length of the ladder?

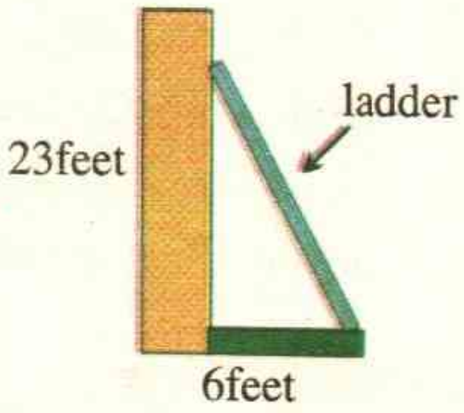
Solution:

$$(\text{length of ladder})^2 = (23 \text{ feet})^2 + (6\text{feet})^2$$

$$x^2 = 529 \text{ feet}^2 + 36 \text{ feet}^2$$

$$x^2 = 565 \text{ feet}^2$$

$$x = 23.76 \text{ feet}$$



THEOREM 15.2

(Converse of Pythagoras' theorem)

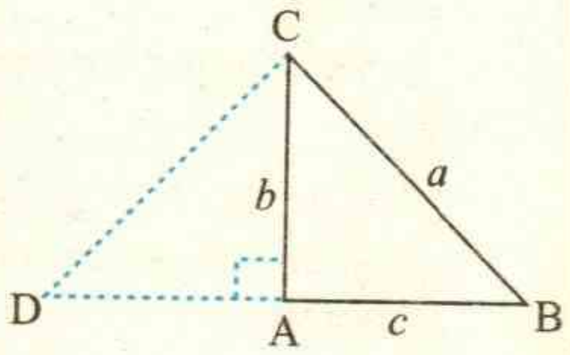
If the square of one side of a triangle is equal to the sum of the squares of the other two sides, then the triangle is a right-angled triangle.

Given:

In ΔABC, measures of $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ are a, b and c respectively.

Also $a^2 = b^2 + c^2$.

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To Prove:

ΔABC is a right triangle. i.e. $m\angle CAB = 90°$

Construction:

Draw $\overline{DA} \perp \overline{CA}$ such that $DA \cong BA$. Join D to C.

Proof:

ACD is a right-angled triangle.		($\because m\angle CAD = 90°$)
$\therefore$	$(m\overline{CD})^2 = (m\overline{AD})^2 + (m\overline{AC})^2$	(By Pythagoras theorem)
or	$(m\overline{CD})^2 = c^2 + b^2$ ------ (i)	($\because m\overline{DA} = m\overline{BA} = c$)
But	$a^2 = b^2 + c^2$ ------ (ii)	(Given)
$\therefore$	$(m\overline{CD})^2 = a^2$	(From (i) and (ii))
or	$m\overline{CD} = a = m\overline{CB}$	
Now in	$\Delta DAC \leftrightarrow \Delta BAC$	
CD= CB		(Proved)

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$CA = CA$ (Common)

$DA = BA$ (Construction)

$\therefore \quad \Delta DAC \cong \Delta BAC$ (S.S.S $\cong$ S.S.S)

Hence $\angle CAD \cong \angle CAB$ (Corresponding angles of congruent triangles)

But $m\angle CAD = 90°$ (Construction / $\overline{DA} \perp \overline{CA}$)

$\therefore \quad m\angle CAB = 90°$

or ΔABC is a right-triangle.

Classifying Triangles

The converse of the Pythagoras theorem is used to verify that a given triangle is a right triangle.

If $c^2 = a^2 + b^2$, then $m<C = 90°$ and ΔABC is a right triangle.

If $c^2 < a^2 + b^2$, then $m<C < 90°$ and ΔABC is an acute triangle.

If $c^2 > a^2 + b^2$, then $m<C > 90°$ and ΔABC is an obtuse triangle.

Example 3

Is the triangle whose sides are 3cm, 4cm and 5cm is a right-angle triangle?

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Solution:

A triangle is a right angled if the square of the longest side is equal to the sum of the squares of the other two sides. (Converse of Pythagoras theorem)

$(5cm)^2 = (3cm)^2 + (4cm)^2$

$25cm^2 = 9cm^2 + 16\ cm^2$

$25cm^2 = 25cm^2$

Therefore, the given triangle is a right-angled triangle.

Example 4

Calculate the value of x

Solution:

$12^2 + x^2 = 20^2$

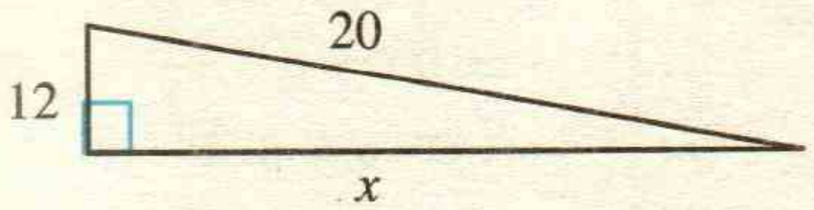
$x^2 = 20^2 - 12^2$

$= 400 - 144$

$= 256$

$x = \sqrt{256}$

$x = 16$



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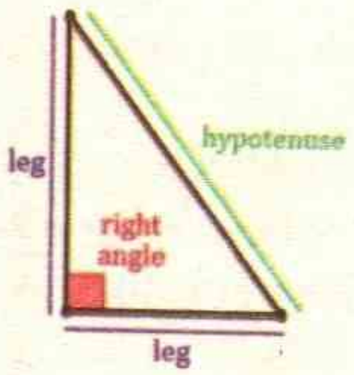
Exercise 15.1

1. Find the hypotenuse of a right-angled triangle when the sides containing the right angle are:

i) 12cm and 5cm

ii) 6cm and 2.5cm

iii) 15cm and 8cm



2. Find the diagonal of a rectangular field whose sides are:

i) 7.2m and 3m

ii) 15.6m and 6.5m

iii) 6.72m and 5.04m

3. A ladder whose foot is 2.5m from the front of a house reaches a window 6m above the ground. Calculate the length of the ladder.

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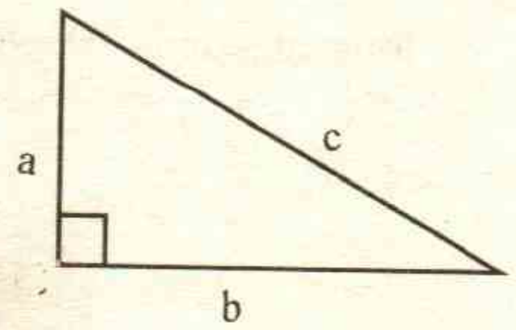
4. A ladder 2.9m long just reaches the top of a wall 2.1m high. How far from the foot of the wall is the foot of the ladder.

5. Find the length of the unknown side.

i) a = 4 cm, b = 7 cm

ii) a = 7 cm, c = 25 cm

iii) b = 6 cm, c = 9 cm



6. Verify whether a triangle with the given side lengths is a right-angled triangle?

i) 9,12, and 15

ii) 9,10, and 15

iii) 10,12 and 20

iv) 36,48, and 60

> Pythagorean Triples is a set of three positive integers a, b, and c that satisfy the equation $c^2 = a^2 + b^2$
> Common Pythagorean Triples are
> 3,4,5
> 5,12,13
> 8,15,17
> 7,24,25

7. The sides of a rectangular swimming pool are 50m and 30m. What is the length between the opposite corners.

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REVIEW EXERCISE 15

1. Circle the correct option.

i) Diagonal of a rectangle measures 6.5cm. If its width is 2.5cm, its length is
a) 4cm b) 9cm c) 6cm d) 3cm

ii) Which of the following are the sides of a right-angled triangle?
a) 2, 3, 4 b) 3, 4, 5 c) 4, 5, 6 d) 5, 6, 7

2. Measure of each of the sides of an equilateral triangle is 8cm. Find the length of the altitude.

3. Measure of the base of an isosceles triangle is 10 cm. If the perpendicular drawn from the vertex to the base is12cm long, find the measures of the congruent sides of the triangle.

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4. Diagonals of a quadrilateral ABCD are perpendicular to each other. Prove that $(m\overline{AB})^2 + (m\overline{CD})^2 = (m\overline{AD})^2 + (m\overline{BC})^2$.

5. The sides of a triangle have lengths x, $x+4$, and 20. If the length of the longest side is 20, what values of x make the right triangle?

SUMMARY

- In a right-angled triangle, the square of the length of hypotenuse is equal to the sum of the squares of the lengths of the other two sides.
- $(\text{Hypotenuse})^2 = (\text{perpendicular})^2 + (\text{base})^2$
- If the square of one side of a triangle is equal to the sum of the squares of the other two sides, then the triangle is a right-angled triangle.

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UNIT 16

THEOREMS RELATED WITH AREA

After reading this unit, the students will be able to:

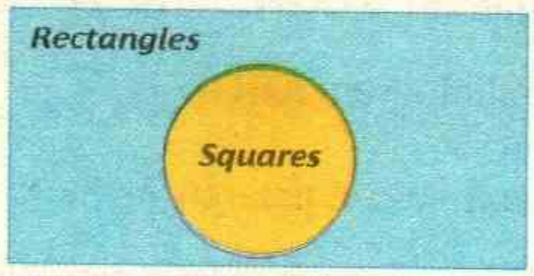
Prove the following theorems along with corollaries and apply them to solve appropriate problems.

- Parallelograms on the same base and lying between the same parallel lines (or of the same altitude) are equal in area.
- Parallelograms on equal bases and having the same altitude are equal in area.
- Triangles on the same base and of the same altitude are equal in area.
- Triangles on equal bases and of the same altitude are equal in area.

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Remember

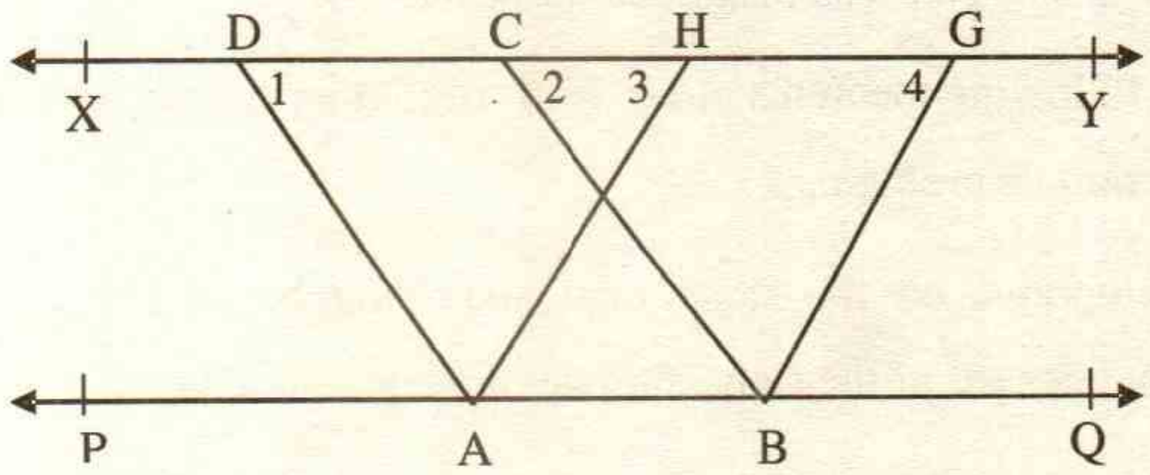
- If any side of a triangle is taken as its base, the perpendicular to that side from the opposite vertex is called the altitude of the triangle.
- Perpendicular distance between two parallel lines remains the same and does not change.
- If two figures are of equal area, it is not necessary that the two will be congruent as well.



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THEOREM 16.1

Parallelograms on the same base and lying between the same parallel lines (or of the same altitude) are equal in area.



Given:

ABCD and ABGH are two parallelograms having the same base $\overline{AB}$ and lying between two parallel lines $\overleftrightarrow{XY}$ and $\overleftrightarrow{PQ}$ (i.e. both have the same altitude).

To Prove:

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Parallelograms ABCD and ABGH are equal in area.

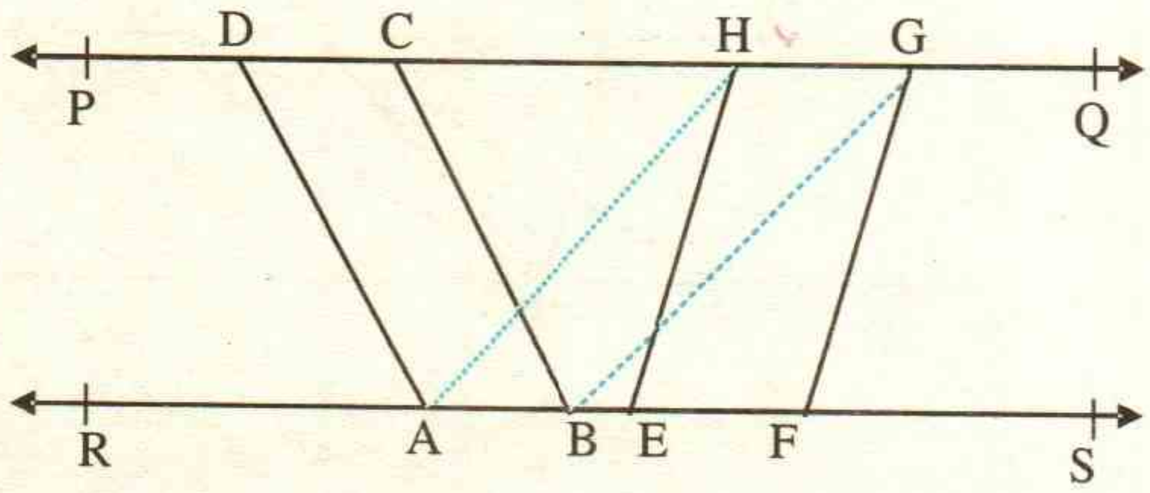
Proof:

Statement	Reason
In $\Delta ADH \leftrightarrow \Delta BCG$	
$\overline{AH} \cong \overline{BG}$	(Opposite sides of parallelogram)
$\angle 1 \cong \angle 2$	(Corresponding angles)
$\angle 3 \cong \angle 4$	(Corresponding angles)
$\therefore\ \Delta ADH \cong \Delta BCG$	(A.A.S $\cong$ A.A.S)
or, ΔADH and ΔBCG are equal in area.	
Subtracting each triangle from the whole figure ABCH, we get Quadrilateral ABCH – $\Delta ADH \cong$ Quadrilateral ABCH – ΔBCG	
Parallelogram ABCD $\cong$ Parallelogram ABGH	
or, both the parallelograms are equal in area.	

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THEOREM 16.2

Parallelograms on equal bases and having the same altitude are equal in area.



Given:

ABCD and EFGH are two parallelograms having the same altitude (i.e. lying between the same parallel lines) and $\overline{AB} \cong \overline{EF}$.

To Prove:

The parallelograms are equal in area.

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Construction:

Join A to H and B to G.

Proof:

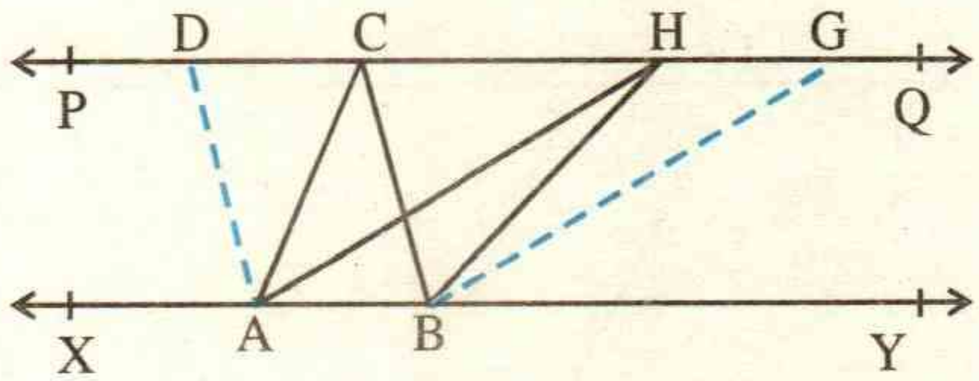
Statements	Reasons
∵ $\overline{HG} \cong \overline{EF}$	(Opposite sides of $\|\|^{gm}$)
and $\overline{EF} \cong \overline{AB}$	(Given)
∴ $\overline{AB} \cong \overline{HG}$	(Transitive Property)
But $\overline{AB} \parallel \overline{HG}$	(Lying on two parallel lines)
∴ ABGH is a parallelogram	(Opposite sides parallel and congruent)
Hence	
Area of $\|\|^{gm}$ ABCD = Area of $\|\|^{gm}$ ABGH	
Also Area of $\|\|^{gm}$ ABGH = Area of $\|\|^{gm}$ EFGH	
∴ Area of $\|\|^{gm}$ ABCD = Area of $\|\|^{gm}$ EFGH	(Transitive Property)

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THEOREM 16.3

Triangles on the same base and of the same altitude are equal in area.



Given:

ΔABC and ΔABH have the same base $\overline{AB}$ and are of the same altitude (i.e. lying between two parallel lines $\overleftrightarrow{PQ}$ and $\overleftrightarrow{XY}$)

To Prove:

Area of ΔABC = Area of ΔABH

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Construction:

Draw $\overline{AD} \parallel \overline{BC}$ and $\overline{BG} \parallel \overline{AH}$ to cut $\overleftrightarrow{PQ}$ at D and G.

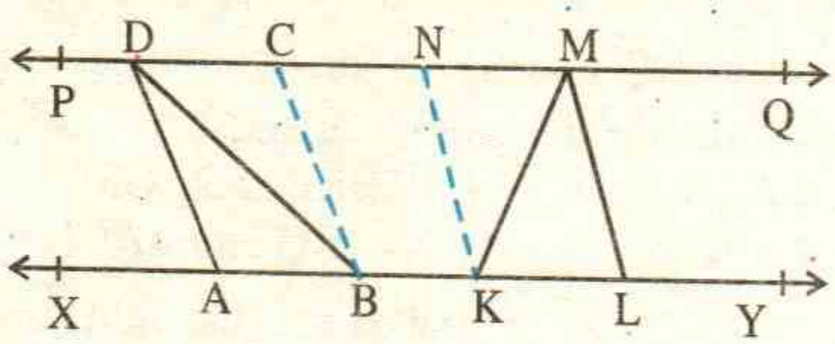
Proof:

Statements	Reasons
ABCD and ABGH are $\parallel^{gm}$	(Opposite sides are parallel)
$\therefore$ Area of $\parallel^{gm}$ ABCD = Area of $\parallel^{gm}$ ABGH	
But Area of $\Delta ABC = \frac{1}{2}$ (Area of $\parallel^{gm}$ ABCD)	(Diagonal $\overline{AC}$ bisects $\parallel^{gm}$)
Similarly,	
Area of $\Delta ABH = \frac{1}{2}$ (Area of $\parallel^{gm}$ ABGH)	(Diagonal $\overline{BH}$ bisects $\parallel^{gm}$)
Hence Area of ΔABC = Area of ΔABH	

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THEOREM 16.4

Triangles on equal bases and of the same altitudes are equal in area.



Given:

ΔABD and ΔKLM are between the same parallel lines $\overleftrightarrow{PQ}$ and $\overleftrightarrow{XY}$.

Also $\overline{AB} \cong \overline{KL}$.

To Prove:

Area of ΔABD = Area of ΔKLM

Construction:

Draw $\overline{BC} \parallel \overline{AD}$ and $\overline{KN} \parallel \overline{LM}$ to intersect $\overleftrightarrow{PQ}$ at C and N.

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Proof:

Statements	Reasons
ABCD and KLMN are parallelograms.	(Opposite sides parallel)
$\because$ $\overline{AB} \cong \overline{KL}$.	(Given)
and both the $\parallel^{gm}$ are having the same altitude.	(Given)
$\therefore$ Area of $\parallel^{gm}$ ABCD = Area of $\parallel^{gm}$ KLMN	
But Area of $\Delta ABD = \frac{1}{2}$ Area of $\parallel^{gm}$ ABCD	(Diagonal bisects the $\parallel^{gm}$)
and Area of $\Delta KLM = \frac{1}{2}$ Area of $\parallel^{gm}$ KLMN	(Diagonal bisects the $\parallel^{gm}$)
Hence Area of ΔABD = Area of ΔKLM	

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Exercise 16.1

1. In ΔABC, $\overline{PQ}$ is drawn parallel to $\overline{BC}$, cutting $\overline{AB}$ and $\overline{AC}$ at point P and Q respectively. $\overline{BQ}$ and $\overline{CP}$ are drawn to meet at R.
Prove that the following are equal in areas.
   i) ΔPBC and ΔQBC ii) ΔBRP and ΔCRQ
   iii) ΔPQB and ΔPQC iv) ΔABQ and ΔACP

2. PQRS is a parallelogram. A and B are the mid points of $\overline{PQ}$ and $\overline{PS}$ respectively. What fraction of the area of the parallelogram is:
   i) ΔPAB ii) ΔQBR iii) ΔBAR

3. ABCD is a quadrilateral. The diagonals $\overline{AC}$ and $\overline{BD}$ meet at E. If ΔABE and ΔCDE are equal in area then prove that $\overline{AD} \parallel \overline{BC}$.

4. How many tiles, each 8 inches square, will be required to pave a rectangular space 18 ×30 feet?

REVIEW EXERCISE 16

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1. Select the correct answer and write the corresponding letter a, b, c or d in the box.

i) Perpendicular distance between two lines is the same. The lines are
   a) Perpendicular to each other b) Parallel to each other
   c) Intersecting d) None of the above

ii) If two triangles have equal area then they will _____ be congruent as well.
   a) Not necessarily b) Necessarily
   c) Definitely d) None of the above

iii) Perpendicular from a vertex of a triangle to its opposite side is called _____.
   a) Median b) Perpendicular bisector
   c) Altitude d) Angle bisector

iv) Parallelograms having same base and same altitude are _____.
   a) Congruent b) Equal in area
   c) Similar d) All of the above.

v) Two parallelograms have equal bases. They will be having the same area if _____.
   a) Their altitudes are equal b) Their altitude is the same
   c) They lie between the same parallel lines d) All of the above

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vi) If two triangles have equal bases and equal altitudes, what else will they have equal?

a) area b) perimeter c) size d) angles

vii) Suppose a triangle has a base length of 4 feet and a height of 4 feet. Its interior area is

a) 4 square feet
b) 8 square feet
c) 16 square feet
d) Impossible to determine without more information.

viii) Suppose a square has a diagonal measure of 10 units. The area of the square is

a) 25 square feet
b) 50 square feet
c) 100 square feet
d) Impossible to determine without more information.

ix) Find the area of a triangle with base $\frac{15}{4}$ inches and altitude $\frac{8}{5}$ inches.

a) 2 sq. in b) 6 sq. in c) 3 sq. in d) 4.35 sq. in

2. Prove that a median of a triangle divides it into two parts of equal area.

3. In ΔABC, D and E are the mid points of $\overline{AB}$ and $\overline{AC}$ respectively. Prove that ΔADE, ΔBDE and ΔCDE are equal in area.

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4. P is any point on the median $\overline{AD}$ of ΔABC. Prove that ΔABP and ΔACP are equal in area.

5. Find the area of a triangle with base 20 inches and altitude 12 inches.

SUMMARY

- If any side of a triangle is taken as its base, the perpendicular to that side from the opposite vertex is called the altitude of the triangle.
- Perpendicular distance between two parallel lines remains the same and does not change.
- If two figures are of equal area, it is not necessary that the two will be congruent as well.
- Parallelograms on the same base and lying between the same parallel lines (or of the same altitude) are equal in area.
- Parallelograms on equal bases and having the same altitude are equal in area.

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UNIT 17

PRACTICAL GEOMETRY TRIANGLES

After reading this unit, the students will be able to:

Construct a triangle having given:

- two sides and the included angle,
- one side and two of the angles,
- two of its sides and the angle opposite to one of them (with all the three possibilities).



Draw:

- angle bisectors,
- altitudes,
- perpendicular bisectors,
- medians,

of a given triangle and verify their concurrency.

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Construct:

- a triangle equal in area to a given quadrilateral,
- a rectangle equal in area to a given triangle,
- a square equal in area to a given rectangle,
- a triangle of equivalent area on a base of given length.

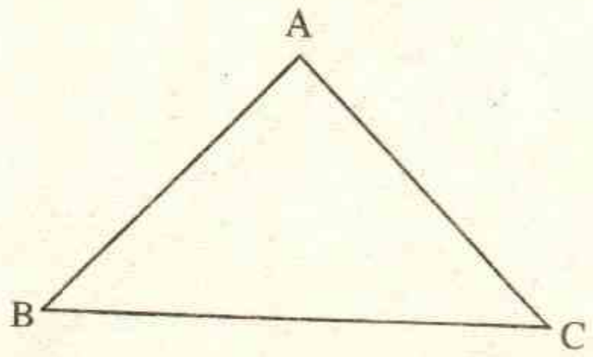
Practical geometry is the relationship between theory and practice. Engineers, architects and builders make use of practical geometry before they build buildings, bridges and other structures. Geometric awareness, such as the use of shapes, lines and figures, is necessary to design different projects. The knowledge of basic constructions such as construction of triangles, rectangles, squares is very useful in everyday life.

In practical geometry, a construction is a drawing made with the straight edge and compass. Constructions are a powerful learning technique, because they force you to think about the properties of geometric objects. Constructions are like challenging intellectual games.

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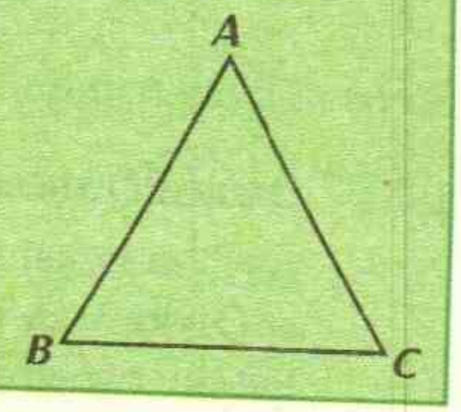
17.1 Elements of a Triangle

A triangle has six elements, three sides and three angles. For example the six elements of triangle the ABC are $\overline{AB}$, $\overline{BC}$, $\overline{AC}$, $\angle ABC$ or $\angle B$, $\angle ACB$ or $\angle C$, and $\angle BAC$ or $\angle A$.



The six elements of a triangle are:

1. ***Side AB.***
2. ***Side BC.***
3. ***Side AC.***
4. ***Angle ABC.***
5. ***Angle ACB.***
6. ***Angle BAC.***



17.2 Construction of a Triangle in the following three cases

a) When two sides and the included angle are given.
b) When one side and two of the angles are given.
c) When two of its sides and the angle opposite to one of them are given.

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17.2.1 Case - 1

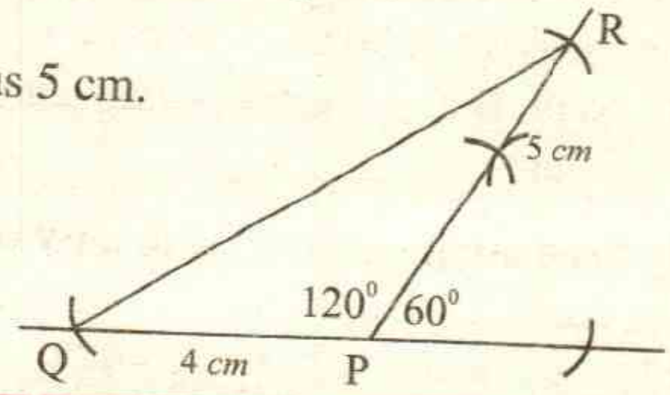
To construct a triangle, when the lengths of two sides and measure of the included angle are given.

Example 1

Construct a triangle PQR given that $\overline{PQ} = 4$ cm, $\overline{PR} = 5$ cm and $m\angle P = 120°$.

Steps of Construction:

1. Draw $\overline{PQ}$ of measure 4 cm.
2. At P, construct an angle of 120° with the help of a compass.
3. With P as centre, draw an arc $\overline{PR}$ of radius 5 cm.
   i.e, $\overline{PR} = 5$ cm.
4. Join R to Q.
   PQR is the required triangle.



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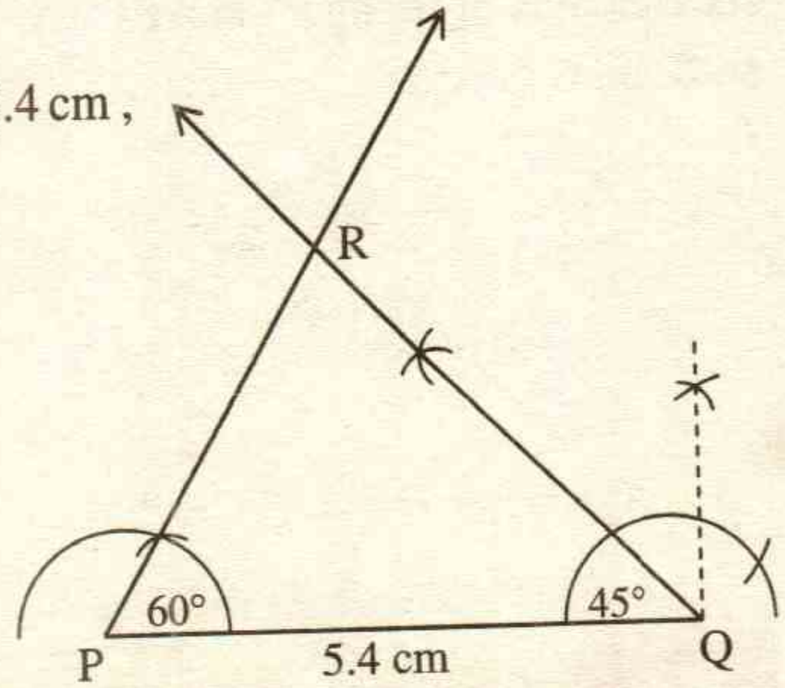
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17.2.2 Case - 2

To construct a triangle when length of one side and measures of two of its angles are given.

Example 2

Construct ΔPQR such that, $m\overline{PQ} = 5.4\,cm$, $m\angle PQR = 45°$ and $m\angle RPQ = 60°$.



Steps of Construction:

1. Draw $\overline{PQ}$ measuring 5.4cm.
2. At P, draw an angle of 60° and at Q, draw an angle of 45°.
3. Let the two arms meet at R.
4. PQR is the required triangle.

17.2.3 Case - 3

To construct a triangle when lengths of two of its sides and the measure of an angle opposite to one of these sides is given.

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In this case, three possibilities can emerge.

a) When only one triangle can be constructed with the help of given data.

b) When two triangles can be constructed with the help of same data.

c) When construction of a triangle is not possible with the help of given data.

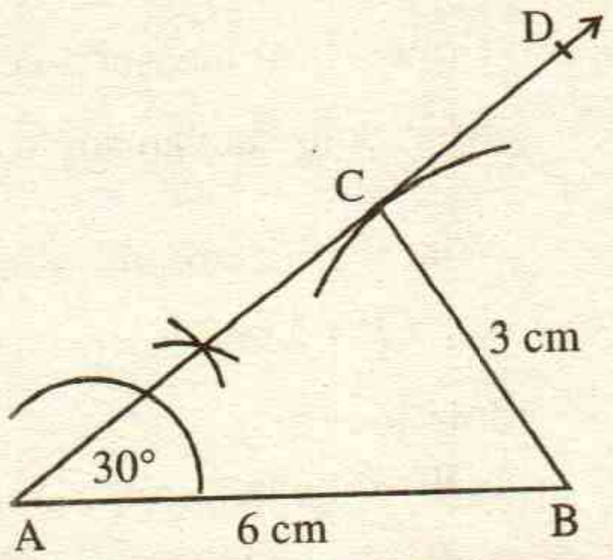
These possibilities are illustrated with the help of following example;

Example 3

Construct ΔABC such that $m\overline{BC} = 3\,cm$, $m\overline{AB} = 6\,cm$ and $m\angle A = 30°$.

Steps of Construction:

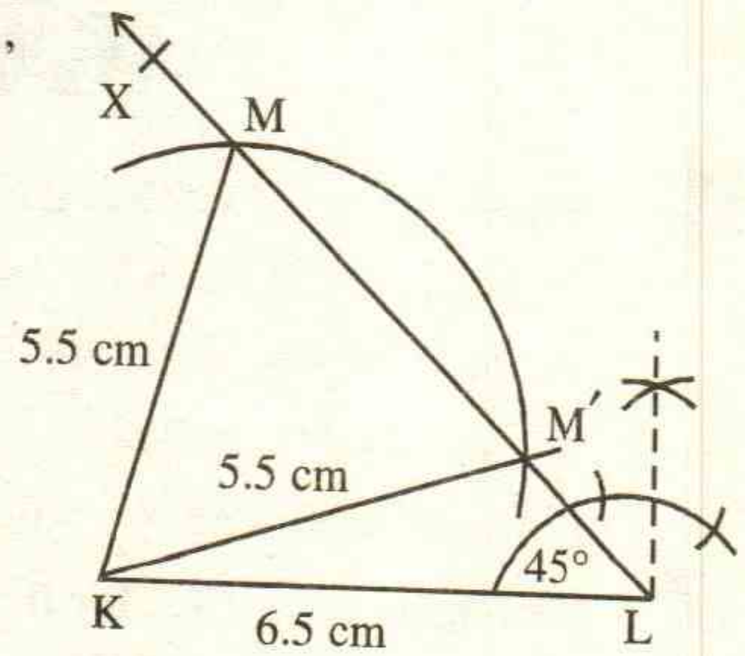
1) Draw $\overline{AB}$ measuring 6cm.
2) At A, construct $\angle BAD$ such that its measure is 30°.
3) With B as center, draw an arc of radius 3cm.
4) The arc intersects $\overrightarrow{AD}$ at only one-point C.
5) Join C to B.
6) ABC is the required triangle.



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Example 4

Construct ΔKLM, such that, $m\overline{KL} = 6.5\,cm$, $m\overline{KM} = 5.5\,cm$ and $m\angle L = 45°$.



Steps of Construction:

1) Draw $\overline{KL}$ such that $m\overline{KL} = 6.5\,cm$.
2) At L construct an $\angle KLX = 45°$.
3) With K as centre, draw an arc of radius 5.5cm to intersect $\overrightarrow{LX}$ at M and M′.
4) Join K to M and M′.
5) KLM and KLM′ are the required triangles.

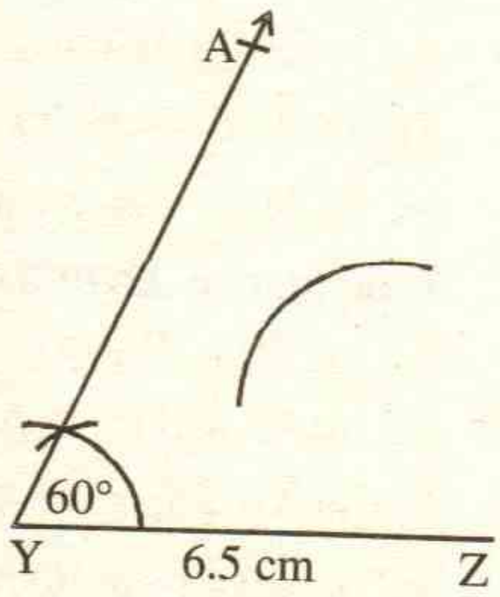
Example 5

Construct ΔXYZ when $m\overline{ZX} = 4\,cm$, $m\overline{YZ} = 6.5\,cm$ and $m\angle Y = 60°$.

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Steps of Construction:

1) Draw $\overline{YZ}$ measuring 6.5cm.
2) At Y, construct $\angle ZYA$ such that $m\angle ZYA = 60°$.
3) With Z as center, draw an arc of radius 4cm ($= m\overline{ZX}$).
4) The arc does not intersect the arm $\overrightarrow{YA}$ of $\angle Y$.
   In this case, no triangle can be constructed to satisfy the given data.



From these examples it can be concluded that if lengths of two sides and an angle (which is not included angle) are given, then there can be:

i) One triangle, or
ii) Two triangles, or
iii) no triangle at all.

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Exercise 17.1

1. **Construct a Δ XYZ, for the following assumptions:**

i. $m\angle X = 30°$, $m\overline{XY} = 3.5\,cm$, and $m\overline{XZ} = 4\,cm$

ii. $m\angle Y = 45°$, $m\overline{XY} = 4.2\,cm$, and $m\overline{YZ} = 4.5\,cm$

iii. $m\angle Z = 60°$, $m\overline{XZ} = 3.8\,cm$, and $m\overline{YZ} = 4.4\,cm$

iv. $m\angle Y = 90°$, $m\overline{XY} = 4.6\,cm$ and $m\overline{YZ} = 2.9\,cm$

2. **Construct a Δ ABC , for the following assumptions:**

i. $m\overline{AB} = 4.5\,cm$, $m\angle A = 45°$, and $m\angle B = 60°$

ii. $m\overline{BC} = 5\,cm$, $m\angle B = 30°$, and $m\angle C = 75°$

iii. $m\overline{AC} = 4.8\,cm$, $m\angle A = 120°$, and $m\angle C = 30°$

iv. $m\overline{AB} = 3.6\,cm$, $m\angle A = 75°$, and $m\angle B = 45°$

3. **Construct a Δ KLM, for the following assumptions:**

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i. $m\overline{KL} = 4.8\,cm$, $m\angle K = 45°$, and $m\angle M = 60°$

ii. $m\overline{LM} = 3.8\,cm$, $m\angle K = 30°$, and $m\angle M = 75°$

iii. $m\overline{KM} = 5\,cm$, $m\angle K = 105°$, and $m\angle L = 45°$

iv. $m\overline{KM} = 5.4cm$, $m\angle K = 75°$, and $m\angle M = 45°$

4. **Construct a ΔABC(whenever possible), for the following assumptions:**

i. $m\angle B = 30°$, $m\overline{AB} = 8\,cm$, and $m\overline{AC} = 4cm$

ii. $m\overline{AB} = 7cm$, $m\overline{AC} = 5.5\,cm$, and $m\angle B = 45°$

iii. $m\overline{AB} = 6cm$, $m\overline{AC} = 5.6\,cm$, and $m\angle B = 60°$

iv. $m\overline{AB} = 6\,cm$, $m\overline{AC} = 2.5\,cm$, and $m\angle A = 60°$

v. $m\overline{AC} = 5$ cm, $m\angle A = 75°$, and $m\angle C = 60°$

5. Construct ΔLMNsuch that, $m\overline{LM} = 5.4\,cm$, $m\angle L = 75°$ and $m\angle M = 45°$.

6. Construct ΔPQR such that, $m\overline{QR} = 8\,cm$, $m\overline{RS} = 12\,cm$ and $m\angle R = 90°$.

Construct ΔKLM such that, $m\overline{KL} = 4.8cm$, $m\overline{LM} = 3.9\,cm$ and $m\angle L = 30°$.

8. Construct ΔPQR such that, $m\overline{QR} = 6.5\,cm$, $m\angle P = 30°$ and $m\angle Q = 60°$.

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17.2.4 Concurrent lines and point of Concurrency

i) Two coplanar lines are either parallel or intersecting.

ii) Two lines intersect each other at one and only one point, which is called their point of intersection.

Now, if a third line also passes through the same point of intersection, then the three lines are said to be concurrent. Point through which these three lines pass is called the point of concurrency.

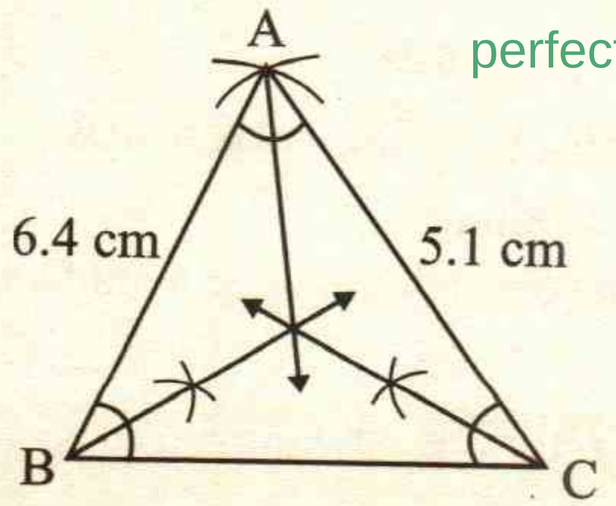
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Three or more lines are said to be concurrent, if they pass through the same point.

17.2.5 To draw angle bisector of a triangle and to verify their Concurrency

Example 6

Construct ΔABC whose $m\overline{AB} = 4.6\,cm$, $m\overline{BC} = 5\,cm$ and $m\overline{CA} = 5.1\,cm$. Draw angle bisectors of the triangle, and verify that these are concurrent.



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Remember

A line, which divides a given angle into two angles of equal measure, is called 'angle bisector'.

Steps of Construction:

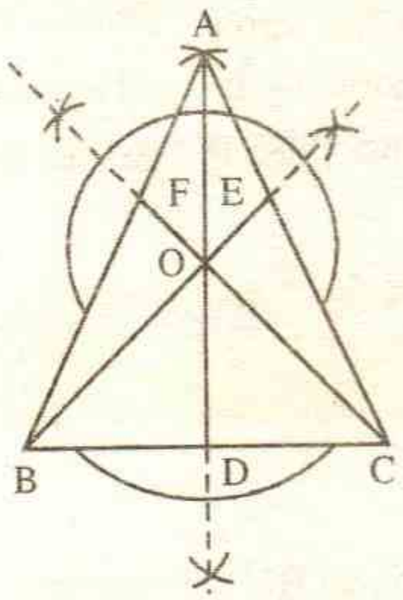
1. Draw $\overline{BC}$ measuring 5cm.
2. With B and C as centers and radii respectively equal to 4.6 and 5.1, draw arcs intersecting at A.
3. Draw $\overline{AB}$ and $\overline{AC}$.
4. Draw bisector of angles A, B and C.
   These bisectors pass through the same point I. Thus, angle bisectors of a triangle are concurrent.

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17.2.6 To draw altitudes of a triangle and to verify their Concurrency

Example 7

Construct ΔABC such that $m\overline{AB} = 5.6\,cm$, $m\overline{BC} = 6\,cm$ and $m\overline{CA} = 5\,cm$. Draw its altitudes and verify that these are concurrent.



Remember

A perpendicular drawn from a vertex of a triangle to its opposite side is called 'altitude' of the triangle.

Steps of Construction:

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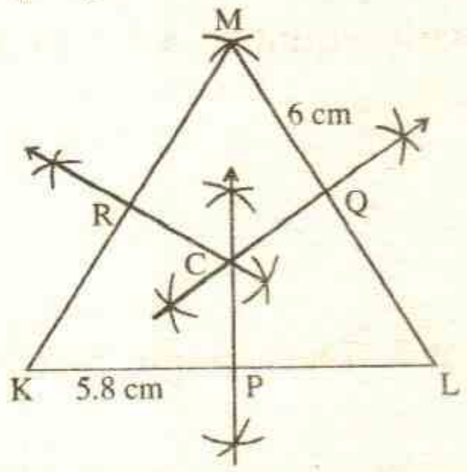
1. Construct ΔABC with the help of given data.
2. Draw perpendiculars from A to $\overline{BC}$, B to $\overline{CA}$ and C to $\overline{AB}$.
3. $\overline{AD}$, $\overline{BE}$ and $\overline{CF}$ are the required altitudes.

We see that the three altitudes pass through the same point O. i.e. altitudes of a triangle are concurrent.

17.2.7 To draw perpendicular bisectors of a triangle and to verify their concurrency

Example 8

Construct ΔKLM such that $m\overline{KL} = 5.8\,cm$, $m\overline{LM} = 6\,cm$ and $m\angle L = 60°$. Draw its perpendicular bisectors and verify their concurrency.



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A line, which is perpendicular to a line segment and divides it into two equal parts, is called its 'Perpendicular Bisector' or Right Bisector.

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Steps of Construction:

1. Construct ΔKLM according to the given data.
2. Draw the right bisectors of $\overline{KL}$, $\overline{LM}$ and $\overline{MK}$.
3. $\overleftrightarrow{CP}$, $\overleftrightarrow{CQ}$ and $\overleftrightarrow{CR}$ are the three perpendicular bisectors of $\overline{KL}$, $\overline{LM}$ and $\overline{MK}$ respectively.

All the three perpendicular bisectors pass through C. i.e. they are concurrent.

17.2.8 To draw medians of a triangle and to verify their Concurrency

Example 9

Construct ΔABC such that $m\overline{AB} = 6\,cm$, $m\angle A = 70°$ and $m\angle C = 50°$. Draw its medians and verify their concurrency.

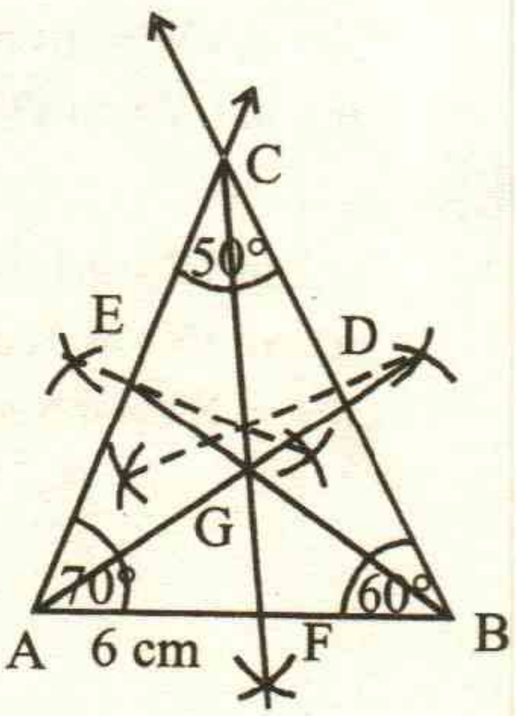
$\because$ $m\angle A = 70°$ and $m\angle C = 50°$

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$\therefore$ $m\angle B = 180° - (70° + 50°)$

$= 180° - 120°$

$= 60°$



Remember

Line segment joining the midpoint of one side of a triangle to its opposite vertex is called 'median'.

Steps of Construction:

1. Construct ΔABC with the help of given data.
2. Find the mid points D, E and F of $\overline{BC}$, $\overline{AC}$ and $\overline{AB}$ respectively.
3. Draw the medians $\overline{AD}$, $\overline{BE}$ and $\overline{CF}$.
4. All the three medians pass through G.

   Therefore the medians of a triangle are concurrent.

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Exercise 17.2

1. Construct ΔABC, draw their angle bisectors and verify their concurrency:

i) $m\overline{AB} = 4.5\,cm$, $m\overline{BC} = 3.1\,cm$ and $m\overline{CA} = 5.2\,cm$.

ii) $m\overline{AB} = m\overline{BC} = m\overline{CA} = 12$.

iii) $m\overline{CA} = 5.8\,cm$, $m\angle A = 45°$ and $m\angle C = 75°$.

2. Construct ΔPQR, draw their altitudes and verify their concurrency:

i) $m\overline{PQ} = 6\,cm$, $m\overline{QR} = 4.5\,cm$ and $m\overline{PR} = 5.5\,cm$.

ii) $m\overline{PQ} = 4.5\,cm$, $m\overline{QR} = 3.9\,cm$ and $m\angle R = 45°$.

iii) $m\overline{PQ} = 6\,cm$, $m\angle P = 75°$ and $m\angle Q = 60°$

3. Construct ΔUVW, draw their perpendicular bisectors and verify their concurrency:

i) $m\overline{UV} = 7\,cm$, $m\overline{VW} = 6.5\,cm$ and $m\overline{WU} = 5.8\,cm$.

ii) $m\overline{VW} = 10\,cm$, $m\overline{WU} = 4.2\,cm$ and $m\angle W = 120°$.

iii) $m\overline{UV} = m\overline{VW} = m\overline{WU} = 6.8\,m$.

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4. Construct ΔXYZ, draw their medians and verify their concurrency.

i) $m\overline{YZ} = 4.1\,cm$, $m\angle Y = 60°$ and $m\angle X = 75°$.

ii) $m\overline{ZX} = 4.3\,cm$, $m\angle X = 75°$ and $m\angle Y = 45°$.

iii) $m\overline{XY} = 4.5\,cm$, $m\overline{YZ} = 3.4\,cm$ and $m\overline{ZX} = 5.6\,cm$.

$0 \times 9 + 1 = 1$
$1 \times 9 + 2 = 11$
$12 \times 9 + 3 = 111$
$123 \times 9 + 4 = 1111$
$1234 \times 9 + 5 = 11111$
$12345 \times 9 + 6 = 111111$
$123456 \times 9 + 7 = 1111111$
$1234567 \times 9 + 8 = 11111111$
$12345678 \times 9 + 9 = 111111111$
$123456789 \times 9 + 10 = 1111111111$

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17.3 FIGURES WITH EQUAL AREAS

17.3.1 Construct a triangle equal in area to a given quadrilateral

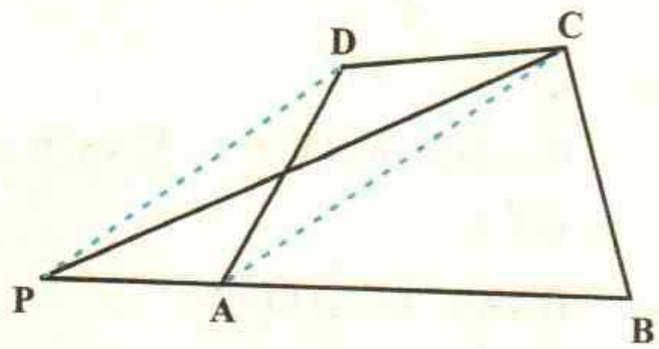
Given:

A quadrilateral ABCD.

Required:

To construct a triangle equal in area to the quadrilateral ABCD.

Steps of Construction:



1. Join C to A.
2. Through D draw DP $\| \overline{CA}$, meeting BA produced at P.
3. Join C to P.
4. $\because \Delta APC$ and ΔADC stand on the same base AC and between the same parallel lines $\overline{AC}$ and $\overline{PD}$.
   $\therefore$ Area of ΔAPC = Area of ΔADC
5. Area of ΔAPC + Area of ΔABC = Area of ΔADC + Area of ΔABC
6. Area of ΔPBC = Area of quadrilateral ABCD
   ΔPBC is the required triangle.

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17.3.2 Construct a rectangle equal in area to a given triangle

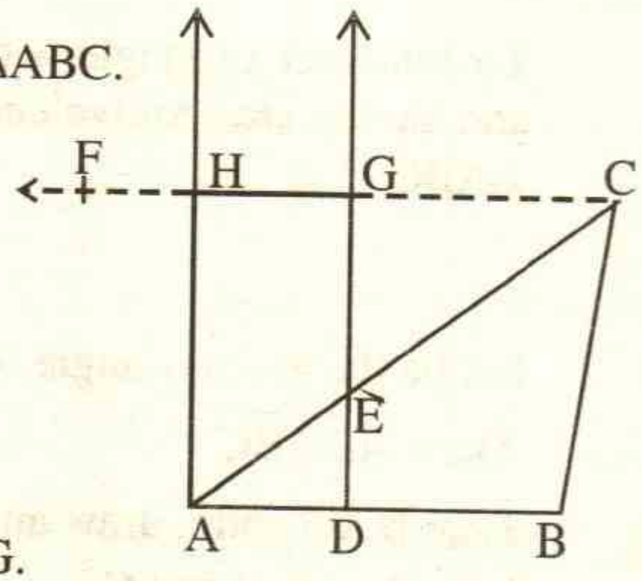
Given:

ΔABC.

Required:

To construct a rectangle equal in area to ΔABC.

ps of Construction:



1) Let ABC be the given triangle.
2) Bisect $\overline{AB}$ at D.
3) At D, construct $\angle ADE$ such that
4) $m\angle ADE = 90°$.
5) Through C, draw $\overleftarrow{CF} \| \overline{BA}$, meeting $\overleftrightarrow{DE}$ at G.
6) Through A, draw $\overleftarrow{AH} \| \overline{DG}$, meeting $\overleftarrow{CF}$ at H.
   ADGH is the required rectangle.

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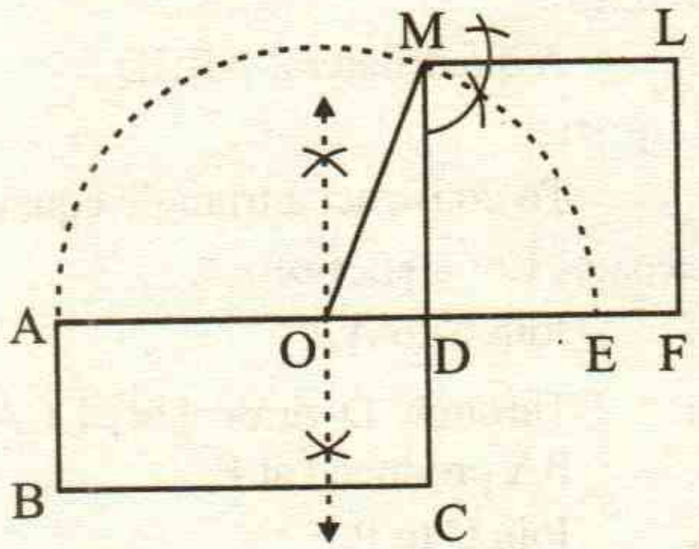
17.3.3 To construct a square equal in area to a given rectangle

Given:

Rectangle ABCD.

Required:

To construct a square equal in area to rectangle ABCD.



Steps of Construction:

1. Produce to $\overline{AD}$ to E such that $m\overline{DE} = m\overline{CD}$.
2. Bisect $\overline{AE}$ at O.
3. With centre as O and radius $\overline{OA}$ draw semicircle.
4. Produce $\overline{CD}$ to meet the semicircle at M.
5. On DM as a side complete a square DFLM.

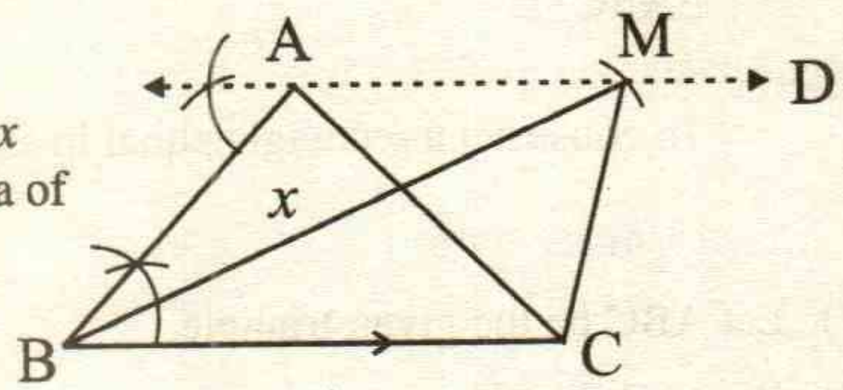
This is the required square.

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17.3.4 To construct a triangle of equivalent area on a base of given length

Given:

ΔABC with $\overline{BC}$ as its base.



Required:

To construct a triangle with base x and having area equivalent to area of ΔABC.

Steps of Construction:

1. Let be the given triangle. ABC□
2. Draw $\overleftrightarrow{AD} \parallel \overline{BC}$
3. Take B as centre, draw an arc of radius x cutting $\overleftrightarrow{AD}$ at M.
4. Join M with B and C.

Then BCM is the required triangle with base $BM = x$ and area equivalent to ΔABC.

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Exercise 17.3

1. Draw a quadrilateral ABCD, such that $m\overline{AB} = 3$ cm, $m\angle B = 60°$, $m\angle A = 120°$, $m\overline{BC} = 3.5$ cm and $m\overline{AD} = 4$ cm. Construct a triangle equal in area to the quadrilateral ABCD.

2. Draw a rectangle PQRS such that $m\overline{PQ} = 5$ cm and $m\overline{QR} = 3.5$ cm. Construct a square equal in area to rectangle PQRS.

3. Draw a triangle ABC such that $m\overline{AB} = 5$ cm, $m\overline{BC} = 4$ cm and $m\overline{CA} = 4.5$ cm. Construct a rectangle equal in area to the given triangle.

4. Construct a square having area equal to the given rectangle.

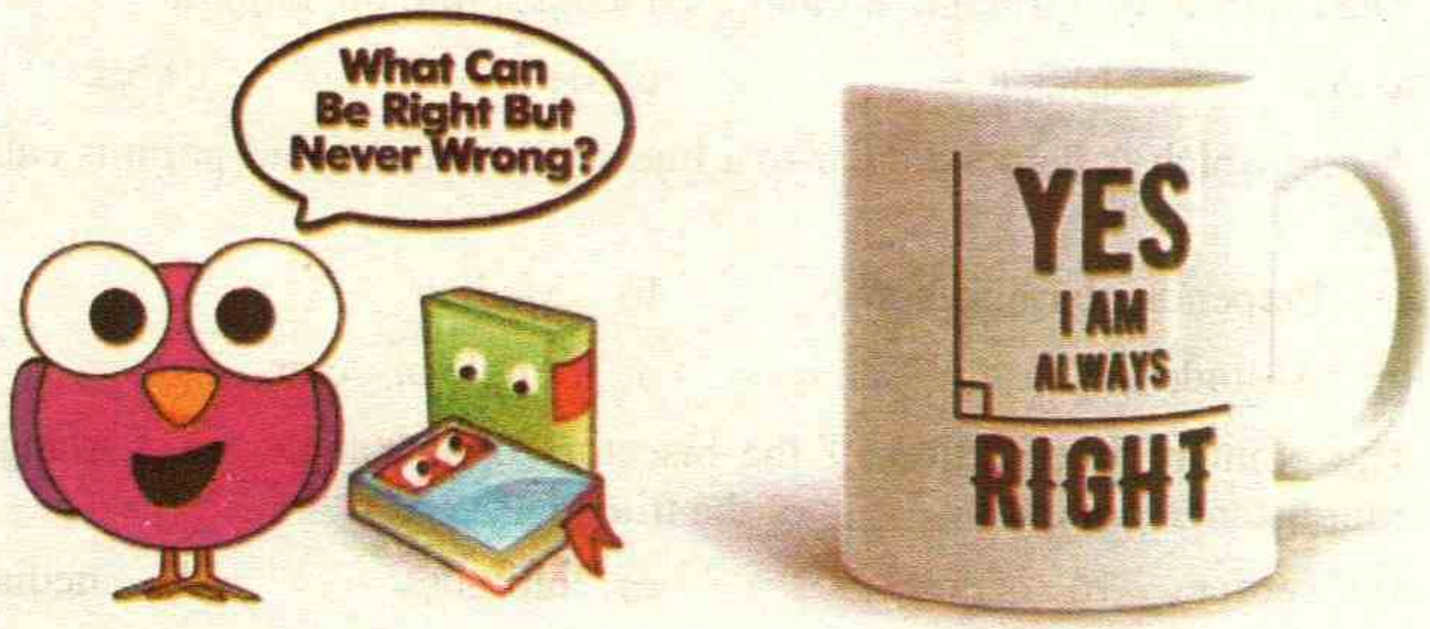
5. Construct a square equal in area to a rectangle whose adjacent sides are 4.5 cm and 2.2 cm respectively. Measure the sides of the square and find its area and compare with the area of the rectangle.

6. Construct a square equal in area to the sum of two squares having sides 3 cm and 4 cm respectively.

7. Construct a triangle having base 3.5 cm and other two sides equal to 3.4 cm and 3.8 cm respectively. Transform it into an equal square.

8. Construct a triangle having base 5 cm and other sides equal to 5 cm and 6 cm. Also construct a square equal in area to the given triangle.

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REVIEW EXERCISE 17

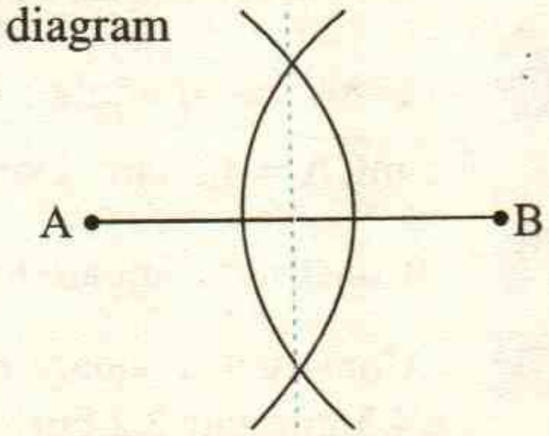
1 Select the correct answer.

i) What is the first step in constructing an angle bisector?

a) Draw a ray b) Label the points of intersection

c) Measure the line d) Place the compass point on the vertex.

ii) What geometric construction is shown in the diagram below.



a) A line parallel to a given line

b) An angle bisector

c) An angle congruent to a given angle

d) A perpendicular

iii) A line segment joining the midpoint of one side of a triangle to its opposite vertex is called

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a) Perpendicular bisector b) Median

c) Altitude d) Angle bisector

iv) You are looking at a triangle whose orthocenter, centroid and circumcenter are all the same point. What type of triangle are you looking at?

a) Scalene b) Isosceles c) Equilateral d) Right

v) The centroid of a triangle divides the medians into the ratio of

a) 2:1 b) 3:1 c) 4:1 d) 5:1

vi) A line which is perpendicular to a line segment at its mid-point is called a / an

a) Perpendicular bisector b) Median

c) Altitude d) Angle bisector

vii) The point of intersection of the bisectors of the angles of a triangle is equidistant from the _____ of the triangle.

a) vertices b) sides c) altitudes d) medians

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viii) Altitudes of a triangle are _____.

a) equal in length
b) equidistant from the vertices
c) concurrent
d) perpendicular bisectors

ix) If measures of three angles of a triangle are known, how many triangles can be constructed?

a) Only one triangle
b) Two triangles
c) No triangle
d) Infinite triangles.

x) The point of intersection of the perpendicular bisectors of the sides of a triangle is equidistant from the _____ of the triangle.

a) altitudes b) medians c) sides d) vertices

2. Construct ΔABC such that, $m\overline{AB} = 3.7\,cm$, $m\overline{BC} = 2.5\,cm$ and $m\angle B = 60^\circ$.

3. Construct a ΔABC such that, $m\overline{BC} = 5.8\,cm$, $m\angle A = 30^\circ$ and $m\angle B = 45^\circ$

4. Construct ΔABC, such that, $m\overline{AC} = 4.5\,cm$, $m\overline{BC} = 4.1\,cm$ and $m\angle B = 75^\circ$.

5. Construct ΔABC, draw their angle bisectors and verify their concurrency:
$m\overline{AB} = 5.3\,cm$, $m\angle A = 45^\circ$ and $m\angle B = 45^\circ$.

6. Construct ΔPQR, draw their altitudes and verify their concurrency:
$m\overline{PR} = 5.8\,cm$, $m\angle P = 45^\circ$ and $m\angle Q = 105^\circ$.

7. Construct ΔUVW, draw their perpendicular bisectors and verify their concurrency:
$m\overline{UW} = 5.8\,cm$, $m\angle U = 45^\circ$ and $m\angle V = 105^\circ$.

8. Construct ΔXYZ, draw their medians and verify their concurrency.
$m\overline{ZX} = 6\,cm$, $m\angle Y = 60^\circ$ and $m\angle Z = 75^\circ$.

9. Draw a triangle PQR such that $m\overline{PQ} = 5.6$ cm, $m\overline{QR} = 4.5$ cm and $m\overline{RP} = 3.4$ cm. Construct a triangle SPQ equivalent in area to the triangle PQR.

10. Construct a rectangle whose adjacent sides are 2.5 cm and 5 cm respectively. Transform it into an equal square.

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SUMMARY

- Practical geometry is the relationship between theory and practice.
- Engineers, architects and builders make use of practical geometry before they build buildings, bridges and other structures.
- In practical geometry, a construction is a drawing made with the straight edge and compass. Parallelograms on the same base and lying between the same parallel lines(or of the same altitude) are equal in area.
- A triangle has six elements, three sides and three angles.
- Two coplanar lines are either parallel or intersecting.
- Two lines intersect each other at one and only one point, which is called their point of intersection.

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- Angle bisectors of a triangle are concurrent.
- Altitudes of a triangle are concurrent.
- A line, which is perpendicular to a line segment and divides it into two equal parts, is called its 'Perpendicular Bisector' or Right Bisector.
- Line segment joining the mid-point of one side of a triangle to its opposite vertex is called 'median'.
- Three or more than three lines are said to be concurrent if these pass through the same point and that point is called the point of concurrency.
- The point where the internal bisectors of the angles of a triangle meet is called incentre of a triangle.



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Answers

Exercise 1.1

1. (i) square (ii) rectangular (iii) square (iv) square (v) rectangular (vi) rectangular
2. (i) 2×3 (ii) 1×1 (iii) 2×3 (iv) 3×2 (v) 1×2 (vi) 3×3
3. (i) 2 (ii) 0 (iii) 1 (iv) 6 (v) –4 (vi) 0
4. $A = D,\ B = C.$
5. $u = 5, v = 2, w = 0.$
6. $a = -2, b = -7, c = -1, x = -3, y = -5, z = 2.$
7. $a = 1, b = -2, c = 3, d = 2.$

Exercise 1.2

1. (i) $P^t = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$ (ii) $Q^t = \begin{bmatrix} l & n \\ m & p \end{bmatrix}$ (iii) $R^t = [6]$

(iv) $S^t = \begin{bmatrix} -5 & -2 & 4 \\ 1 & 1 & 4 \end{bmatrix}$ (v) $T^t = \begin{bmatrix} 6 & 13 & 2 \\ 7 & 1 & 4 \\ 8 & 3 & 5 \end{bmatrix}$

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2. i) and ii), iii) and iv) are transpose of each other.
3. Symmetric matrices: *ii)*.
4. Skew-symmetric matrices: *i),ii),iv)*.

Exercise 1.3

1. (i) 2-by-3 (ii) 2-by-2
2. (i) $\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$ (ii) $\begin{bmatrix} pa & pb & pc \\ pd & pe & pf \end{bmatrix}$
3. $\begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 1 & \frac{1}{2} & \frac{3}{4} \\ -\frac{1}{4} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$
4. $\begin{bmatrix} 6 & 8 \\ 8 & 17 \\ 11 & 15 \end{bmatrix}$
5. $\begin{bmatrix} 7 & 0 & 61 \\ 13 & 4 & 12 \\ 5 & 5 & 1 \end{bmatrix}$
6. $\begin{bmatrix} 12 & \frac{4}{3} \\ 4 & -\frac{14}{3} \\ \frac{25}{3} & \frac{28}{3} \end{bmatrix}$
7. $x = 2, y = 4, z = \frac{5}{3}, w = 3$
8. $X = \begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix}, Y = \begin{bmatrix} 1 & -2 \\ 0 & 5 \end{bmatrix}$

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10. i) $\begin{bmatrix} 5 & 0 & 7 \\ -6 & 8 & 8 \\ -9 & -6 & 5 \end{bmatrix}$ ii) $\begin{bmatrix} -15 & 10 & 1 \\ 32 & -6 & -16 \\ 3 & 22 & 15 \end{bmatrix}$ iii) $\begin{bmatrix} 0 & 4 & -1 \\ -1 & 1 & 5 \\ -1 & -3 & 2 \end{bmatrix}$ iv) $\begin{bmatrix} 4 & -2 & 11 \\ -1 & 9 & 3 \\ -11 & -1 & 8 \end{bmatrix}$

13. i) $\begin{bmatrix} -3 & -4 \\ -6 & -2 \end{bmatrix}$ ii) $\begin{bmatrix} -a & a & -b \\ c & -a & b \\ -l & -m & -n \end{bmatrix}$

Exercise 1.4

1. AB, BA, CA, BC.

2. (i) Yes (ii) No (iii) $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$

3. (i) I (ii) $\begin{bmatrix} 1 & -\frac{10}{3} \\ 0 & -\frac{2}{3} \end{bmatrix}$

4. (i) $\begin{bmatrix} 1 & 2 \\ 3 & 5 \\ 0 & 2 \end{bmatrix}$ (ii) No

6. A

7. No.

8. (i) $(AB)C = \begin{bmatrix} -8 & 2 \\ 8 & -2 \end{bmatrix}$, $A(BC) = \begin{bmatrix} -8 & 2 \\ 8 & -2 \end{bmatrix}$. (ii) Yes (iii) Associative law

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10. (i) A (ii) B.

Exercise 1.5

1. (i) 29 (ii) 62 (iii) 97 (iv) 3 (v) $-2ps+3qr$ (vi) 1
(vii) -29 (viii) abc

2. Singular matrices: (i), (ii), (iv) Non-singular matrices: (iii)

3. (i) $adj A = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$ (ii) $adj\, B = \begin{bmatrix} 3 & 1 \\ -2 & -3 \end{bmatrix}$
(iii) $adj\, C = \begin{bmatrix} b & 2b \\ -2a & 3a \end{bmatrix}$ (iv) $adj\, D = \begin{bmatrix} -4 & -6 \\ -2 & -3 \end{bmatrix}$

4. (i) $A^{-1} = \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix}$ (ii) $B^{-1} = \frac{1}{2}\begin{bmatrix} 2 & -4 \\ -1 & 3 \end{bmatrix}$ (iii) $C^{-1} = \frac{1}{5}\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$
(iv) $D^{-1} = \frac{1}{6}\begin{bmatrix} 4 & 3 \\ -2 & 0 \end{bmatrix}$ (v) $I^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ 5. (i) $AB = \begin{bmatrix} 2 & -2 \\ -4 & 6 \end{bmatrix}$
(ii) $BA = \begin{bmatrix} 5 & -1 \\ -11 & 3 \end{bmatrix}$ (iii) $A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}, B^{-1} = \frac{1}{2}\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$

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Exercise 1.6

1. (i) $x = 1$, $y = -1$ (ii) $x = 3$, y = 6. (iii) $x = 2$, $y = -\frac{1}{2}$

(iv) $x = \frac{7}{5}$, $y = \frac{1}{5}$

(iii) $x = 2$, y = –1

2. (i) $x = \frac{7}{3}$, $y = -\frac{4}{3}$ (ii) $x = 3$, y = 1

(iv) $x = 0$, y = 1

3. 8 , 4

4. Length = 10, Width = 5 5. bag = Rs. 75. Pen = Rs. 8

6. Father's age= 40 years, Son's age = 15 years

Review Exercise 1

1. (i) c (ii) a (iii) a (iv) c (v) b (vi) a (vii) d (viii) d

2. $x = 1, y = -5$ 3. Not possible. 4. $\begin{bmatrix} 11 & 13 \\ 3 & 5 \end{bmatrix}$ 5. (–2, 5)

6. Cost of small box = Rs. 7, Cost of large box = Rs. 13.

Exercise 2.1

1) $3, 0, \sqrt{36}, 1$ 2) $3, 0, \sqrt{36}, -9, 1$ 3) $\sqrt{3}, \sqrt{7}, -\sqrt{14}, \pi$ 4) $3, \sqrt{36}, 1$

5) $2.5, 3, \frac{5}{7}, -1.96, 0, \sqrt{36}, -\frac{7}{6}, -9, 1, 4\frac{2}{3}, 0.333\ldots$ 6) All of them

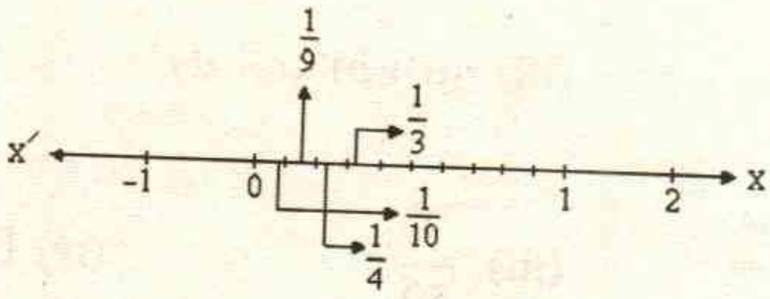
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7) $2.5, \frac{5}{7}, -1.96, -\frac{7}{6}, 4\frac{2}{3}, 0.333\ldots$ 8) –9 9) 0, –9

10) $2.5, \frac{5}{7}, -1.96, -\frac{7}{6}, \sqrt{3}, \sqrt{7}, -\sqrt{14}, \pi, 4\frac{2}{3}, 0.333\ldots$

11) $\frac{1}{6} = 0.1666\ldots$, $\frac{5}{6} = 0.8333\ldots$, $\frac{2}{9} = 0.222\ldots$, $\frac{1}{8} = 0.125$

12)



Exercise 2.2

1) (i) Associative property w.r.t addition
(ii) Distributive property of multiplication over addition
(iii) Additive identity
(iv) Multiplicative inverse

2) (i) 6 (ii) 13 (iii) 108 (iv) 8

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3) (i) d (ii) b (iii) c 4) (i) 3900 (ii) 81000 (iii) 4000
(iv) 30,000

Exercise 2.3

1) (i) Index = 2 , Radicand = $\frac{11}{y}$ (ii) Index = 3 , Radicand = $\frac{13}{3x}$
(iii) Index = 5 , Radicand = ab^2

2) (i) $(36)^{\frac{1}{2}}$ (ii) $(1000)^{\frac{1}{2}}$ (iii) $(8)^{\frac{1}{3}}$ (iv) $(q)^{\frac{1}{n}}$
(v) $\left(5-6a^2\right)^{\frac{3}{2}}$ (vi) $(-64)^{\frac{1}{3}}$

3) (i) $-\sqrt[3]{7}$ (ii) $\sqrt{x^{-3}}$ (iii) $\sqrt[5]{-8}$ (iv) $\sqrt[4]{y^3}$
(v) $\sqrt[5]{b^4}$ (vi) $\sqrt[q]{3x}$

4) (i) $5\sqrt[3]{x}$ (ii) $\frac{2}{3}$ (iii) $5xy$ (iv) $3y-5$
(v) $18\sqrt{2}$ (vi) $3xy\sqrt[3]{2z^2}$

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Exercise 2.4

1) (i) Base = 2, Exponent = −9 , value = $\frac{1}{1024}$ (ii) Base = $\frac{a}{b}$, Exponent = p, value= $\frac{a^p}{b^p}$ (iii) Base = −4 , Exponent = 21 , value = 16

2) (i) a^8 (ii) $\left(\frac{b}{a}\right)^5$ (iii) $-a^7$ (iv) $-8a^6b^9$
(v) $4a^3b^2$ (vi) a^4b^2 (vii) $\frac{1}{2}$ (viii) $9a^4b^4$
(ix) a^3b^{-6}

3) (i) 7^2 (ii) 20 (iii) $(a+b)^3(c+d)^3$
(iv) $a^{\frac{1}{6}}$ (v) x^2

4) (i) $\frac{5}{9}$ (ii) $a^{\frac{a+b}{ab}} \cdot b^{\frac{a+b}{ab}}$ (iii) $\frac{1}{50}$ (iv) 1

Exercise 2.5

1) (i) $13+11i$ (ii) $9-2i$ (iii) $11-3i$ (iv) -8
2) (i) $8-6i$ (ii) $-12i$ (iii) $7+2i$ (iv) $14i$
3) (i) $19-2i$ (ii) $14+8i$ (iii) $-2-26i$ (iv) $2\sqrt{2}-i$

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4) (i) $\frac{9}{26}+\frac{7}{26}i$ (ii) $\frac{1}{2}+\frac{7}{2}i$

5) (i) $6-6i$ (ii) $1+5i$ (iii) $10+8i$ (iv) $\frac{23}{41}-\frac{2}{41}i$

6) (i) $-8+3i$ (ii) $-4-9i$ (iii) $7-6i$ (iv) $\sqrt{5}+i$

Review Exercise 2

1. (i) F (ii) T (iii) F (iv) F

2. (i) a (ii) c (iii) c (iv) c (v) a
(vi) a (vii) b (viii) c (ix) b (x) a (xi) d

3. (i) $-\frac{8}{27}$ (ii) -72 (iii) $-12\sqrt{3}$ (iv) $\frac{5\sqrt[3]{3}}{3}$

4. 64 5. $\frac{32}{37}+\frac{7}{37}i$ 6. Multiplicative inverse 7. 6

Exercise 3.1

1. (i) 4.05×10^{5} (ii) 1.67×10^{6} (iii) 3.9×10^{-7}
(iv) 9.2×10^{-4} (v) 2.346×10^{11} (vi) 8.904×10^{9}
(vii) 1.04×10^{-3} (viii) 5.14×10^{-9} (ix) 5×10^{-5}

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2. (i) 0.000083 (ii) 4,100,000 (iii) 20,700,000
(iv) 0.00000315 (v) 0.000000000627 (vi) 0.0000000541
(vii) 0.0007632 (viii) 940,000 (ix) –2600000000

3. 8 min and 20 sec.

Exercise 3.2

1. (i) $\log_4 256 = 4$ (ii) $\log_2 \frac{1}{64} = -6$ (iii) $\log_x y = \frac{3}{4}$
(iv) $\log_{10} 1 = 0$ (v) $\log_3 \frac{1}{81} = -4$ (vi) $\log_{64} 16 = \frac{2}{3}$

2. (i) $a^{-2} = \frac{1}{a^2}$ (ii) $2^{-7} = \frac{1}{128}$ (iii) $b^{64} = 3$
(iv) $a^{1} = a$ (v) $a^{0} = 1$ (vi) $4^{\frac{-3}{2}} = \frac{1}{8}$

3. (i) 4 (ii) $\frac{1}{64}$ (iii) $\frac{1}{2}$ (iv) $\frac{8}{5}$ (v) 128 (vi) $\frac{1}{2}$
(vii) 10 (viii) 8 (ix) 6560

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Exercise 3.3

1. i) 1 ii) 0 iii) 0 iv) -1 v) 2
   vi) 3 vii) 5 viii) -3
2. i) 1.941 ii) 4.572 iii) 2.877 iv) 0.9642
   v) $\bar{3}.2014$ vi) $\bar{2}.4083$ vii) 0.8295
3. i) 3.3938 ii) 0.3802 iii) 1.9661 iv) 2.6836
   v) $\bar{1}.8938$ vi) $\bar{2}.9808$ vii) $\bar{3}.8283$ viii) 2.8451

Exercise 3.4

1. (i) 17.81 (ii) 6.920 (iii) 346.7 (iv) 0.01781
   (v) 0.3518 (vi) 3570.0
2. (i) 0.6920 (ii) 156.0 (iii) 39000.0 (iv) 0.001060
   (v) 74.40 (vi) 0.06871

Exercise 3.5

1. (i) 1/2 (ii) $-1/3$ (iii) 3/2 (iv) 2
   (v) $\overline{10}.1763$ (vi) 1.6532
2. (i) $\log\frac{8}{81}$ (ii) log(0.144) (iii) log0.5 (iv) $\log\frac{\sqrt{x}z^3}{4y^2}$
3. (i) 42 (ii) 20 (iii) t (iv) 5
4. 3

Exercise 3.6

1. (i) 165.340 (ii) 0.1745 (iii) 1.1980
   (iv) 0.5285 (v) 3614930.899
2. (i) 2.0212 (ii) 2.0333 (iii) 0.6190
   (iv) 0.3801 (v) -2.0916

Review Exercise 3

1. (i) b (ii) c (iii) a (iv) c (v) b (vi) c
   (vii) d (viii) a (ix) b (x) b (xi) c (xii) b (xiii) d
2. 9.4732×10^3
3. 0.000041
4. $\log_3\frac{1}{27}=-3$
5. $1=5^0$
6. $x=2$
7. -3
8. 1.0934
9. $a=2$
10. 0.0004

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Exercise 4.1

1. (i) is a polynomial. (ii), (iii) are not polynomials.
2. (i) and (iii) are in not lowest terms and (ii) is in lowest term.
3. (i) $\frac{1}{x}$ (ii) $\frac{t^3}{t+5}$ (iii) It cannot be reduced further. (iv) $\frac{2}{a-3}$
4. (i) $6x^2$ (ii) $\frac{-6y+16}{y^2+3}$ (iii) $\frac{3y^2+4y}{y^2-16}$ (iv) $\frac{3t^2-14t}{t^2-25}$
5. (i) $7y^2-4y+17$ (ii) $\frac{14}{x^2+1}$ (iii) $\frac{1}{a+3}$ (iv) $\frac{2(x+3)}{3(x-2)}$
6. (i) $\frac{4}{3(x+1)}$ (ii) $\frac{x+3}{4-x}$ (iii) $\frac{3(x-3)}{2(x+5)}$
7. (i) $\frac{2}{3y}$ (ii) $\frac{p^2}{rq}$ (iii) $\frac{a+3}{a-6}$

Exercise 4.2

1. (i) 5 (ii) 7 (iii) 13
2. (i) 37 (ii) 19 (iii) 161
3. (i) -24 (ii) $20\sqrt{13}$ (iii) $\frac{5}{29}$
4. $\frac{3}{4}, \frac{1}{4}$ 5. 54

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Exercise 4.3

1. (i) 73/2, 55/4 (ii) 149/2, 51/4 (iii) 73, 24 (iv) $\frac{65}{2}, \frac{33}{4}$
2. (i) 29 , 10 (ii) 41 , 20
3. (i) 68 (ii) 2
4. 264 5. 14,560 6. ± 1

Exercise 4.4

1. (i) 33 (ii) 29 2. (i) ± 6 (ii) $\pm 4\sqrt{2}$ 3. (i) 44 (ii) $\frac{13}{2}$
5. $(x-y)^2+(y-z)^2+(z-x)^2$ 6. $\frac{29}{2}$

Exercise 4.5

1. (i) 4 (ii) -153 (iii) 40
2. (i) 230 (ii) 98 (iii) 469

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3. (i) $\frac{65}{8}$ (ii) 2 4. (i) $\frac{63}{8}$ (ii) $\frac{532}{27}$ (iii) $\frac{4095}{64}$

5. 28 6. 225 9. $a^2+\frac{1}{a^2}=a^4+\frac{1}{a^4}=a^3+\frac{1}{a^3}=2$

Exercise 4.6

1. (i) a^3-1 (ii) $27-b^3$ (iii) $512+b^3$ (iv) a^3+8

2. (i) $8p^3+\frac{1}{8p^3}$ (ii) $\frac{27}{8}p^3-\frac{8}{27p^3}$ (iii) $27p^3-\frac{1}{27p^3}$

(iv) $125p^3+\frac{1}{125p^3}$

3. (i) x^6-y^6 (ii) x^8-y^8 (iii) $64x^6-y^6$ (iv) x^6-64

4. $x^{\frac{3}{2}}-y^{\frac{3}{2}}$ 5. $x^{3p}+y^{3q}$

Exercise 4.7

1. (i) ,(iv) 2. (i) $2\sqrt{3}$ (ii) $4\sqrt{3}$ (iii) $4\sqrt{3}.\sqrt{5}$

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3. (i) $6+2\sqrt{5}-3\sqrt{3}-\sqrt{15}$ (ii) $\sqrt{6}+\sqrt{3}-4\sqrt{2}-4$

(iii) $\sqrt{10}+2+\sqrt{15}+\sqrt{6}$ (iv) -3

4. (i) $\frac{\sqrt{7}}{7}$ (ii) $\frac{\sqrt{5}}{5}$ (iii) $\sqrt{2}+1$ (iv) $-5(2-\sqrt{5})$ (v) $2\sqrt{5}$

5. $2\sqrt{5}$,18 6. $2\sqrt{2}$,10 7. 10, 98 8. 2,6 9. 6, 38 10. 194

Review Exercise 4

1. i) a ii) a iii) c iv) d v) a vi) c

vii) d viii) c ix) c x) b

2. $\frac{9a^2y^3}{20x^3}$ 3. $\frac{1}{3}$ 4. 29,10 5. ± 7 6. 6 7. 800

8. 488 9. $\left(\frac{64}{125}x^3-\frac{125}{64x^3}\right)$ 10. $-\frac{8}{x^4-16}$

Exercise 5.1

1. $3s^2t(3s+5t^2-t)$ 2. $5a^2b^2c^2(2bc^2-3a+6a^2b)$ 3. $(x-1)(a-1)$

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4. $(x+y)(x-2y^2)$

5. $\left(2x+\frac{1}{x}\right)^2$

6. $[2(x+y)+52]^2$

7. $\left(\frac{x^2}{y^2}+\frac{y^2}{x^2}\right)\left(\frac{x}{y}+\frac{y}{x}\right)\left(\frac{x}{y}-\frac{y}{x}\right)$

8. $2(x+12)(x-12)$

9. $(1+u-v)(1-u+v)$

10. $(5ab-2c+4d)(5ab-2c-4d)$

Exercise 5.2

1. $(x^2+4x+8)(x^2-4x+8)$

2. $(2x^2+6x+9)(2x^2-6x+9)$

3. $(a^2+b^2+ab)(a^2+b^2-ab)$

4. $(x^2+x-1)(x^2-x+1)$

5. $(x^4-x^2+1)(x^2+x-1)(x^2-x+1)$

6. $\left(x^2+\frac{1}{x^2}+3\right)\left(x^2+\frac{1}{x^2}-3\right)$

7. $(9x^2+\frac{1}{9x^2}+4)(9x^2+\frac{1}{9x^2}-4)$

8. $4(x^2+3xy+4y^2)(x^2-3xy+4y^2)$

9. $(4m^2+2mn+n^2)(4m^2-2mn+n^2)$

10. $xy(2x^2+xy+3y^2)(2x^2-xy+3y^2)$

Exercise 5.3

1. $(x-3)(x-4)$

2. $(x+4)(x-3)$

3. $(5+x)(4-x)$

4. $(2y-1)(y-3)$

5. $(2x+1)(2x+3)$

6. $(5y+1)(2y-1)$

7. $3x(x-3)(2x+1)$

8. $2x(y+6)(y-2)$

9. $-(3t-2)(4t+1)$

10. $-2xy(4x+3y)(2x+y)$

11. $(x+3)(x+2)$

12. $4x^2y^4(x^3y^3-7)(x^3y^3-3)$

13. $4x+48$

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Exercise 5.4

1. $(4x^2-16x+11)^2$

2. $(9x^2+9x+2)(9x^2+9x-16)$

3. $(x+5)^2(x^2+10x+15)$

4. $(x^2+3x+1)^2$

5. $x^2\left(x+\frac{6}{x}+8\right)\left(x+\frac{6}{x}+4\right)$

6. $(4x-3y)^3$

7. $\left(\frac{a}{2}-\frac{b}{3}\right)^3$

8. $\left(\frac{x}{a}+\frac{a}{x}\right)^3$

9. $(3a+7b)^3$

10. $\left(2x-\frac{1}{3x}\right)^3$

Exercise 5.5

1. $(a-3)(a^2+3a+9)$

2. $(a^2+b^2)(a^4-a^2b^2+b^4)$

3. $3(2x+1)(4x^2-2x+1)$

4. $(1-3r)(1+3r+9r^2)$

5. $2(x-4)(x^2+4x+16)$

6. $4x^2(x-4)(x^2+4x+16)$

7. $18(x-y-2a+2b)\{(x-y)^2+2(x-y)(a-b)+2(a-b)^2\}$

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8. $(x+1)(x^2-x+1)(x^6-x^3+1)$ 9. $(a+c+d)(a^2+c^2+d^2-ac-ad+2cd)$

10. $(3x-y)(9x^2+3xy+y^2)$

Exercise 5.6

1. (i) 4 (ii). -44 (iii) -4 2. a = 3 3. $b=5$

4. (i) $(x-1)(x+2)(x-3)$
(ii) $(x+1)(x+2)(x-2)$
(iii) $(x-1)(x-2)(x+3)$
(iv) $(x-1)(x-3)(x-5)$
(v) $(x+2)(x-3)^2$
(vi) $(x+1)(x-4)(x+5)$
(vii) $(x-2)(x-3)(x+4)$
(viii) $(x+2)(x-4)^2$

Review Exercise 5

1. (i) T (ii) T (iii) T (iv) F (v) F (vi) F

2. (i) $(3x-4y^2)(3x+4y^2)$ (ii) $(x-3y)(x^2+3xy+9y^2)$
(iii) $(x-2)(x-3)$ (iv) $(a-b)^2+2ab$ (v) $(2b+c)(4b^2-2bc+c^2)$

3. (i) b (ii) c (iii) d (iv) a (v) b (vi) b
(vii) a

4. (i) $3x(x-3)(x+2)$ (ii) $(4y+3)(16y^2-12y+9)$
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(iii) $(x+y)(x-y)(x^2+xy+y^2)(x^2-xy+y^2)$
(iv) $(7a-5b)(49a^2+35ab+25b^2+1)$

5. $(x^2+5x+16)(x+6)(x-1)$

6. (i) $(x-2)(x^2+2x-13)$ (ii) $(x+1)(x+2)(x-42)$

7. $(a^2+b^2+ab)(a^2+b^2-ab)$

8. $2x^2-2x+6$, -24 9. a)1 b) 7 10. -5

Exercise 6.1

1. (i) $x+6$ (ii) x^2+y^2 (iii) $x-3$
(iv) $3^2(x-y)^2(x+2y)$ (v) $(x+y)$

2. (i) $x-3$ (ii) y^2-2y+1 (iii) $(x+1)$ (iv) $x+5$

3. (i) x^2-y^2 (ii) x^3-y^3 (iii) $x^3(x+1)(x-1)(x^2+1)(x^2+x+1)$
(iv) $2^3\cdot3^2(x-y)^3(x+2y)^3$

4. (i) $H.C.F=(x^2+3x+2)$, $L.C.M=(x-4)(x^3-2x^2-13x-10)$
(ii) $H.C.F=(2x^2-3)$, $L.C.M=(x^2-x+2)(4x^4-2x^3+3x-9)$

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(iii) $H.C.F = (a^2+a+1),\ L.C.M = (a^4-a^3-a+1)(a^2-a+1)$

(iv) $H.C.F = (x^4-x-1),\ L.C.M = (x-1)(1+2x+x^2-x^4-x^5)$

5. x^2-x-2

6. $x^3+7x^2+8x+14$

7. $2x^2+13x+15$ and x^2+4x-5

8. $x+4$

9. 16 Children

Exercise 6.2

1. (i) $\frac{x+2y}{x+y}$ (ii) $\frac{2x}{3x+2y}$ (iii) $\frac{10}{y^2-4}$ (iv) $\frac{3y^2-2xy}{x^2-y^2}$

(v). $\frac{3x^3+xy}{(x+y)(2x-y)(2x+y)}$ (vi). 0 (vii). $\frac{8x^8y}{x^8-y^8}$

(viii). $\frac{2a+13}{(a+2)(a+5)(a+8)}$ (ix). $\frac{8a^7}{a^8-b^8}$ (x). $\frac{2x-1}{x^2-y^2}$

2. (i) $-(x+5)$ (ii) $\frac{3(y-1)}{2x(x-1)}$ (iii) $\frac{(x-1)(2x-1)}{(x+1)(x-4)(x^2+1)}$

(iv) $\frac{a}{(a-b)^2}$ (v). $\frac{7}{xy(x-2)}$ (vi) $\frac{1}{a+b}$ (vii) $\frac{2}{3}$

(viii). 1 (ix). $\frac{3+x}{x^4}$ (x) $\frac{(a+c)(a-x)}{(x+a)^2}$

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Exercise 6.3

1. (i) $x+2$ (ii) $x+y+3$ (iii) $xy-4$ (iv) . $\pm(x+\frac{1}{x}-4)$

(v). x^2+3x+1 (vi) $\pm\left(x-\frac{1}{x}-\frac{5}{2}\right)$ (vii) $\pm\left(x^2+\frac{1}{x^2}-2\right)$

(viii). $\pm\left(\frac{2x^3-3y^3}{3x^2+4y^2}\right)$

2. (i) $2x^2-x+3$ (ii) $x^2+\frac{x}{2}-4$ (iii) $x+y-1$

(iv) $x^2-\frac{1}{x^2}-6$

3. ₹ = 64

4. (i) $x-2$ (ii) $-x+2$

5. (i) $-2x-3$ (ii) $2x+3$ (iii) x = 2

6. (i) $l=24, m=36$ (ii) $l=-60, m=-36$

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Review Exercise 6

1. (i) a (ii) d (iii) b (iv) c (v) a (vi) d
(vii) d (viii) a (ix) b (x) c

2. (i) $\frac{7s^2-7s+2}{2(s+2)(s+1)(s-2)}$ (ii). 0 (iii). $\frac{2(x+2)}{y(x-2)}$ (iv) $\frac{1}{a+b}$

3. $(x^2-5x+6)(x^3-4x+3)$ 4. (i) $2x-3$ (ii) x^2+2x+1 5. $\frac{x+y}{x^2-xz+z^2}$

Exercise 7. 1

1.
(i) $\{3\}$ (ii) $\left\{\frac{40}{9}\right\}$ (iii) $\left\{\frac{7}{2}\right\}$ (iv) $\{3\}$ (v) $\left\{\frac{-17}{14}\right\}$ (vi) $\left\{20\frac{1}{2}\right\}$

2. 17 3. 1, –5 4. 25, 27, 29. 5. 7 Years 6. 95

7. 46 8. $8m, 2m$

Exercise 7. 2

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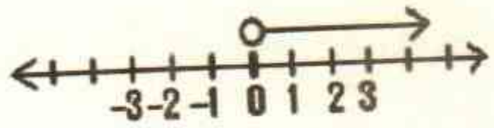
1. $a=25$ 2. $b=16$ 3. $r=8$ 4. $r=4$ 5. $t=36$
6. $x=5$ 7. $x=8$ 8. $y=5$

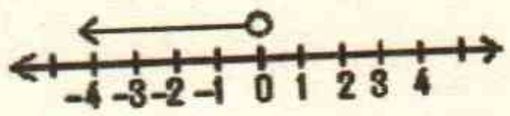
Exercise 7. 3

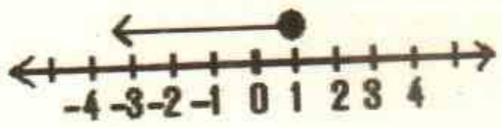
1) $\{2, -8\}$ 2) $\left\{-1, \frac{7}{5}\right\}$ 3) $\left\{12.\frac{28}{3}\right\}$ 4) $\{7, 1\}$ 5) $\left\{-\frac{2}{3}, -2\right\}$

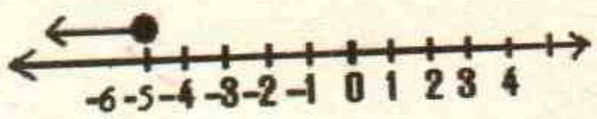
6) $\left\{\frac{9}{2}\right\}$ 7) $\left\{\frac{37}{3}, -11\right\}$ 8) $\left\{0, \frac{4}{5}\right\}$ 9) $\left\{-\frac{17}{4}, \frac{23}{4}\right\}$

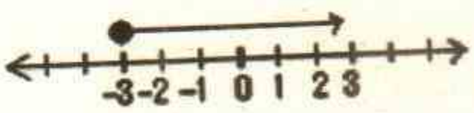
Exercise 7. 4

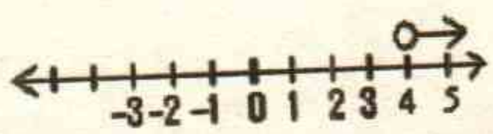
1. i)


ii)


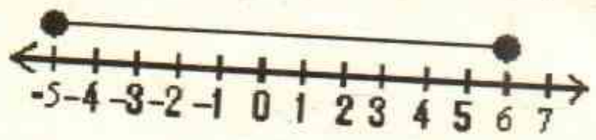
iii)


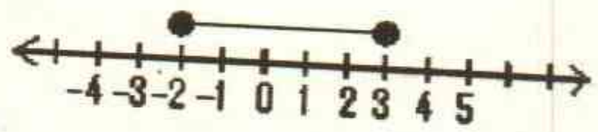
iv)


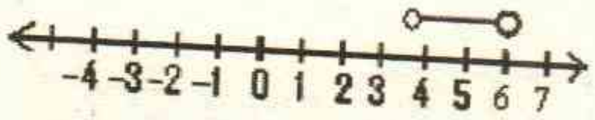
v)


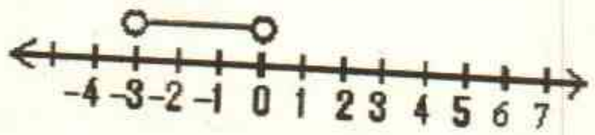
vi)


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vii) 

viii) 

ix) 

x) 

2. i) $\{1,2,3\}$ ii) $\{1, 2, 3, 4, 5\}$ iii) $\{x \mid x \in R, x \leq -1\}$

iv) $\{5, 6\}$ v) $\{2,3,4,5\}$ vi) $\{x \mid x \in R, x \leq -9\}$

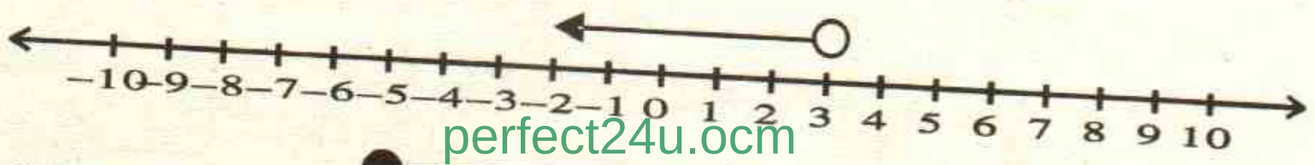
vii) $\{1, 0, -1, -2, -3, \ldots\}$

3. (i) $\{x \mid x \in R, x \leq 3\}$ (ii) $\{x \mid x \in R, x \geq -5\}$ (iii) $\{x \mid x \in R, x > 6\}$

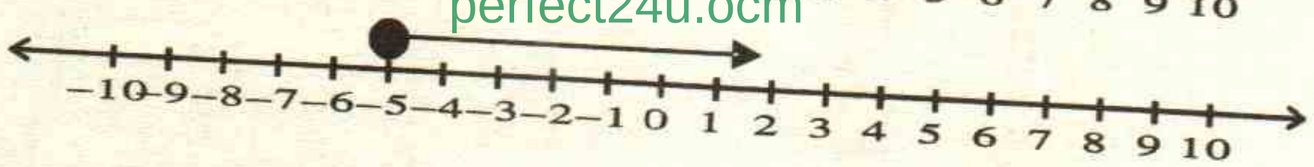
(iv) $\{x \mid x \in R, -8 < x < 2\}$ (v) $\{x 1 x \in R\, x \geq 7\}$

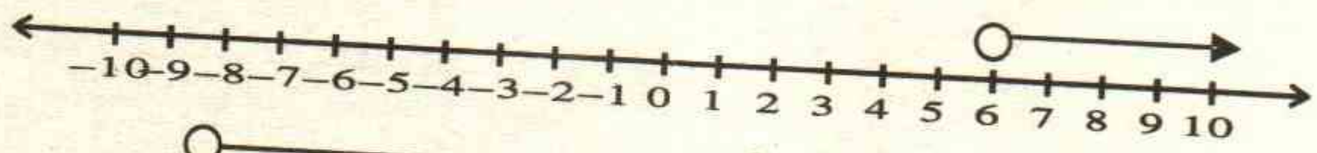
(vi) $\{x \mid x \in R, -2 \leq x < 2\}$

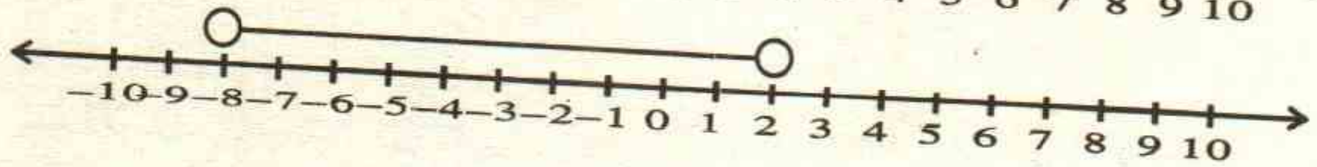
The graphs of i – vi are given below.

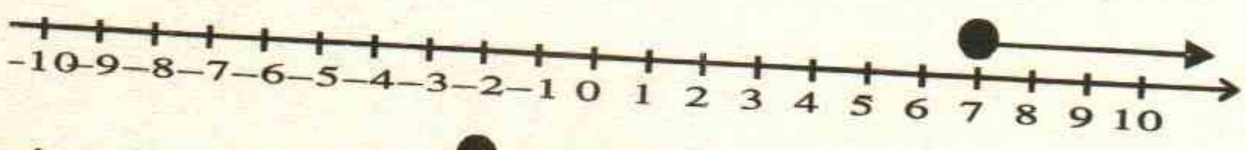
(i) 

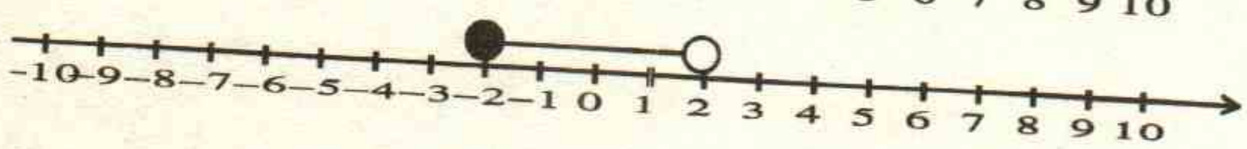
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(ii) 

(iii) 

(iv) 

(v) 

(vi) 

Review Exercise 7

1. (i) c (ii) a (iii) a (iv) b (v) b
(vi) b (vii) a (viii) c (ix) d (x) d

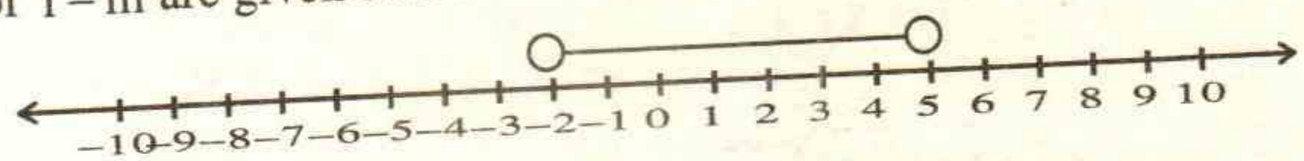
2. (i) $\{-1\}$ (ii) $\{5\}$ (iii) $\{2\}$ (iv) $\{1,-8\}$

3. (i) $\{x 1 x \in R, 1 < x < 3]$ (ii) $\{x \mid x \in R, -1 < x < 4\}$
(iii) $\{x \mid x \in R, -4 \leq x < -2\}$

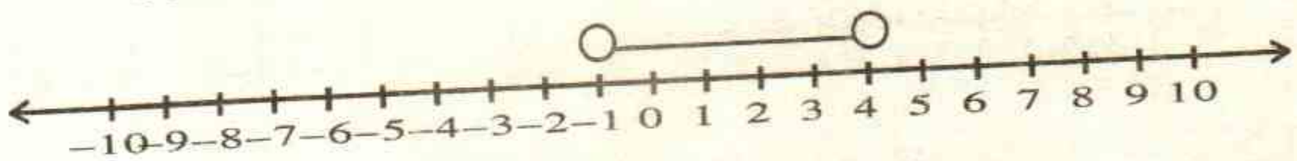
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The graphs of i – iii are given below.

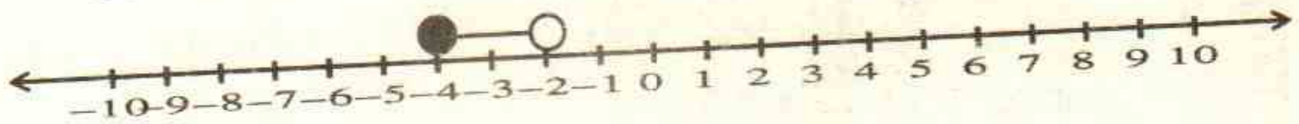
(i)



(ii)



(iii)



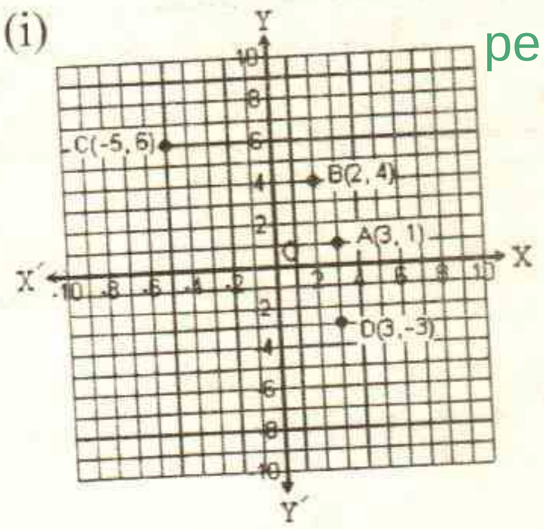
4. Father's age = 40 years, son's age = 10 years.

Exercise 8.1

1) (i) $x = -7$, $y = 5$ (ii) $x = 0$, $y = 7$ (iii) $x = -3$, $y = 8$
(iv) $x = -3$, $y = -3$ (v) $x = 10$, $y = 12$

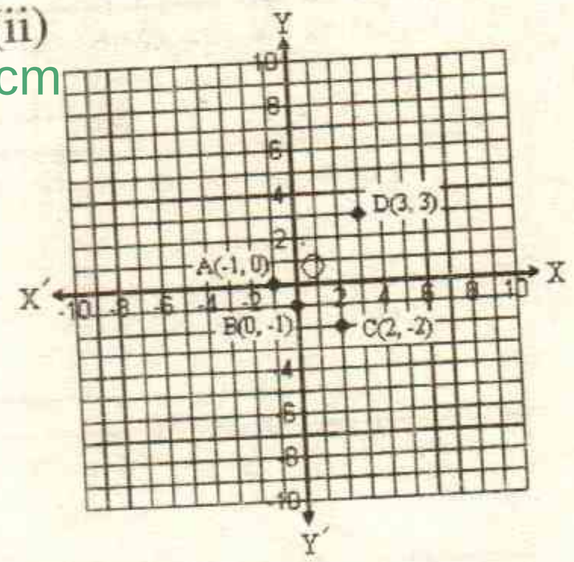
2) (i) Quadrant II (ii) Quadrant III (iii) Quadrant I
(iv) Quadrant IV (v) Quadrant II

3) (i)

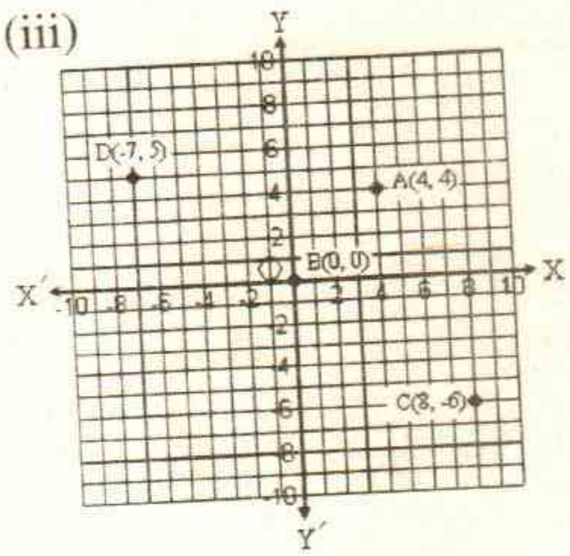


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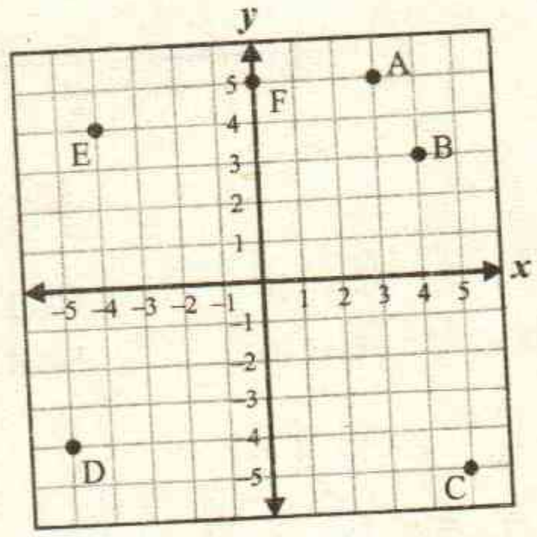
(ii)



(iii)



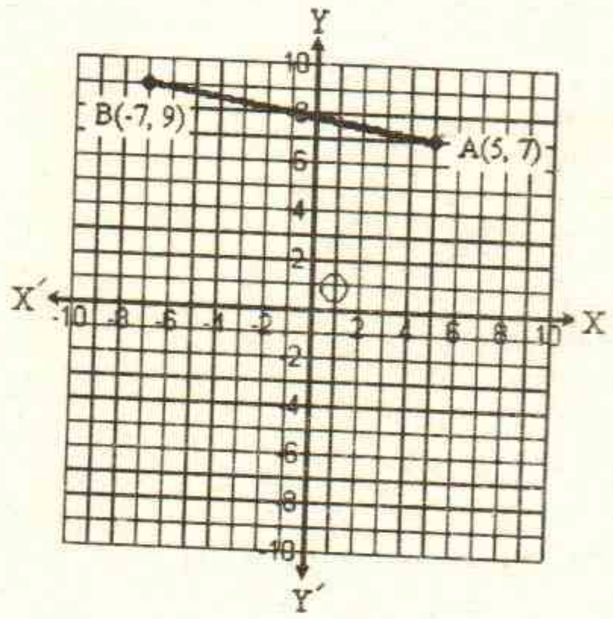
4)



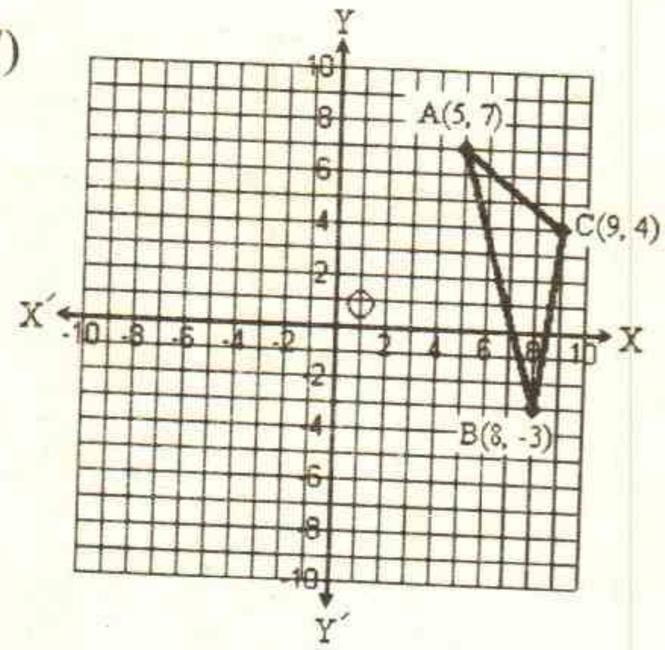
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5) A(3,2), B(4,–2), C(0,–3), D(–3, 4)

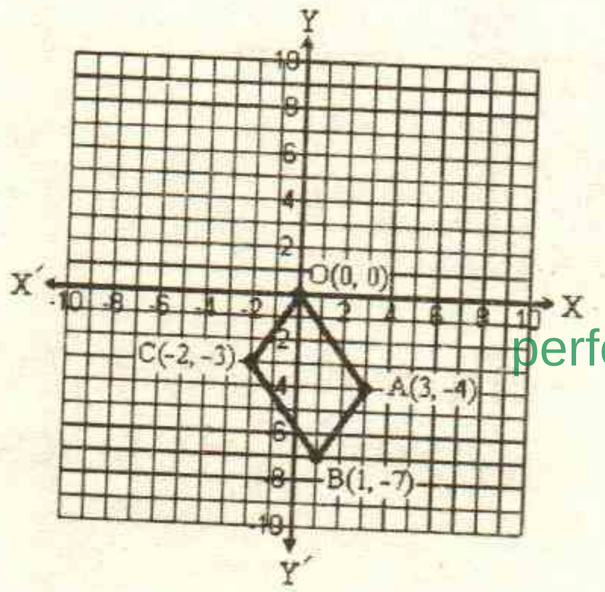
6)



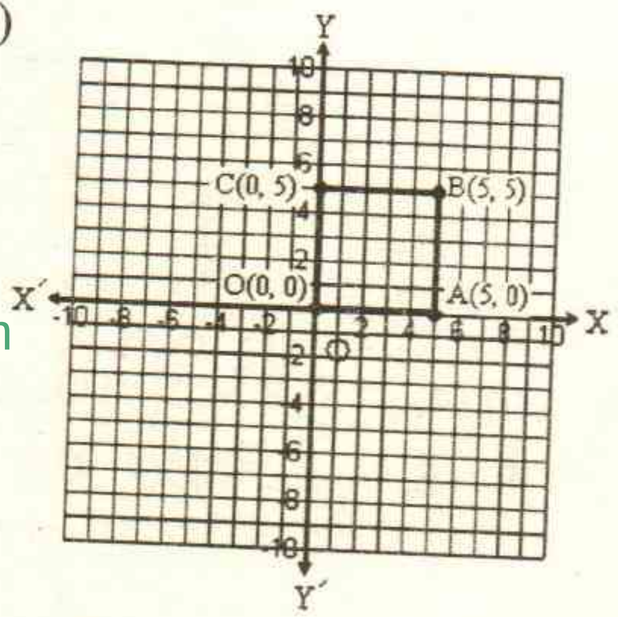
7)



8)



9)



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Exercise 8.2

1) (a) Yes (b) No (c) No d) No

2) (4,–3), (2, 0),(0, 3),(–2, 6)

3) The following is the required table of four values.

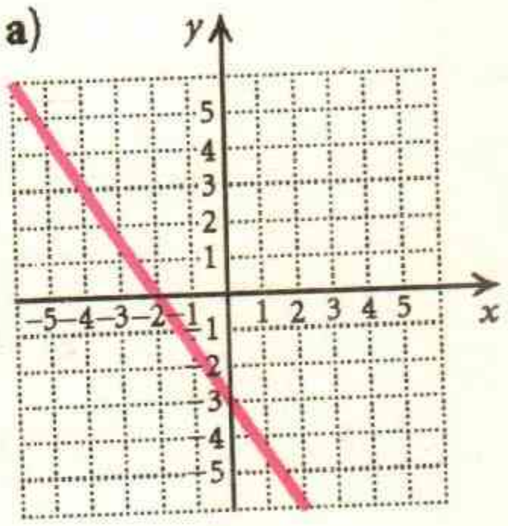
x	y
-1	-5
0	-4
1	-3
2	-2

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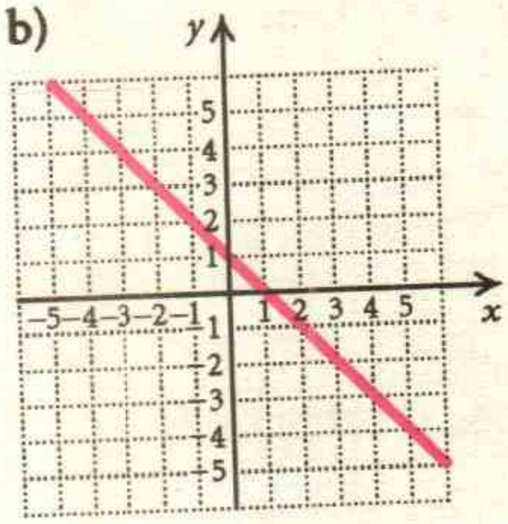
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4)

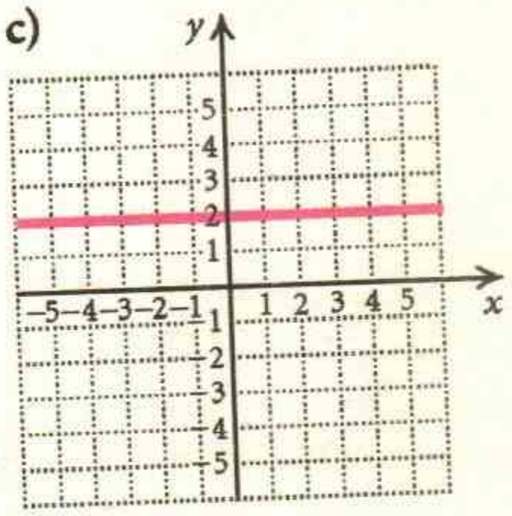
a)



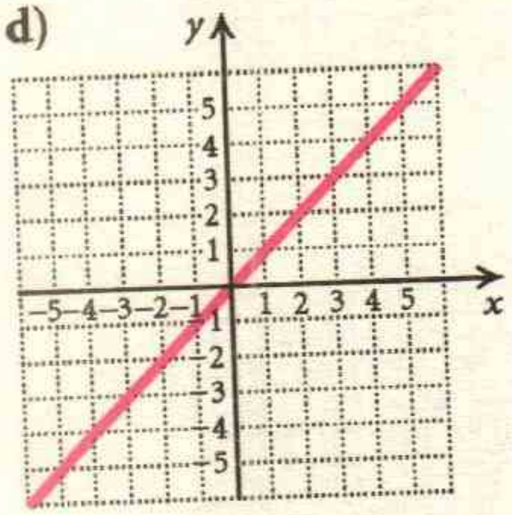
b)



c)



d)



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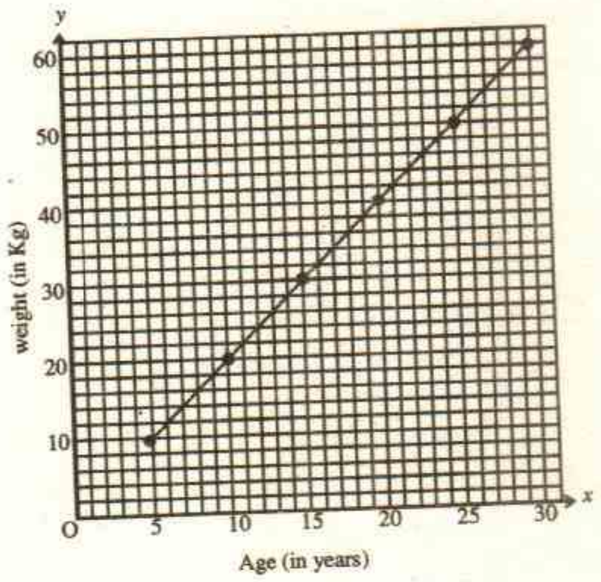
5) i) (42, 15), (14,3), (−2, −27/7)

ii) (−5, 12), (10, −6), (5/3, 4)

iii) (6, 0), (0, 18), (4 , 0)

iv) (21/2, 2), (14, 0), (08)

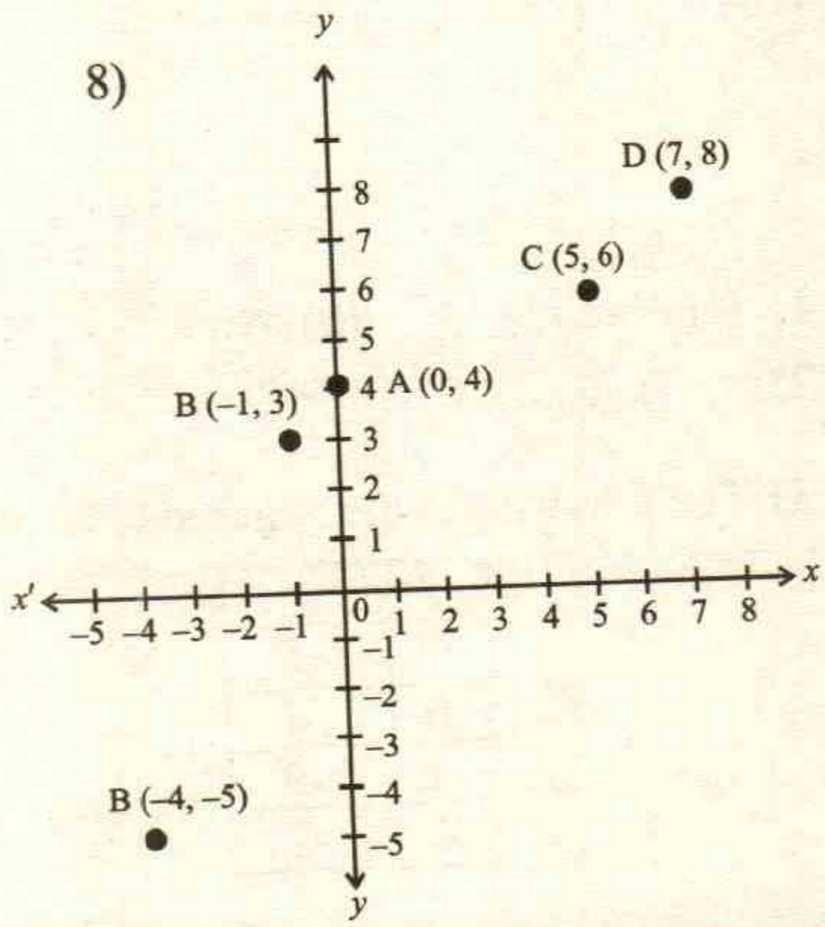
6)



7) (i) C = Rs. 2500

(ii) n = 150 units

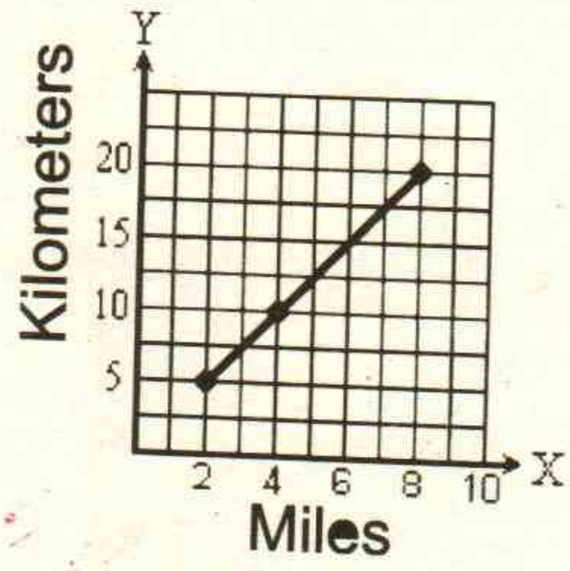
8)



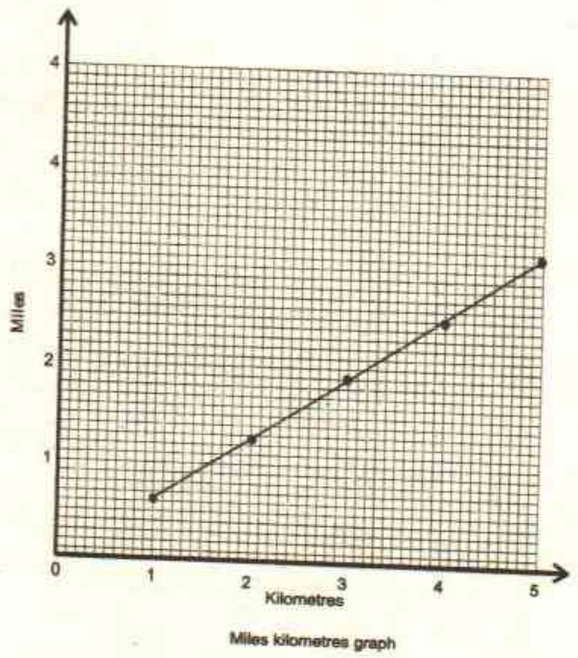
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Exercise 8.3

1)

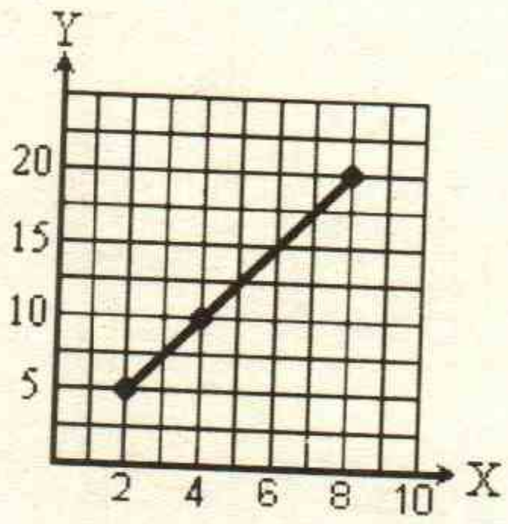


2)

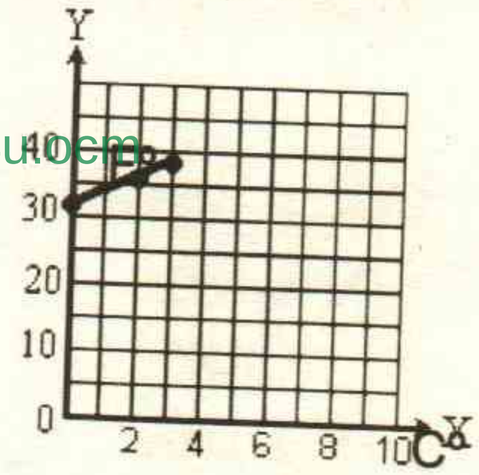


Miles kilometres graph

3)

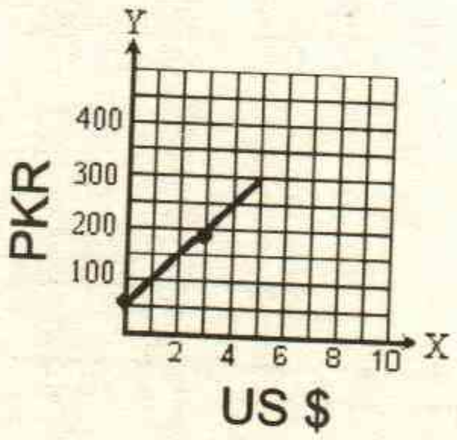


4) Fahrenheit Temperature
$32F^0$, $35.6F^0$, $37.4F^0$.

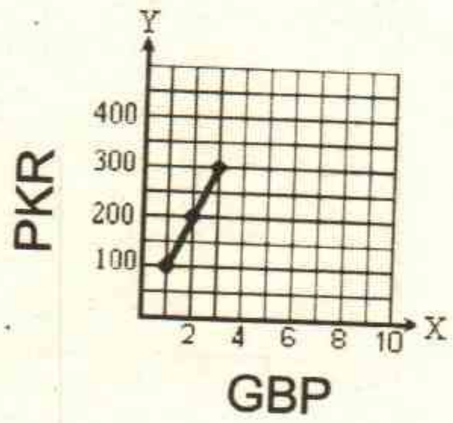


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5) (i)



(ii)



6) (i) {(1, 1)}
(ii) {(–3, –4)}
(iii) {(–3, 2)}
(iv) { }
(v) {(5, 1)}

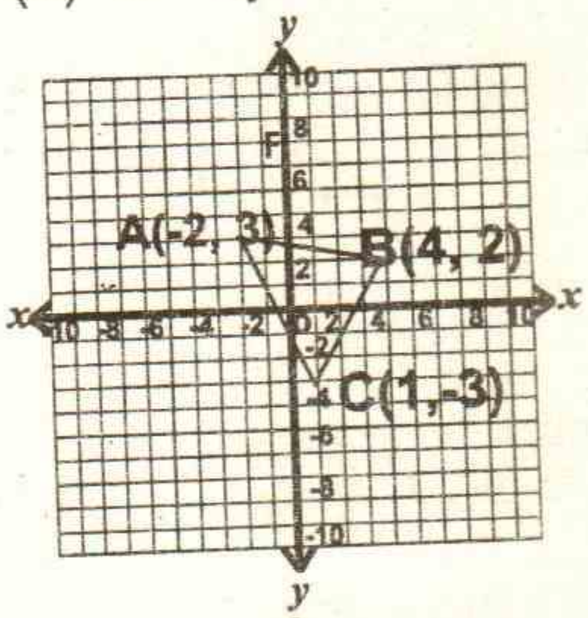
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Review Exercise 8

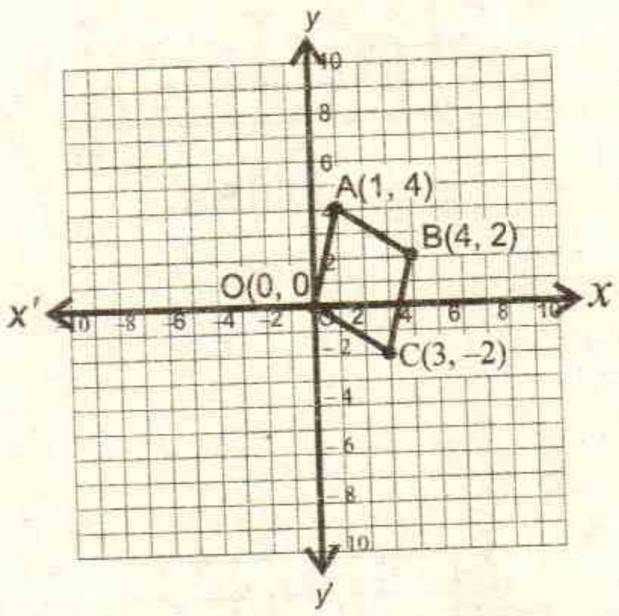
1. (i) c (ii) c (iii) b (iv) a (v) c
(vi) a (vii) a (viii) d (ix) b (x) b

2. (i) $x=2,\ y=3$, Quadrant I (ii) $x=-4,\ y=5$, Quadrant III
(iii) $x=4, y=0$, It lies on X- axis

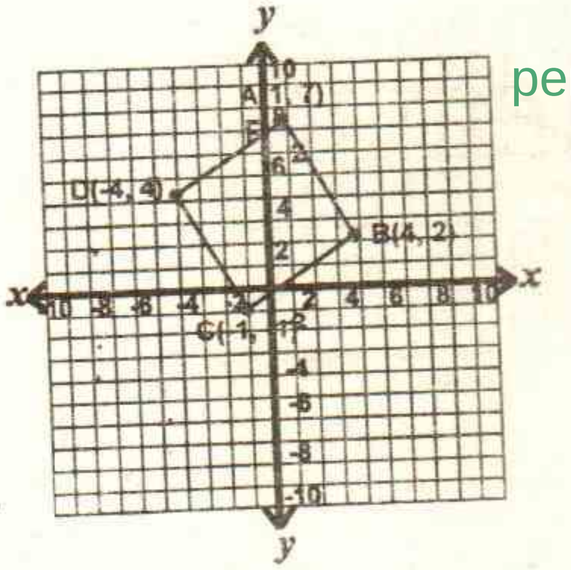
3.



4.

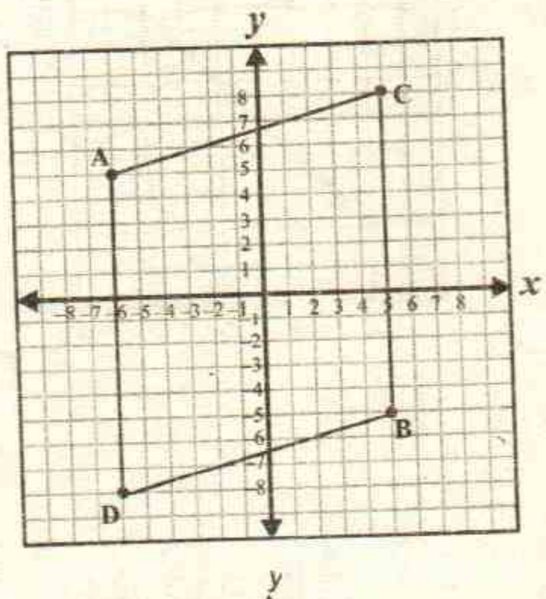


5.

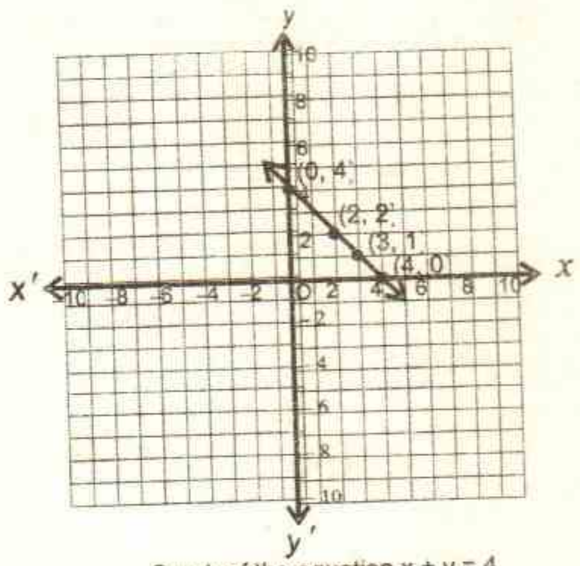


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6.

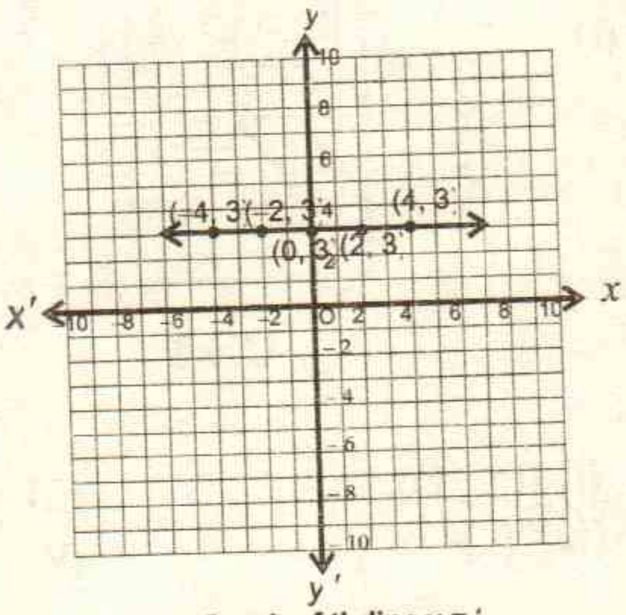


7.



Graph of the equation x + y = 4

8.

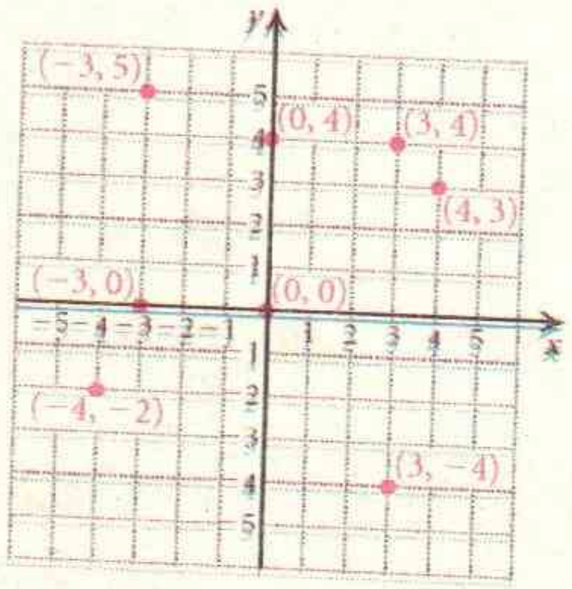


Graph of the line y = 3

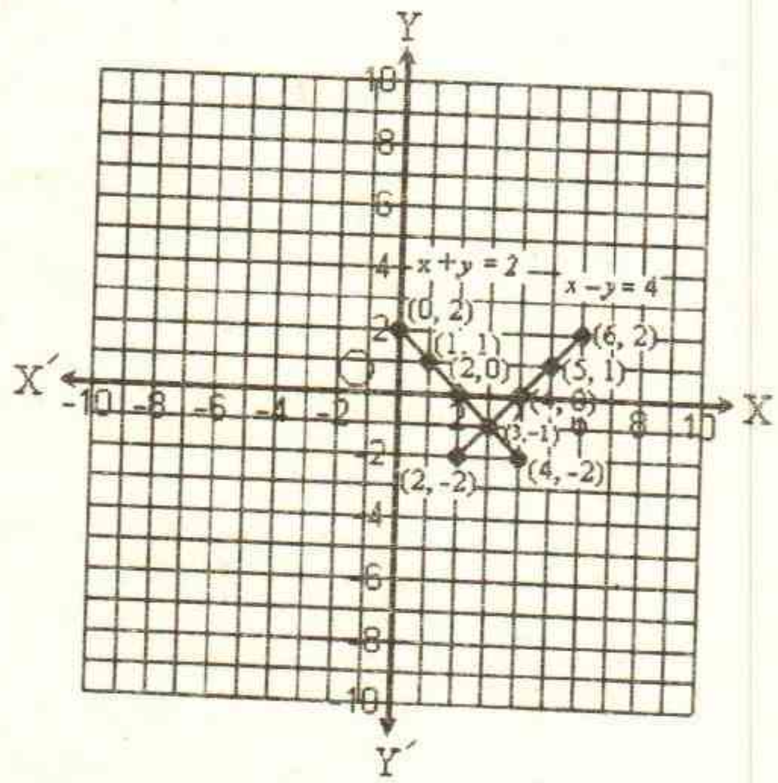
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9.



10.



Exercise 9.1

1. (i) 4 (ii) 2 (iii) 3 (iv) 5 (v) 5
   (vi) 2

2. i) $2\sqrt{2}$ ii) $3\sqrt{2}$ iii) 1 iv) $2\sqrt{2}$

3. (i) 5 (ii) 13 (iii) 17 (iv) $\sqrt{113}$ (v) $8\sqrt{2}$ (vi) $12\sqrt{2}$ (vii) $2\sqrt{68}$

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Exercise 9.2

7. 5 9. -3or8 10. (−2,2)

Exercise 9.3

1. (i) (3, 2) (ii) (5, 4) (iii) (−11/2, −3/2) (iv) (b/2, a/2)

2. (2, 12) 3. (−5/2, 5/2), (9/2, 15/2), (9/2, −3/2) 4. 5, −3

Review Exercise 9

1. (i) c (ii) b (iii) c (iv) d (v) d
   (vi) a (vii) b (viii) d (ix) a (x) c

2. 10 3. $6\sqrt{5}$ 5. $\left(\frac{3}{2}, 2\right)$ 6. $(-5, 14)$

Exercise 10.1

8. $x = 4$

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Review Exercise 10

1. i) d ii) d iii) d iv) c v) b

Exercise 11.1

1. 70°, 110°, 110° 2. 55°, 55°, 125°, 125° 3. 28

4. i) $x = 9,\ y = 15$ ii) $n = 12,\ m = 5$ iii) $p = 60^0$
   iv) $z = 28,\ d = 126$

5. i) $x = 13$ ii) $x = 10$ iii) $x = 3$ 7. 120^0

10. i) $\overline{FC} = 12$ ii) $\overline{BF} = 9$ iii) $\overline{AG} = 10$ iv) $\overline{GE} = 5$

Review Exercise 11

1. i) b ii) c iii) a iv) c v) b
   vi) a vii) a viii) d ix) d x) b
2. $60^0, 120^0, 120^0$ 3. $50^0, 50^0, 130^0, 130^0$ 5. 110^0
6. $130^0, 50^0, 130^0$

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Exercise 12.1

4. AB = 15 5. i) $\overline{AB} = \overline{BC}$ ii) $x = 5$ iii) $AB = 25$
6. i) No ii) Yes iii) No
7. i) 5 ii) 4 iii) 8

Review Exercise 12

1. i) d ii) b iii) b iv) c v) a
   vi) b vii) d viii) b

Exercise 13.1

1. x is greater than 5 and less than 19 2. not possible
4. x is greater than 12 and less than 34
6. i) side opposite to 70^0 ii) side opposite to 45^0 iii) side opposite to 70^0
7. i) ST ii) RS 8. a), b), d), f).

Review Exercise 13

1. i) b ii) c iii) d iv) d v) c
   vi) b vii) a viii) a
2. greater than 4 inches and less than 20 inches 3. No

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Exercise 14.1

1. 20° 2. 80° 5. 2.6 *cm*

6. $x = 10.4$ 7. $\overline{DC} = \frac{120}{19} cm$, $\overline{BD} = \frac{108}{19} cm$

8. $\overline{PQ} = 6cm$, $\overline{PR} = 14cm$ 9. 2:3 10. 3 cm & 3.5 cm (approx)

11. $x = 24$ 12. $\overline{ED} = 3cm$, $\overline{EF} = 7.5cm$ 14. $\overline{RL} = 6$

Review Exercise 14

1. i) d ii) *c* iii) c iv) d v) b

3. $\overline{AC} = 22\,cm$

Exercise 15.1

1. i) 13 cm ii) 6.5 *cm* iii) 17 *cm*
2. i) 7.8 cm ii) 16.9 cm iii) 8.4 cm
3. 6.5 m 4. 2 m
5. i) $\sqrt{65}$cm ii) 24 cm iii) $3\sqrt{5}$cm

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6. i) This a right-angled triangle. ii) This is not a right-angled triangle.
   iii) This is not a right-angled triangle. iv) This a right-angled triangle.
7. $10\sqrt{34}$ m

Review Exercise 15

1. i) c ii) b 2. $4\sqrt{3}\,cm$ 3. 13 cm 5) $x = 12$

Exercise 16.1

2. i) $\frac{1}{8}$th ii) $\frac{1}{2}$ iii) $\frac{3}{8}$th 4. 9720

Review Exercise 16

1. i) b ii) a iii) c iv) d
   v) d vi) a vii) b viii) b ix) c

5. 120 sq. inches

Review Exercise 17

1. i) d ii) d iii) b iv) c v) a
   vi) a vii) b viii) c ix) d x) d

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